A NOVEL ALGORITHM FOR ASYMPTOTIC STABILITY ANALYSIS OF SOME CLASSES OF STOCHASTIC TIME-FRACTIONAL VOLTERRA EQUATIONS

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Abstract. The article presents a regularization method to study global asymptotic stability of stochastic multi-time scale Volterra equations as an alternative to the algorithms based on Lyapunov functionals. Two different concepts of stability were linked and examined. The central idea of the method is based on a parallelism between the Lyapunov stability and a stochastic version of the input-to-state stability, which is well-known in the control theory of deterministic ordinary differential equations. At the first step the algorithm transforms the given delay equation into a Volterra differential equation with stochastic control, following replacement of the Lyapunov stability of the former with the input-to-state stability of the latter. To estimate the norms of the solutions we use a regularization technique based on the concept of inverse-positive matrices. This algorithm could be extended and applied for stability analysis of new classes of stochastic fractional differential equations and their applications.

Keywords: Time-fractional models, time scales, integral transforms, Lyapunov stability, input-to-state stability.

AMS Subject Classification: 93E15, 60H30, 34K50, 34D20.

1. Introduction.

Time-fractional stochastic differential models became popular in applications, and its analysis is presented in multiple highly cited monographs and articles, for example, [3],[9],[11],[20],[22], [24],[27],[30],[31] and [32].

There are several definitions of a derivative to a given fractional order (see references in [28]). The most celebrated fractional derivative operators are those of Riemann-Liouville and Caputo, each of which has pros and cons compared with the other: for example, the Riemann-Liouville derivative is always continuous and analytic with respect to the order of differentiation, while the Caputo derivative gives rise to classical initial conditions rather than fractional ones. In fact, two derivative definitions only require integer-order differentiation and fractional-order integration; so, in a nutshell, the basic operation of fractional calculus is not fractional differentiation but fractional integration. There is a long standing dispute, still going on, about the pros and cons of the different definitions (see, for example, [7],[8],[12],[13]), although, these debates are outside the scope of this paper.

In literature fractional stochastic processes are loosely classified as fractional-in-time (by using either Caputo or Riemann-Liouville time derivatives) or fractional-in-noise (fractional Wiener or Levy noise). In fact, these two models are similar, whereas stability analysis of stochastic multi-time scale delay differential equations could be tricky and challenging. Whereas Caputo-based stochastic analysis is a well-studied area [1],[2],[10],[15],[18],[19],[26],[29]; alternative notions for fractional derivatives versus the classical Riemann-Liouville definition were used in [11],[16],[23]. One of such choice is called a Jumarie-type derivative [13],[14],[17] which is just a slight modification the Riemann-Liouville definition of fractional derivative that has some specific and helpful properties for applications, e.g., in the sense that it removes the effects of the initial value of the considered function and it is defined for arbitrary continuous (non-differentiable) functions.

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The Jumarie-type derivative [13] for $0 < \alpha \le 1$ is defined as

$$f^{\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha}[f(s) - f(0)]ds,$$

and integration is given by

$$\int_0^t f(s)(ds)^{\alpha} = \alpha \int_0^t (t-s)^{\alpha-1} f(s)ds.$$

A multi-time scale integral could be expressed [14] as

$$(I_j f)(t) = \int_0^t f(s)dT_j(s),$$

where $T_j(t)$, $j = 1, 2, ..., m$, is a linearly independent set of time "scales".

The target of this report is a stochastic fractional-in-time Volterra equation defined with multiple deterministic and stochastic time scales:

$$dx(t) = \sum_{j=1}^{m} [f_j(t, (H_{1j}x)(t))(dt)^{\alpha_j} + g_j(t, (H_{2j}x)(t))dB_j(t)] \quad (t \ge 0). \tag{1}$$

Here $f_j(\omega, t, v)$ and $g_j(\omega, t, v)$ are random functions, H_{1j} and H_{2j} are linear delay operators, $0 < \alpha_j \le 1$, $dB_j(t)$ are Itô differentials generated by the standard scalar Wiener processes (Brownian motions) B_j, m is the number of the deterministic/stochastic time-scales and $x(t)$ is an unknown stochastic process on $\Re$ satisfying, in addition to (1), the initial condition

$$x(s) = \varphi(s) \quad (s \le 0), \tag{2}$$

where $\varphi(\omega, s)$ is some random function (not necessarily continuous).
Throughout the paper we tacitly assume that

$$f_j(\cdot, \cdot, 0) = 0 \quad \text{and} \quad g_j(\cdot, \cdot, 0) = 0 \quad (P \otimes \mu) - \text{almost everywhere} \tag{3}$$

(μ is the Lebesgue measure on $\Re$), which simply means that $x \equiv 0$ satisfies Eq. (1) and the initial condition (2) with $\varphi \equiv 0$. A solution of the initial value problem (1)-(2) is a progressively measurable stochastic process x almost surely satisfying (2) for μ-almost all $s \in \Re_-$ and the integral equation

$$x(t) - \varphi(0) = \sum_{j=1}^{m} \left[\int_0^t \alpha_j (t-s)^{\alpha_j - 1} f_j(s, (H_{1j}x)(s))ds + \int_0^t g_j(s, (H_{2j}x)(s))dB_j(s) \right] \tag{4}$$

for all $t \in \Re_+$.

Novel aspects of this report:

- A general class of stochastic Volterra equations in multiple time scales is introduced providing a framework for studying its global stability.
- Two distinct types of asymptotic stability notions for stochastic models were linked and examined.
- Nonlinear regularization technique based on the concept of inverse-positive matrices was used.
- We believe that our techniques is new and is of independent interest for testing stability of general classes of fractional stochastic models.

The paper is structured as follows. Section 2 contains basic notations and definition. In Section 3 two types of stability are linked, examined and discussed. In Section 4 the regularization method for nonlinear models is formulated and discussed. Finally, some concluding remarks and future venue are oulined in Section 5.

2. Preliminaries

Basic Notations:

- $\Re = (-\infty, \infty)$, $\Re_+ = [0, \infty)$, $\Re_- = (-\infty, 0)$.
- μ is the Lebesgue measure defined on $\Re$ or its subintervals.
- E is the expectation.
- $|.|$ is the fixed norm in $\Re^n$ and $\|.\|$ is the associated matrix norm $\|.\|$.
- $B_j(t)$ $(t \in \Re_+,\ j = 1, ..., m)$ are the standard scalar Brownian motions (Wiener processes).

Fixed Constants:

- $n \in N$ is the dimension of the phase space, i.e. the size of the solution vector.
- $m \in N$ is the number of the deterministic/stochastic time-scales .
- The indices i, j satisfy $1 \le i \le 2$, $1 \le j \le m$.
- $0 < \alpha_j \le 1$ define the time scales.
- p is a fixed real constant appearing in the p-stability we assume that $p \ge 2$ and $p > \alpha_j^{-1}$.

We keep fixed the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\Re}, P)$ satisfying the standard conditions [22] assuming, in addition, that $\mathcal{F}_t = \mathcal{F}_0$ for all $t \le 0$. All stochastic processes in this paper are supposed to be progressively measurable w.r.t. this stochastic basis or parts of it.

3. Two concepts of stability

Let $J \subset \Re_+$. The following spaces of random variables and stochastic processes are used:

- The space k_p^n consists of all n-dimensional, $\mathcal{F}_0$-measurable random variables $\{\xi : E|\xi|^p < \infty\}$.
- $\mathcal{L}_p(J, \Re^l)$ contains all progressively measurable l-dimensional stochastic processes $x(t)$ $(t \in J)$ such that $\int\limits_J E|x(t)|^p dt < \infty$.
- For a given positive continuous function $\gamma(t)$, $t \in J$, the space $\mathcal{M}_p^\gamma(J, \Re^l)$ consists of all progressively measurable l-dimensional stochastic processes $x(t)$ $(t \in J)$ such that
$$\sup_{t\in J} E|\gamma(t)x(t)|^p < \infty.$$
- For $l = n$ and $J = \Re_+$ we define $\mathcal{M}_p^\gamma \equiv \mathcal{M}_p^\gamma(\Re_+, \Re^n)$, and if, in addition, $\gamma = 1$, then we put $\mathcal{M}_p \equiv \mathcal{M}_p^1(\Re_+, \Re^n)$.

In the sequel, the natural norms on the spaces k_p^n, $\mathcal{L}_p(J, \Re^l)$ and $\mathcal{M}_p^\gamma(J, \Re^l)$ are used. We will also treat the last two spaces as subsets of $\mathcal{L}_p(J', \Re^l)$ and $\mathcal{M}_p^\gamma(J', \Re^l)$, respectively, where $J' \supset J$, by putting $x = 0$ outside J. In what follows, we assume that (1)-(2) has a unique solution $x(t, \varphi)$, $t \in \Re$. Specific existence and uniqueness conditions for Eq. (1) are presented in Appendix (Theorem A.4). In the well-known definition of the stochastic Lyapunov stability below we assume that $\varphi \in \mathcal{M}_p(\Re_- \cup \{0\}, \Re^n)$.

Definition 3.1. *Eq. (1) is called globally*

- *p-stable if there exists $c > 0$ such that $E|x(t, \varphi)|^p \le c \sup\limits_{s\le 0} E|\varphi(s)|^p$ for all $t \in \Re_+$;*
- *asymptotically p-stable if it is p-stable and, in addition, $\lim\limits_{t\to\infty} E|x(t, \varphi)|^p = 0$;*
- *exponentially p-stable if there exist $c > 0$ and $\beta > 0$ such that the inequality $E|x(t, \varphi)|^p \le c \exp\{-\beta t\} \sup\limits_{s\le 0} E|\varphi(s)|^p$ holds for all $t \in \Re_+$.*

To define the second kind of stability we introduce a multi-time scale stochastic Volterra equation with predefined controls:

$$dy(t) = \sum_{j=1}^{m} [(F_j(y, u_{1j}))(t)(dt)^{\alpha_j} + (G_j(y, u_{2j}))(t)dB_j(t)] \quad (t \ge 0), \tag{5}$$

where controllers $u_{ij} = u_{ij}(t, \omega)$ $(t \in \Re_+)$ belong to the space $\mathcal{M}_p(\Re_+, \Re^l)$, F_j and G_j are some nonlinear Volterra mappings [4]. Note that Eq. (5) only requires the initial condition for $t = 0$

$$y(0) = y_0 \in k_p^n. \tag{6}$$

Given $u_{ij} \in \mathcal{M}_p(\Re_+, \Re^l)$, by a solution of the control problem (5)-(6) we understand a progressively measurable stochastic process $y(t)$ almost surely satisfying the initial condition (6) and the integral equation

$$y(t) - y_0 = \sum_{j=1}^{m} \left[\int_0^t \alpha_j (t-s)^{\alpha_j - 1} F_j(y, u_{1j})(s) ds + \int_0^t G_j(y, u_{2j})(s) dB_j(s) \right] \tag{7}$$

for all $t \in \Re_+$. Two integrals here are understood in the sense of Lebesgue and Itô, respectively. In the sequel, we will tacitly assume that the restrictions on the operators F_j and G_j ensure the existence of these integrals.

It is convenient to introduce the total control space $\mathcal{U}$. Let $u = (u_{ij})$ for $u_{ij} \in \mathcal{M}_p(\Re_+, \Re^l)$ and define $\mathcal{U}$ to be the direct product of $2m$ copies of the space $\mathcal{M}_p(\Re_+, \Re^l)$, equipped with the natural norm. Let denote by $y(\cdot, y_0, u)$ a unique solution of Eq. (5) satisfying (6) on $\Re_+$. Specific conditions ensuring existence and uniqueness can be found in Appendix (Theorem A.3).

Definition 3.2. *We say that Eq. (5) is $\mathcal{M}_p^\gamma$-stable if for all $y_0 \in k_p^n$ and $u_{ij} \in \mathcal{M}_p(\Re_+, \Re^l)$*

1. *$y(\cdot, y_0, u) \in \mathcal{M}_p^\gamma$;*
2. *there exists $K > 0$ such that $||y(\cdot, y_0, u)||_{\mathcal{M}_p^\gamma} \le K(||y_0||_{k_p^n} + ||u||_{\mathcal{U}})$.*

The link between the two definitions of stability is described in Theorem 3.1, and to explore it we expose a relation between Eq. (1) and Eq. (5). For two given stochastic processes $y \in \mathcal{M}_p$ and $\varphi \in \mathcal{M}_p(\Re_- \cup \{0\})$ let us define

$$y_+(t) = \begin{cases} y(t) & (t \in \Re_+) \\ 0 & (t \in \Re_-) \end{cases} \quad \text{and} \quad \varphi_-(t) = \begin{cases} 0 & (t \in \Re_+) \\ \varphi(t) & (t \in \Re_-). \end{cases}$$

Based on Definition 3.1 and Definition 3.2 we deduce the following result.

Proposition 3.1. *Let F_j and G_j be defined by*

$$F_j(y, u) = f_j(\cdot, H_{1j} y_+ + u_{1j}), \quad G_j(y, u) = g_j(\cdot, H_{2j} y_+ + u_{2j}), \tag{8}$$

the linear operators H_{ij} map $\mathcal{M}_p(\Re, \Re^n)$ to $\mathcal{M}_p(\Re_+, \Re^l)$ and $u_{ij} = H_{ij}\varphi_-$. Then the stochastic process

$$x(t) = \begin{cases} y(t, \varphi(0), u) & (t \in \Re_+) \\ \varphi(t) & (t \in \Re_-) \end{cases} \tag{9}$$

is the solution of the initial value problem (1)-(2) if and only if y is the solution of the initial value problem (5)-(6) with $y(0) = \varphi(0)$.

Proof. Let $y(t)$ be a solution of the problem (5)-(6) and Then (9) can be rewritten as $x(t) = y_+(t, \varphi(0), u) + \varphi_-(t)$ $(t \in \Re)$, and for all $t \in \Re_+$ we obtain $x(t) = y(t)$ and $H_{ij}y_+ + H_{ij}\varphi_- = H_{ij}x$ due to linearity of H_{ij}. Hence $x(t)$ satisfies Eq. (1). In addition, $x(t) = \varphi(t)$ for $t \le 0$. Assume now that $x(t)$ is a solution of the problem (1)-(2) and put $y = x|_J$. Then $x(t) = y_+(t) + \varphi_-(t)$ $(t \in \Re_- \cup J)$, so that $H_{ij}x = H_{ij}y_+ + H_{ij}\varphi_-$, which means that $y(t)$ satisfies Eq. (5) if F_j and G_j are defined as in (8). The result then follows from the assumption $y(0) = \varphi(0)$. □

The next example will be useful both for interpreting problem (1)-(2) and understanding its transition to the control problem (5)-(6).

Example 3.1. *Let the distributed delay operators H_{ij} is given by*

$$(H_{ij}x)(t) = \int_{-\infty}^{t} d_s \mathcal{R}_{ij}(t, s) x(s),$$

where $\mathcal{R}_{ij}(t,s)$ are $n\times l$-matrix valued, Borel measurable functions defined on $\{(t,s) : t \in \Re_+,\ -\infty < s \le t\}$. The control problem for Eq. (1) reads as

$$dy(t) = \sum_{j=1}^{m}\left(f_j(t, \int_0^t d_s\mathcal{R}_{1j}(t,s)y(s) + u_{1j})(dt)^{\alpha_j} + g_j(t, \int_0^t d_s\mathcal{R}_{2j}(t,s)y(s) + u_{2j})dB_j(t)\right).$$

In particular, with time-dependent delays is given as

$$(H_{ij}x)(t) = x(h_{ij}(t)),$$

where $h_{ij}(t) \le t$ are Borel measurable functions, Eq. (1) has the following form:

$$dy(t) = \sum_{j=1}^{m}\left[f_j(t,(S_{1j}y)(s) + u_{1j})(dt)^{\alpha_j} + g_j(t,(S_{2j}y)(s) + u_{2j})dB_j(t)\right],$$

where S_{ij} are inner superposition operators (see e.g., [4]) given by

$$(S_{ij}y)(t) = \begin{cases} y(h_{ij}(t)) & (t \in \Re_+) \\ 0 & (t \in \Re_-). \end{cases}$$

A few conditions for the above delay operators satisfying the assumptions of Proposition 3.1 can be found in Appendix and in [25].

We are now in position to connect two stability concepts.

Theorem 3.1. *Assume that*

- *the linear operators $H_{ij} : \mathcal{M}_p(\Re,\Re^n) \to \mathcal{M}_p(\Re_+,\Re^l)$ to the subspace $\mathcal{M}_p(\Re_-,\Re^n)$ are bounded,*
- *the nonlinear operators F_j and G_j are defined by (8) and*
- *there exists a positive continuous function $\gamma(t)$ ($t \in \Re_+$) such that Eq. (5) is $\mathcal{M}_p^\gamma$-stable.*

Then

1. *if $\gamma(t) = 1$ ($t \in \Re_+$), then Eq. (1) is globally p-stable;*
2. *if $\lim_{t\to\infty}\gamma(t) = \infty$, $\gamma(t) \ge \delta$, $t \in \Re_+$ for some $\delta > 0$, then Eq. (1) is globally asymptotically p-stable;*
3. *if $\gamma(t) = \exp\{\beta t\}$ ($t \in \Re_+$) for some $\beta > 0$, then Eq. (1) is globally exponentially p-stable.*

The proof follows directly from Definitions 3.1 and 3.2, the formula $u_{ij} = H_{ij}\varphi_-$ and the representation (9).

4. The regularization method for nonlinear equations

To introduce and motivate our regularization technique, we briefly consider what perhaps is more "natural" approach in examination of global stability for nonlinear models. A major advantage in replacing Lyapunov stability by the input-to-state stability ($\mathcal{M}_p^\gamma$-stability in our setting) is a possibility to avoid using Lyapunov-Krasovskii functionals on the space of continuous prehistory functions or similar ones.

Despite the fact that existence of such functionals mainly sufficient for the stability, sometimes called a Lyapunov stability certificate, construction of specific functionals limited by the lack of a computable technique for generating Lyapunov functions. For instance, well-known equations with time-variable or unbounded delays are particularly difficult to treat by this method. The regularization, or the method of auxiliary equations, is an alternative to the method of Lyapunov-Krasovskii functionals. It is proved to be efficient for many classes of deterministic linear delay differential equations (see, for example, [5]). Although, most of the models treated by this method are linear and based on estimations of the norms of certain linear integral operators. Here we extend regularization method to the class of nonlinear equations. The crucial step in this generalization is the links to control theory which were established in the previous section, and possibility of using component-wise estimation of solutions instead of calculating the norm of linear operators. The key step in this method consists in choosing an auxiliary linear equation

$$dy(t) = \sum_{j=1}^{m} [((Q_j y)(t) + z_{1j}(t))(dt)^{\alpha_j} + z_{2j}(t)dB_j(t)] \quad (t \in \Re_+), \tag{10}$$

where $Q_j : \mathcal{M}_p \to \mathcal{L}_{p_j}(\Re_+, \Re^n)$ $(p_j > \frac{1}{\alpha_j})$ are k_p^1-linear operators, $z_{1j} \in \mathcal{L}_{p_j}(\Re_+, \Re^n)$ and $z_{2j} \in \mathcal{L}_2(\Re_+, \Re^n)$. Assuming the existence and uniqueness property for Eq. (10) for any initial condition (6) and using the linearity of Q_j, we obtain the following representation of its solutions:

$$y(t) = U(t)\chi(0) + (Wz)(t), \tag{11}$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, which is an $n \times n$-matrix whose columns satisfy this homogeneous equation and $U(0) = I_n$ and

$$W : \prod_{j=1}^{m} (\mathcal{L}_{p_j}(\Re_+, \Re^n) \times \mathcal{L}_2(\Re_+, \Re^n)) \to \mathcal{M}_p$$

is the Green operator for (10), $(Wz)(0) = 0$ and Wz is a solution of Eq. (10) for any z from the domain of W. Using the solutions representation of the auxiliary equation we can regularize Eq. (5) by rewriting it as

$$y(t) = U(t)y_0 + \sum_{j=1}^{m} \Big[(W_{1j}(-Q_j y + F_j(y, u_{1j})))(t) + \sum_{j=1}^{m}(W_{2j} G_j(y, u_{2j}))(t)\Big] \quad (t \in R_+). \tag{12}$$

We shall use the following definition of an M-matrix [6].

Definition 4.1. *An invertible matrix $B = (b_{\kappa\lambda})_{\kappa,\lambda=1}^{n}$ is called inverse-positive if all entries of the matrix B^{-1} are nonnegative.*

B is inverse-positive if $b_{\kappa\lambda} \leq 0$ $(1 \leq \kappa, \lambda \leq n,\ \kappa \neq \lambda)$, and one of the following conditions is satisfied:

(1) the leading principal minors of the matrix B are positive;
(2) there exist numbers $v_\kappa > 0$ $(\kappa = 1, ..., n)$ such that either

$$v_\kappa b_{\kappa\kappa} > \sum_{\lambda=1\ (\kappa\neq\lambda)}^{n} v_\lambda |b_{\kappa\lambda}| \quad (\kappa = 1, ..., n), \quad \text{or} \quad v_\lambda b_{\lambda\lambda} > \sum_{\kappa=1\ (\kappa\neq\lambda)}^{n} v_\kappa |b_{\kappa\lambda}| \quad (\lambda = 1, ..., n).$$

In particular, if $v_\kappa = 1$, $\kappa = 1, ..., n$, then we obtain the class of matrices with strict diagonal dominance and non-positive off-diagonal entries.

Recall (see e.g. [22]) that a stopping time on the given stochastic basis is a random variable $\eta : \Omega \to [-\infty, \infty]$ satisfying $\{\omega \in \Omega : \eta(\omega) \leq t\} \subset \mathcal{F}_t$ for all $t \in \Re_+$. Denote

$$z^\eta(t) = \begin{cases} z(t) & (t < \eta) \\ z(\eta) & (t \geq \eta). \end{cases}$$

If z is progressively measurable, then so is z^η.

Given a continuous function $\gamma : \Re_+ \to (0, \infty)$, an initial value $y_0 = [y_{01}, ..., y_{0n}]^T \in k_p^n$, a control $u = (u_{ij} : i = 1, 2, j = 1, ..., m)$, $u_{ij} \in \mathcal{M}_p(\Re_+, \Re^l)$, which produce the solution of Eq. (5)

$$y(t, y_0, u) = [y_1(t, y_0, u), ..., y_n(t, y_0, u)]^T$$

and a nonnegative stopping time η, we define

- $\bar{y}_0 = [\bar{y}_{01}, ..., \bar{y}_{0n}]^T$, where $\bar{y}_{0\nu} = (E|y_{0\nu}|^p)^{1/p} \equiv ||y_{0\nu}||_{k_p^1}$, and
- $\bar{y}^\eta = [\bar{y}_1^\eta, ..., \bar{y}_n^\eta]^T$, where $\bar{y}_\nu^\eta = \sup_{0 \leq t \leq \eta} (E|\gamma(t) y_\nu(t, y_0, u)|^p)^{1/p}$,

so that $\bar{y}_\nu^\eta = \bar{y}_\nu^\eta(\gamma, p)$, $\bar{y}^\eta = \bar{y}^\eta(\gamma, p)$ and $\bar{y}_\nu^\eta = \bar{y}_\nu^\eta(\gamma, p) for\ \nu = 1, ..., n$. These notations allow us to formulate and prove the following crucial result for nonlinear fractional stochastic equations in multiple-time scales.

Theorem 4.1. *Suppose there exist a real $n\times n$-matrix C and two constants $K_1>0$ and $K_2>0$ such that I_n-C is inverse-positive and for any stopping time $0\le\eta<\infty$ the vector $\bar{y}^\eta=\bar{y}^\eta(\gamma,p)$ satisfies the matrix inequality*

$$\bar{y}^\eta\le C\bar{y}^\eta+K_1\bar{y}_0+K_2\|u\|_{\mathcal{U}}e_n\qquad (e_n=[1,...,1]^T\in\Re^n). \tag{13}$$

Then Eq. (5) is $\mathcal{M}_p^\gamma$-stable.

Proof. For any $r>0$, $y_0\in k_p^n$, $u\in M_p(\Re_+,\Re^l)$ let us define the stopping time η_r by

$$\eta_r=\inf\{t>0,1\le\nu\le n:\ |\gamma(t)y_\nu(t,y_0,u)|>r\}.$$

Then $|\gamma(t)y_\nu^{\eta_r}|\le r$ almost surely for all $1\le\nu\le n$. Therefore, $\bar{y}^{\eta_r}$ is finite for all $r>0$. By (13),

$$K_1\bar{y}_0+K_2\|u\|_{\mathcal{U}}e_n-(I_n-C)\bar{y}^{\eta_r}\ge 0.$$

Multiplying this vector inequality by the matrix $(I_n-C)^{-1}$ with nonnegative entries yields

$$\bar{y}^{\eta_r}\le (I_n-C)^{-1}(K_1\bar{y}_0+K_2\|u\|_{\mathcal{U}}e_n).$$

Therefore,

$$|\bar{y}^{\eta_r}|\le K'(|\bar{y}_0|+\|u\|_{\mathcal{U}}), \tag{14}$$

where $K'=\|(I_n-C)^{-1}\|\max\{K_1,K_2|e_n|\}$ and $|.|$ is some norm in $\Re^n$.

Let us now employ the estimates in (14) and Definition 3.2 . For some fixed number ν_0 ($1\le\nu_0\le n$) and constants C_i we have

$$\begin{aligned}|\bar{y}_0|^p &\le C_1|\bar{y}_0|_\infty^p=C_1\max_\nu E|y_{0\nu}|^p=C_1E|y_{0\nu_0}|^p\le C_1E\sum_{\nu=1}^n|y_{0\nu}|^p\\ &=C_1E|y_0|_p^p\le C_2E|y_0|^p=C_2\|y_0\|_{k_p^n},\end{aligned}$$

so that

$$|\bar{y}_0|\le \text{const}\,\|y_0\|_{k_p^n}. \tag{15}$$

If $z(t)=\gamma(t)y^{\eta_r}(t,y_0,u)$ we get

$$\begin{aligned}\|y^{\eta_r}\|_{\mathcal{M}_p^\gamma}^p &=\sup_{t\ge0}E|z(t)|^p\le C_1'\sup_{t\ge0}E|z(t)|_1^p=C_1'\sup_{t\ge0}E\left(\sum_{\nu=1}^n|z_\nu(t)|\right)^p\\ &\le C_1'C_2'\sup_{t\ge0}E\sum_{\nu=1}^n|z_\nu(t)|^p\le C_2'\sum_{\nu=1}^n\sup_{t\ge0}E|z_\nu(t)|^p\\ &=C_2'|\bar{y}^{\eta_r}|_p^p\le C_3'|\bar{y}^{\eta_r}|^p\end{aligned}$$

for some constants C_i' , so that

$$\|y^{\eta_r}\|_{\mathcal{M}_p^\gamma}\le \text{const}\,|\bar{y}^{\eta_r}|. \tag{16}$$

Combination of (14) with (15) and (16) yields

$$||y^{\eta_r}||_{\mathcal{M}_p^\gamma}\le K(\|y_0\|_{k_p^n}+\|u\|_{\mathcal{U}}),$$

where K depends only on K', C_2 and C_3'. By the assumptions of the theorem the solution $y(t,y_0,u)$ is defined for all $t\in\Re_+$. Hence $\lim_{r\to\infty}\eta_r=\infty$ almost surely and

$$||y||_{\mathcal{M}_p^\gamma}=\lim_{r\to\infty}||y^{\eta_r}||_{\mathcal{M}_p^\gamma}\le K(\|y_0\|_{k_p^n}+\|u\|_{\mathcal{U}}),$$

which implies $\mathcal{M}_p^\gamma$-stability of Eq. (5). □

The estimates below may be useful for stability tests. For an arbitrary scalar, progressive measurable stochastic process $f(s)$ on $\Re_+$ and a measurable function $g:\Re_+\to\Re$ we have

$$E\left|\int_0^t f(s)dB(s)\right|^{2p}\le c_p^{2p}E\left(\int_0^t|f(s)|^2ds\right)^p\qquad (t\in\Re_+,\ p\ge1), \tag{17}$$

where $B(t)$ $(t \in \Re_+)$ is the standard scalar Brownian motion and c_p is a certain constant dependent on p, but independent of f; some explicit formulae for c_p can be found in the literature, for instance, in [21], where $c_p = 2\sqrt{12}p$, which, however, is not best possible, as evidently, $c_1 = 1$,

$$\left|\int_0^t (t-s)^{\alpha-1} g(s)ds\right|^q \leq d_q^q\, t^{q\alpha-1} \int_0^t |g(s)|^q\, ds \quad (t \in \Re_+,\ q > \alpha^{-1}), \tag{18}$$

where $d_q = \left(\frac{q-1}{q\alpha-1}\right)^{1-1/q}$,

$$E\left|\int_0^t g(s)f(s)ds\right|^q \leq \left(\int_0^t |g(s)|ds\right)^q \sup_{0\leq s\leq t} (E\,|f(s)|^q) \quad (t \in \Re_+, q \geq 1), \tag{19}$$

and

$$E\left|\int_0^t g^2(s)f^2(s)ds\right|^q \leq \left(\int_0^t g^2(s)ds\right)^{q/2} \sup_{0\leq s\leq t} (E\,|f(s)|^q) \quad (t \in \Re_+, q \geq 2). \tag{20}$$

The proofs of (18)-(20) are straightforward and based on Hölder's inequality.

The next illustrative example demonstrates applications of Theorem 4.1.

Example 4.1. *Let $1 \leq p < \infty$. Consider the following system of linear equations*

$$dx(t) = -\sum_{j=1}^{m} \left[A^{(j)} x(h_j(t))(dt)^{\alpha_j} + \sum_{\tau=1}^{m_j} A^{(j,\tau)} x(h_{j\tau}(t)) d\mathcal{B}_j(t)\right] \quad (t \geq 0), \tag{21}$$

where $A^{(j)} = (a_{sl}^{(j)})_{s,l=1}^n$, $j = 1, ..., m$, $A^{(j,\tau)} = (a_{sl}^{(j,\tau)})_{s,l=1}^n$, $j = 1, ..., m$, $\tau = 1, ..., m_i$ are real $n \times n$-matrices and $h_j, h_{j\tau}$, $j = 1, ..., m, \tau = 1, ..., m_j$ are continuous functions such that $h_j(t) \leq t, h_{j\tau} \leq t$, $t \geq 0, j = 1, ..., m, \tau = 1, ..., m_j$, $0 < \alpha_j \leq 1, j = 1, ..., m$, A^1 is a diagonal matrix with the positive diagonal entries $a_\nu^{(1)}$ and $\alpha_1 = 1$.

Let C be the $n \times n$-matrix with the entries

$$\begin{aligned} c_{\nu\kappa} &= \sum_{j=2}^{m} \left[|a_{\nu\kappa}^{(j)}|(\exp\{-\alpha_j\}(\alpha_j/a_{\nu\nu}^{(1)})^{\alpha_j} + \Gamma(\alpha_j+1)/(a_{\nu\nu}^{(1)})^{\alpha_j})\right] \\ &+ \sum_{j=1}^{m}\sum_{\tau=1}^{m_j} c_p \left[|a_{\nu\kappa}^{(j,\tau)}|/\sqrt{2a_{\nu\nu}}\right] \quad (\nu, \kappa = 1, ..., n). \end{aligned} \tag{22}$$

Then the system (21) will be globally $2p$-stable if the matrix $I_n - C$ defined by (22) is inverse-positive. Here c_p is the universal constant from the estimate (17).

We briefly sketch construction of the proof of Example 4.1.

- *The desired property of the Lyapunov stability follows from the $\mathcal{M}_p^\gamma$-stability of the associated equation (5) with predefined controls; in this example $\gamma = 1$.*
- *Eq. (5) is then regularized by means of an auxiliary equation (10); in this example we choose $Q_1 = A^{(1)}$ (see Eq. (21)) and Q_j to be the zero matrices for all $j = 2, ..., m$.*
- *The regularized counterpart of Eq. (5) becomes (in the component form)*

$$\begin{aligned} y_\nu(t) &= \exp\{-a_\nu^{(1)} t\} y_{0\nu} + \sum_{j=2}^{m} \int_0^t \alpha_j (t-s)^{\alpha_j - 1} \exp\{-a_\nu^{(1)}(t-s)\}(F_{j\nu}(y, u_{1j}))(s)ds \\ &+ \sum_{j=1}^{m} \int_0^t \exp\{-a_\nu^{(1)}(t-s)\}(G_{j\nu}(y, u_{2j}))(s)d\mathcal{B}_j(s) \quad (\nu = 1, ..., n). \end{aligned}$$

- *Using the estimates (17)-(20) for each component of y we arrive, after some technical steps, at the vector inequality (13), where the matrix C is defined by (22). Applying Theorem 4.1 concludes the proof.*

5. Concluding remarks and open problems.

This report is primarily focused on the global asymptotic stability of stochastic Volterra equations of time-fractional order, with two main motivations: examine general class of stochastic fractional models and design a new algorithm for stability analysis. The emphasis is placed on the combination of two different stability analysis methods: a regularization method for qualitative global asymptotic stability analysis, and the method based on a parallelism between the Lyapunov stability and a stochastic version of the input-to-state stability, which is well-known in the control theory of deterministic ordinary differential equations. A nonlinear analogue of the classical Bohl-Perron theorem for certain classes of fractional stochastic models was obtained. An important point for nonlinear equations stability analysis is the use of inverse-positive matrices instead of estimating the norms of operators which is more natural and useful. We illustrate efficiency of the new algorithm for examination of global asymptotic stability.

It would be interesting to use our algorithm and obtain asymptotic stability tests for stochastic differential equations of fractional order, as well as other extensions with suitable modifications. By adopting the ideas developed in this paper, it would be desirable to establish stability theorems for time-fractional (Caputo-type) stochastic differential equations driven by fractional Brownian motion.

We believe that results of this paper might be a starting point for developing alternatives to conventional Lyapunov-type theorems for stability of fractional stochastic nonlinear differential and integro-differential equations.

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Appendix A. Supplementary Results

All proofs of the results presented in this section can be found in [25].

Theorem A.1. *Let $J = [0, T]$ and $1 \le p < \infty$. Assume that $h(t)$ $(t \in J)$ is a Borel function such that $h(t) \le t$ μ-almost everywhere on J. Then the operator*

$$(Hx)(t) = x(h(t)), \tag{23}$$

is a linear bounded operator $\mathcal{M}_p(\Re_- \cup J, \Re^n) \to \mathcal{M}_p(\Re_+, \Re^n) \equiv \mathcal{M}_p$.

Theorem A.2. *Let $J = [0, T]$ and $1 < p < \infty$. Assume that the values of $\mathcal{R}(t, s)$ $(t \in J,\ -\infty < s \le t)$ are $l \times n$-matrices and $\mathcal{R}$ satisfies the following conditions:*

1. *$\mathcal{R}$ is Borel measurable on its domain;*
2. $\sup\limits_{t \in J} Var_{-\infty}^{t} \mathcal{R}(t, \cdot) < \infty$.

Then the operator

$$(Hx)(t) = \int_{-\infty}^{t} d_s \mathcal{R}(t, s) x(s). \tag{24}$$

is a linear bounded operator $\mathcal{M}_p(\Re_- \cup J) \to \mathcal{M}_p(\Re_+, \Re^l)$.

The delay operator (23) can be regarded as a particular case of the delay operator (24) if $\mathcal{R}(t, s) = \operatorname{diag}[\mathbf{1}_h, ..., \mathbf{1}_h]$ to be the $n \times n$ diagonal matrix containing the indicator $\mathbf{1}_h$ of the set $\{(t, s) :\ s \le h(t)\}$. Moreover, if we define $\mathcal{R}(t, s)$ is an $(rn) \times n$-matrix of the form

$$\mathcal{R}(t, s) = \left(\operatorname{diag}[\mathbf{1}_{h_1}, ..., \mathbf{1}_{h_1}], ..., \operatorname{diag}[\mathbf{1}_{h_r}, ..., \mathbf{1}_{h_r}]\right),$$

then the multiple delay operator $x(t) \mapsto (x(h_1(t)), ..., x(h_r(t))$.

Theorem A.3. *Let $J = [0, T]$ and assume that*

1. $0 < \alpha_j \le 1$, $p_j \ge 2$, ${\alpha_j}^{-1} < p_j \le p$ $(1 \le j \le m)$.
2. *The superposition operators generated by two non-anticipating Lipschitz operators F_j and G_j map the space $\mathcal{M}_p(J, \Re^n)$ into the spaces $\mathcal{L}_{p_j}(J, \Re^n)$ and $\mathcal{L}_2(J, \Re^n)$, respectively.*

Then the initial value problem (5)-(6) has a unique (up to the natural equivalence of indistinguishable processes) solution $y(\cdot, y_0) \in \mathcal{M}_p(J, \Re^n)$.

If the constants ℓ and b are independent of J, then the solution $y(t, y_0)$ is defined for all $t \in \Re_+$.

Theorem A.4. *Let $J = [0, T]$ and assume that*

1. $0 < \alpha_j \le 1$, $p_j \ge 2$, ${\alpha_j}^{-1} < p_j \le p$ $(1 \le j \le m)$.
2. *For all $j = 1, ..., m$ the random functions $f_j, g_j : \Omega \times \Re_+ \times \Re^l \to \Re^n$ are such that $f_j(\cdot, \cdot, v)$ and $g_j(\cdot, \cdot, v)$ are progressively measurable for any $v \in \Re^l$ and $f_j(\omega, t, \cdot)$ and $g_j(\omega, t, \cdot)$ are continuous for $P \otimes \mu$-almost all (ω, t), satisfy the Lipschitz condition*

$$|f_j(\omega, t, x_1) - f_j(\omega, t, x_2)| \le \ell |x_1 - x_2|, \quad |g_j(\omega, t, x_1) - g_j(\omega, t, x_2)| \le \ell |x_1 - x_2| \quad a.s. \tag{25}$$

   for some constant ℓ and $x_1, x_2 \in \Re^l$ for $t \in J$.

3. *The $k_{p_1}^1$-linear operators $H_{ij} : \mathcal{M}_p(\Re_- \cup J, \Re^n) \to \mathcal{L}_p(J, \Re^l)$ are bounded for all $i = 1, 2, j = 1, ..., m$.*

Then for any $\varphi \in \mathcal{M}_p(\Re_- \cup \{0\}, \Re^n)$ the initial value problem (1)-(2) has a unique (up to the natural equivalence of indistinguishable processes) solution $x(\cdot, \varphi) \in \mathcal{M}_p(J, \Re^n)$.

If the constant ℓ is independent of J, then the solution $x(t, \varphi)$ is defined for all $t \in \Re$.