EXISTENCE AND UNIQUENESS RESULTS FOR A CLASS OF STOCHASTIC FRACTIONAL DIFFERENTIAL EQUATIONS IN MULTIPLE TIME SCALES

ARCADY PONOSOV

Abstract. The paper studies existence and uniqueness of solutions to nonlinear delay differential equations driven by Jumarie and Itô differentials. The analysis is based on the properties of the singular integral operators in specially designed spaces of stochastic processes and Picard's iterative method.

Keywords: Jumarie derivatives, Brownian motion, multiple time scales.

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sec-Intro

1. Introduction.

Processes operating in a multi-time scale modus arise in a number of fields including finance, science and engineering. In [5] it was suggested to use the fractional Jumarie derivative introduced in [2] $v = \frac{df}{(dt)^\alpha}$ $(0 < \alpha \leq 1)$ and the classical white noise $g = w\frac{dB}{dt}$ to model the deterministic and the stochastic parts of the multi-time scale processes, respectively. In the integral form this reads as

$$f(t) - f(0) = \alpha \int_0^t (t-s)^{\alpha-1} v(s)ds \quad \text{and} \quad g(t) - g(0) = \int_0^t w(s)dB(s). \tag{1}$$

eq-frac-int

Adopting this approach we study the following fractional stochastic delay differential equation in multiple time scales:

$$dx(t) = \sum_{j=1}^m \left(f_j(t, (H_{1j}x)(t))(dt)^{\alpha_j} + g_j(t, (H_{2j}x)(t))dB_j(t)\right) \quad (t \in R). \tag{2}$$

SDDE

Here $f_j(t, v)$ and $g_j(t, v)$ are random functions and H_{1j}, H_{2j} are linear delay operators, $(dt)^{\alpha_j}$ are the fractional Jumarie differentials and $dB_j(t)$ are the Itô differentials generated by the standard scalar Wiener processes (Brownian motions) B_j. The initial condition for (2) is

$$x(s) = \varphi(s) \quad (s \leq 0), \tag{3}$$

prehist

where $\varphi(\omega, s)$ is some random function (not necessarily continuous).

A solution of the initial value problem (2)-(3) is a stochastic process x satisfying (3) for $s \leq 0$ and the integral equation

$$x(t) - \varphi(0) = \sum_{j=1}^m \left(\int_0^t \alpha_j (t-s)^{\alpha_j - 1} f_j(s, (H_{1j}x)(s))ds + \int_0^t g_j(s, (H_{2j}x)(s))dB_j(s) \right) \tag{4}$$

int

for all $t \geq 0$. The main result of the paper is a generalization of the existence and uniqueness theorem from [5] to the case of Eq. (2) and its operator counterpart.

sec-notation

2. Preliminaries

We keep fixed a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in R}, P)$ satisfying the standard conditions [4] assuming, in addition, that $\mathcal{F}_t = \mathcal{F}_0$ for all $t \leq 0$. All stochastic processes in this paper are supposed to be progressively measurable w.r.t. this stochastic basis or parts of it [4].

The following notation is used throughout the paper:

- $R = (-\infty, \infty)$, $R_+ = [0, \infty)$, $R_- = (-\infty, 0)$.
- μ is the Lebesgue measure defined on R or its subintervals.
- E is the expectation corresponding to the probability measure P.
- $B_j(t)$ $(t \in R_+,\ j = 1, ..., m)$ are the standard scalar Wiener processes.

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Department of Sciences and Technology, Norwegian University of Life Sciences, P. O. Box 5003, NO-1432 Ås, Norway, E-mail: arkadi@nmbu.no.

- The space $\mathcal{L}_q(J, R^l)$ $(1 \le q < \infty,\ J \subset R$ is a subinterval), contains all progressively measurable l-dimensional stochastic processes $x(t)$ $(t \in J)$ such that $\int\limits_J E|x(t)|^q dt < \infty$.
- The space $\mathcal{M}_p(J, R^l)$ $(1 \le p < \infty)$ consists of all progressively measurable, l-dimensional stochastic processes $x(t)$ $(t \in J)$ such that
$$\sup_{t \in J} E|x(t)|^p < \infty.$$
- The space k^n consists of all n-dimensional, $\mathcal{F}_0$-measurable random variables, and $k = k^1$ is a commutative ring of all scalar $\mathcal{F}_0$-measurable random variables.
- The space $k^n_p = \{\xi : \xi \in k^n, E|\xi|^p < \infty\}$ $\quad (1 \le p < \infty)$ is a linear subspace of k^n.

The spaces $\mathcal{L}_q(J, R^l)$, $\mathcal{M}_p(J, R^l)$ and k^n_p are supposed to be equipped with the natural norms. Clearly also that for $q \le p$ and finite intervals J we have $\mathcal{M}_p(J, R^l) \subset \mathcal{L}_q(J, R^l)$, but not in the topological sense.

3. Properties of some delay operators

Consider the delay operator
$$(Hx)(t) = x(h(t)), \tag{5}$$
oper-delay-1

Theorem 3.1. *Let $J = [0, T]$ and $1 \le q < \infty$. Assume that $h(t)$ $(t \in J)$ is a Borel function such that $h(t) \le t$ μ-almost everywhere on J. Then the operator (5) is a linear bounded operator from $\mathcal{M}_q(R_- \cup J, R^n)$ to $\mathcal{M}_q(J, R^n)$.*

Proof. Evidently, H is linear and maps progressively measurable processes defined on $R_- \cup J$ to the ones defined on J. In addition, $\sup\limits_{t \in J} E|x(h(t))|^q \le \sup\limits_{t \le T} E|x(t)|^q$, which proves boundedness of H from $\mathcal{M}_q(R_- \cup J, R^n)$ to $\mathcal{M}_q(J, R^n)$. The last statement of the theorem follows from the inclusion $\mathcal{M}_q(R_- \cup \{0\}, R^n) \subset \mathcal{M}_q(R_- \cup J, R^n)$. □

Next, consider the distributed delay operator
$$(Hx)(t) = \int_{-\infty}^t d_s \mathcal{R}(t, s) x(s). \tag{6}$$
oper-distr-d

Theorem 3.2. *Let $J = [0, T]$ and $1 < q < \infty$. Assume that the values of $\mathcal{R}(t, s)$ $(t \in J,\ -\infty < s \le t)$ are $l \times n$-matrices and $\mathcal{R}$ satisfies the following conditions:*

1. *$\mathcal{R}$ is Borel measurable on its domain;*
2. $\sup\limits_{t \in J} Var^t_{-\infty} \mathcal{R}(t, \cdot) < \infty$.

Then the operator (6) is a linear bounded operator from $\mathcal{M}_r(R_- \cup J)$ to $\mathcal{M}_q(J, R^l)$.

Proof. Using the componentwise description of the operator (6) we may assume, without loss of generality, that $l = n = 1$, so that $(Hx)(t) = \int_{-\infty}^t x(s) d_s \mathcal{R}(t, s)$. Evidently, the operator H maps progressively measurable processes defined on $R_- \cup J$ to the ones defined on J. Putting $\mathrm{Var}^t_{-\infty}[\mathcal{R}(t, \cdot)](s) \equiv \bar{\mathcal{R}}(t, s)$ we get
$$\begin{aligned}
\sup_{t \in J} E|\textstyle\int_{-\infty}^t x(s) d_s \mathcal{R}(t, s)|^q &\le \sup_{t \in J} \left(E \textstyle\int_{-\infty}^t |x(s)|^q d_s \bar{\mathcal{R}}(t, s) \times \left(\int_{-\infty}^t d_s \bar{\mathcal{R}}(t, s) \right)^{q-1} \right) \\
&\le \sup_{t \in J} \left(\textstyle\int_{-\infty}^t E|x(s)|^q d_s \bar{\mathcal{R}}(t, s) \right) \times \sup_{t \in J} \left(\int_{-\infty}^t d_s \bar{\mathcal{R}}(t, s) \right)^{q-1} \\
&\le \sup_{s \in R_- \cup J} E|x(s)|^q \sup_{t \in J} \left(\textstyle\int_{-\infty}^t d_s \bar{\mathcal{R}}(t, s) \right) \times \sup_{t \in J} \left(\int_{-\infty}^t d_s \bar{\mathcal{R}}(t, s) \right)^{q-1} \\
&\le \sup_{t \in R_- \cup J} E|x(t)|^q \sup_{t \in J} \left(\textstyle\int_{-\infty}^t d_s \bar{\mathcal{R}}(t, s) \right)^q \le \left(\sup_{t \in J} \mathrm{Var}^t_{-\infty} \mathcal{R}(t, \cdot) \right) \sup_{t \in R_- \cup J} E|x(t)|^q
\end{aligned}$$
which proves boundedness of H from $\mathcal{M}_q(R_- \cup J)$ to $\mathcal{M}_q(J, R^l)$. □

Remark 3.1. *The delay operator (5) can be regarded as a particular case of the delay operator (6) if we put $\mathcal{R}(t,s) = diag[\chi_h, ..., \chi_h]$ to be the $n \times n$ diagonal matrix containing the indicator χ_h of the set $\{(t,s):\ s \le h(t)\}$. Moreover, if we define $\mathcal{R}(t,s)$ to be the $(rn) \times n$-matrix of the form*

$$\mathcal{R}(t,s) = \left(diag[\chi_{h_1}, ..., \chi_{h_1}], ..., diag[\chi_{h_r}, ..., \chi_{h_r}]\right),$$

then we get the multiple delay operator $x(t) \mapsto (x(h_1(t)), ..., x(h_r(t)))$.

4. Main results

Let us first consider the following fractional functional differential equation:

$$dy(t) = \sum_{j=1}^{m} \left((F_j(y))(t)(dt)^{\alpha_j} + (G_j(y)x)(t)dB_j(t)\right) \tag{7}$$ control

equipped with the initial condition

$$y(0) = y_0 \in k_p^n. \tag{8}$$ init

The solution of Eq. (7) is understood in the following sense:

$$y(t) - y(0) = \sum_{j=1}^{m} \left(\alpha_j \int_0^t (t-s)^{\alpha_j - 1} F_j(y)(s)ds + \int_0^t G_j(y)(s)dB_j(s)\right) \quad (t \in J). \tag{9}$$ eq-int-app

def-sup-operator **Definition 4.1.** *Let X, Y be two separable metric spaces and $\sigma : \Omega \times X \to Y$ be a random map. The operator $x(\cdot) \to \sigma(\cdot, x(\cdot))$ is called the superposition operator generated by σ.*

def-Volterra **Definition 4.2.**

- *A continuous map $V : X \to Y$, where X, Y are two separable metric spaces of functions defined on an interval $J \subset R$ is called Volterra if*
$$x_1(s) = x_2(s) \quad \Rightarrow \quad (Vx)(s) = (Vy)(s)$$
for all $x_1, x_2 \in \mathcal{X}$, any $t \in J$ and almost all $s \le t, s \in J$.
- *A map $V : \Omega \times X \to Y$ is called a random Volterra map if $V(\omega, \cdot)$ is Volterra for almost all $\omega \in \Omega$ and $V(\cdot, x)$ is $\mathcal{F}$-measurable for all $x \in X$.*
- *The superposition operator generated by a random Volterra map is defined by $x(\cdot) \mapsto V(\cdot, x(\cdot))$.*
- *A random Volterra map $V : \Omega \times X \to Y$, such that $V^t(\cdot, x)$ is $\mathcal{F}_t$-measurable for all $t \in J$ will be called non-anticipating.*

Evidently, any Volterra map V gives rise to a family of Volterra maps $V^t : X_t \to Y_t$ $(t \in J)$, where X_t and Y_t consist of the restrictions of the functions from X and Y, respectively, to $(-\infty, t] \cap J$. It is also easy to check that the superposition operators generated by random Volterra maps are continuous in probability and if V is non-anticipating, then the superposition operator generated by V preserves progressive measurability of stochastic processes.

In the proofs below we use the following inequalities:

$$E\left|\int_0^t f(s)dB(s)\right|^q \le c_q^q E\left(\int_0^t |f(s)|^2 ds\right)^{q/2} \qquad (t \in R_+,\ q \ge 2), \tag{10}$$ estimate-1

where $f(t)$ is an arbitrary scalar, progressive measurable stochastic process on R_+, $B(t)$ is the standard scalar Brownian motion and c_q is a certain constant, which is independent of f;

$$\left|\int_0^t (t-s)^{\alpha-1} g(s)ds\right|^q \le d_q^q\, t^{q\alpha-1} \int_0^t |g(s)|^q\, ds \quad (t \in R_+,\ q > \alpha^{-1}), \tag{11}$$ estimate-2

where $g:\ R_+ \to R$ is a Lebesgue measurable function and $d_q = \left(\frac{q-1}{q\alpha-1}\right)^{1-1/q}$.

Inequality (10) follows from the estimates proved in e.g. [3], while (11) is a direct consequence of Hölder's inequality.

th-exist **Theorem 4.1.** *Let $J = [0, T]$ and assume that*

(1) $0 < \alpha_j \leq 1$, $p_j \geq 2$, $\alpha_j{}^{-1} < p_j \leq p$ $(1 \leq j \leq m)$.

(2) *The superposition operators generated by the random, non-anticipating operators F_j, G_j $(1 \leq j \leq m)$ map the space $\mathcal{M}_p(J, R^n)$ into the spaces $\mathcal{L}_{p_j}(J, R^n)$ and $\mathcal{L}_2(J, R^n)$, respectively, and satisfy the Lipschitz condition*

$$||F_j y_1 - F_j y_2||_{\mathcal{L}_{p_j}(J,R^n)} \leq \ell ||y_1 - y_2||_{\mathcal{M}_p(J,R^n)}, \quad ||G_j y_1 - G_j y_2||_{\mathcal{L}_2(J,R^n)} \leq \ell ||y_1 - y_2||_{\mathcal{M}_p(J,R^n)} \quad (12)$$ eq-Lipsch

for some constant ℓ and the sub-linear growth condition

$$||F_j \xi||_{\mathcal{L}_{p_j}(J,R^n)} \leq b ||\xi||_{k_p^n}, \quad ||G_j \xi||_{\mathcal{L}_2(J,R^n)} \leq b ||\xi||_{k_p^n} \quad (13)$$ eq-bound

for some constant b and any $\xi \in k_p^n$.

Then the initial value problem (7)-(8) has a unique (up to the natural equivalence of indistinguishable processes) solution $y(\cdot, y_0) \in \mathcal{M}_p(J, R^n)$.

If the constants ℓ and b are independent of J, then the solution $y(t, y_0)$ is defined for all $t \in R_+$.

Proof. We prove this theorem for the equivalent integral equation (9). Notice that due to the Volterra property of the operators F_j and G_j we have

$$||F_j y_1 - F_j y_2||_{\mathcal{L}_{p_j}([0,t],R^n)} \leq \ell ||y_1 - y_2||_{\mathcal{M}_p([0,t],R^n)}, \quad ||G_j y_1 - G_j y_2||_{\mathcal{L}_2([0,t],R^n)} \leq \ell ||y_1 - y_2||_{\mathcal{M}_p([0,t],R^n)} \quad (14)$$ eq-Lipsch-1

for any $t \in J$. Now, the proof becomes a standard application of Picard's iterations. Put

$$y^{(\nu)}(t) = y_0 + \sum_{j=1}^{m} \left(\alpha_j \int_0^t (t-s)^{\alpha_j - 1} F_j(y^{(\nu-1)})(s) ds + \int_0^t G_j(y^{(\nu-1)})(s) dB_j(s) \right) \quad (t \in J, \ \nu \in N) \quad (15)$$ eq-Picard

and $y^{(0)} = y_0$. Using (12), (13), (14) and inequalities (10), (11) with $q = p_j$ we obtain

$$E \left| y^{(\nu+1)}(t) - y^{(\nu)}(t) \right|^p \leq K \int_0^t E \left| y^{(\nu)}(s) - y^{(\nu-1)}(s) \right|^p ds \quad (t \in J, \ \nu \in N) \quad (16)$$ eq-exist-0

and

$$E \left| y^{(1)}(t) - y^{(0)}(t) \right|^p \leq K_0 t ||y_0||_{k_p^n} \quad (t \in J). \quad (17)$$ eq-exist-1

Iterating (16) and using (17) yield

$$E \left| y^{(\nu+1)}(t) - y^{(\nu)}(t) \right|^p \leq K_0 \frac{K^\nu t^\nu}{\nu!} \quad (t \in J, \ \nu \in N), \quad (18)$$ eq-exist-2

which ensures convergence of the sequence $\{y^{(\nu)}\}$ to some y in the space $\mathcal{M}_p(J, R^n)$. The stochastic process $y(t)$ satisfies then Eq. (9) due to continuity of the operators $F_j : \mathcal{M}_p(J, R^n) \to \mathcal{L}_{p_j}(J, R^n)$ and $G_j : \mathcal{M}_p(J, R^n) \to \mathcal{L}_2(J, R^n)$ and boundedness of the linear operators $(\mathcal{I}_{1j} y)(t) = \int_0^t (t-s)^{\alpha_j - 1} y(s) ds$ and $(\mathcal{I}_{2j} y)(t) = \int_0^t y(s) dB(s)$ acting from $\mathcal{L}_{p_j}(J, R^n)$ to $\mathcal{M}_p(J, R^n)$ and from $\mathcal{L}_2(J, R^n)$ to $\mathcal{M}_p(J, R^n)$, respectively (see estimates (10)-(11)).

Assume $y_1(t)$ and $y_2(t)$ to be two solutions of Eq. (9). Then we have, exactly as in (16), that

$$E \left| y_1(t) - y_2(t) \right|^p \leq K \int_0^t E \left| y_1(s) - y_2(s) \right|^p ds \quad (t \in J),$$

and the property of uniqueness follows from Grönwall's lemma. □

To prove the existence and uniqueness theorem for (2) we represent it as Eq. (7). This is a standard procedure in the deterministic theory of functional differential equations [1]. To this end, we assume given two stochastic processes $y \in \mathcal{M}_p(J, R^n)$ and $\varphi \in \mathcal{M}_p(R_- \cup \{0\})$, put

$$y_+(t) = \begin{cases} y(t) & (t \in J) \\ 0 & (t \in R_-) \end{cases} \quad \text{and} \quad \varphi_-(t) = \begin{cases} 0 & (t \in J) \\ \varphi(t) & (t \in R_-) \end{cases}$$

and define

$$F_j(y) = f_j(\cdot, H_{1j}(y_+) + H_{1j}(\varphi_-)), \quad G_j(y) = g_j(\cdot, H_{2j}(y_+) + H_{2j}(\varphi_-)), \tag{19}$$ canonical

which yields Eq. (7).

The result below connects Eq. (2) and (7).

prop-link **Proposition 4.1.** *Assume that the k-linear operators $H_{ij} : \mathcal{M}_p(R_- \cup J, R^n) \to \mathcal{L}_p(J, R^l)$ are bounded for all $i = 1, 2, j = 1, ..., m$. Then the stochastic process*

$$x(t) = \begin{cases} y(t, \varphi(0)) & (t \in J) \\ \varphi(t) & (t \in R_-) \end{cases} \tag{20}$$ x-via-y

is the solution of the initial value problem (2)-(3) if and only if y is a solution of the initial value problem (7)-(8).

Proof. Let y be a solution of the problem (7)-(8). Then (20) can be rewritten as $x(t) = y_+(t, \varphi(0)) + \varphi_-(t)$ $(t \in R_- \cup J)$, and for all $t \in J$ we obtain $x(t) = y(t)$ and $H_{ij}(y_+) + H_{ij}(\varphi_-) = H_{ij}x$ due to linearity of H_{ij}. Hence $x(t)$ satisfies Eq. (2). In addition, $x(t) = \varphi(t)$ for $t \le 0$.

Assume now that x is a solution of the problem (2)-(3) and put $y = x|_J$. Then $x(t) = y_+(t) + \varphi_-(t)$ $(t \in R_- \cup J)$, so that $H_{ij}x = H_{ij}(y_+) + H_{ij}(\varphi_-)$, which means that $y(t)$ satisfies Eq. (7) if F_j and G_j are defined as in (19). By construction, $y(0) = \varphi(0)$, and the result follows. □

ex-canonical **Example 4.1.** *The representation (7) of Eq. (2) with the distributed delay operators H_{ij} given by*

$$(H_{ij}x)(t) = \int_{-\infty}^{t} d_s\mathcal{R}_{ij}(t, s)x(s),$$

where $\mathcal{R}_{ij}(t, s)$ are $n \times l$-matrix valued, Borel measurable functions defined on $\{(t, s) : t \in J,\ -\infty < s \le t\}$, reads as

$$dy(t) = \sum_{j=1}^{m} \left(f_j(t, \int_0^t d_s\mathcal{R}_{1j}(t, s)y(s) + u_{1j})(dt)^{\alpha_j} + g_j(t, \int_0^t d_s\mathcal{R}_{2j}(t, s)y(s) + u_{2j})dB_j(t) \right).$$

In particular, Eq. (2) with time-dependent delays given by

$$(H_{ij}x)(t) = x(h_{ij}(t)),$$

where $h_{ij}(t) \le t$ are Borel measurable functions $(i = 1, 2,\ j = 1, ..., m)$, has the following representation:

$$dy(t) = \sum_{j=1}^{m} \left(f_j(t, (S_{1j}y)(s) + u_{1j})(dt)^{\alpha_j} + g_j(t, (S_{2j}y)(s) + u_{2j})dB_j(t) \right),$$

where S_{ij}, known as inner superposition operators (see e.g. [1]), are defined as

$$(S_{ij}y)(t) = \begin{cases} y(h_{ij}(t)) & (t \in J) \\ 0 & (t \in R_-) \end{cases}$$

Now we are ready to prove the existence and uniqueness result for Eq. (2).

th-exist-SDDE **Theorem 4.2.** *Let $J = [0, T]$ and assume that*

1. $0 < \alpha_j \le 1$, $p_j \ge 2$, ${\alpha_j}^{-1} < p_j \le p$ $(1 \le j \le m)$.
2. *for all $j = 1, ..., m$ the random functions $f_j, g_j : \Omega \times R_+ \times R^l \to R^n$ are such that $f_j(\cdot, \cdot, v)$ and $g_j(\cdot, \cdot, v)$ are progressively measurable for any $v \in R^l$ and $f_j(\omega, t, \cdot)$ and $g_j(\omega, t, \cdot)$ are continuous for $P \otimes \mu$-almost all (ω, t), satisfy the Lipschitz condition*

$$|f_j(\omega, t, x_1) - f_j(\omega, t, x_2)| \le \ell|x_1 - x_2|, \quad |g_j(\omega, t, x_1) - g_j(\omega, t, x_2)| \le \ell|x_1 - x_2| \quad a.s. \tag{21}$$ eq-Lipsch-SD

   for some constant ℓ and all $x_1, x_2 \in R^l$, $t \in J$ and the sub-linear growth condition

$$|f_j(\omega, t, x)| \le b|x|, \quad |g_j(\omega, t, x)| \le b|x| \tag{22}$$ eq-bound-SDD

3. *the k-linear operators $H_{ij} : \mathcal{M}_p(R_- \cup J, R^n) \to \mathcal{L}_p(J, R^l)$ are bounded for all $i = 1, 2, j = 1, ..., m$.*

Then for any $\varphi \in \mathcal{M}_p(R_- \cup \{0\}, R^n)$ *the initial value problem (2)-(3) has a unique (up to the natural equivalence of indistinguishable processes) solution* $x(\cdot, \varphi) \in \mathcal{M}_p(J, R^n)$.

If the constant ℓ *is independent of* J, *then the solution* $x(t, \varphi)$ *is defined for all* $t \in R$.

Proof. The proof is based on Theorem 4.1. Define F_j and G_j using the formulas (19). It is easy to see that the superposition operators generated by the non-anticipating operators F_j, G_j $(1 \le j \le m)$ map the space $\mathcal{M}_p(J, R^n)$ into the space $\mathcal{L}_p(J, R^n)$, which contains both $\mathcal{L}_{p_j}(J, R^n)$ and $\mathcal{L}_2(J, R^n)$, because $p \ge \max\{2, p_j : \ j = 1, ..., m\}$. These operators satisfy the Lipschitz condition (12) and the sub-linear growth condition (13) as well. Therefore, Eq. (7) with F_j, G_j so constructed has a unique solution $y \in \mathcal{M}_p(J)$ satisfying the initial condition $y(0) = \varphi(0)$. Applying Proposition 4.1 completes the proof. □

Remark 4.1. *As* $\mathcal{M}_p([0,T], R^n) \subset \mathcal{L}_p([0,T], R^n)$, *the delay operators (5) and (6) satisfy condition (3) of Theorem 4.2.*

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