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*Dedicated to the 100th Birthday Anniversary of Nikolai Viktorovich Azbelev*

# THE $W$-METHOD IN STABILITY ANALYSIS OF STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

RAMAZAN I. KADIEV AND ARCADY PONOSOV

ABSTRACT. The $W$-method was introduced and developed by N.V. Azbelev and his students. The main idea of it is to transform a given functional differential equation to an integral equation, which is easier to handle. Using this transformation, one can study boundary value problems and asymptotic behaviour of solutions. The present review article describes a stochastic version of this method, which can be used in stability analysis of stochastic functional differential equations. The classical setting of the $W$-method, developed primarily for linear equations and based on estimation of the norm of an associated integral operator, is considered first. An alternative version of the method, which, in addition, includes the theory of positive invertible matrices, is described after that. In the latter approach, all equations of the system are transformed in different ways, depending on specific properties of the equations, and the norm of each integral operator component is estimated separately. In the classical framework, all equations are also transformed separately, but the norm of the operator is estimated without taking into account the specific behaviour of each equation. The alternative approach to the $W$-method can be also applied to nonlinear stochastic functional differential equations. Several verifiable stability conditions are given demonstrating the efficiency of the method.



**Keywords:** Integral transforms, Lyapunov stability, stochastic functional differential equations.

**AMS Subject Classification:** 93E15, 60H30, 34K50, 34D20.

## 1. INTRODUCTION.

sec-Intro

This review paper is aimed at describing a general framework for analysis of the Lyapunov stability of stochastic hereditary equations driven by semimartingales. The core

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Dagestan Federal Research Center of the Russian Academy of Sciences & Department of Mathematics, Dagestan State University, Makhachkala 367005, Russia, E-mail: kadiev_r@mail.ru.

Department of Sciences and Technology, Norwegian University of Life Sciences, P. O. Box 5003, NO-1432 Ås, Norway, E-mail: arkadi@nmbu.no.

idea of the method is an alternative description of the Lyapunov stability in terms of stability with respect to (w.r.t.) the inputs of the associated stochastic functional differential equation (SFDE). The latter can be studied by transforming SFDE to a suitable integral equation by using a reference equation, and this is the very kernel of the $W$-method.

The name *the W-method* is due to N.V.Azbelev who used it in connection with boundary value problems. Later on, this method was applied to stability analysis of deterministic functional differential equations [7] and developed further in the series of publications [2], [4], [8], [9], see also the monographs [4], [6] for more references. In [18], the method was for the first time applied to linear stochastic functional differential equations and developed further in a number of publications, the complete list of which published before 2017 can be found in the review article [19]. An essential feature of the $W$-method in the above publications is the estimation of the norm of integral operators. This idea proved to be efficient in the stability analysis of deterministic and stochastic functional differential equations. For instance, in the stochastic case we managed to treat equations, for which the techniques based on the Lyapunov-Razumikhin-type functionals [28], [29] may be difficult to apply, e.g. equations with random delays and coefficients, unbounded delays etc.

As an alternative to the classical framework of the $W$-method, A. Domoshnitsky and his colleagues has developed a method utilising the idea of positivity of solutions. The first attempt in this direction was made in [11]. Later on, this technique was successfully applied to linear deterministic problems [12]-[14]. In [15], a overview of many important results obtained by this method is offered.

In some cases the conditions for positivity of solutions can be formulated in terms of positive invertible matrices. For instance, the property of absolute stability was studied in this way in the monograph [26]. Recently, this technique was applied to stability of neural networks in [10], while in [1] and [16] it was used to study stability of high-order deterministic and stochastic differential equations with delay.

The aim of the present review article is to offer a concise presentation of the modification of the stochastic $W$–method which is based on the theory of positive invertible matrices. The main source of examples will be the area of stochastic functional differential equations. The particular stability conditions obtained by the method and cited in the paper without the proofs can be found in the publications [20]-[25].

The paper is organized as follows.

Section 2 contains a description of general SDFE to be studied and an overview of the normed spaces to be used in the study. In Section 3 we describe connections between two types of stochastic stability that are crucial for the $W$-method: the stochastic Lyapunov stability and the input-output stability. Section 4 provides a short description of the stochastic $W$-method. We start with the classical approach, which is based on estimation of the norm of some linear operator, and then discuss an alternative approach which, in addition, utilizes the theory of positive invertible matrices. Section 5 contains three examples and concludes the article.

sec1

## 2. Main equations and spaces



Throughout the paper we use an arbitrary yet fixed norm $|.|$ in $R^n$ and the associated matrix norm $\|.\|$. By $I_n$ we mean the identity matrix of the size $n \times n$, while $e_n$ stands for the $n$-dimensional vector whose components are equal to 1.

In addition, we assume given a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P)$, where $\Omega$ is a set of elementary probability events, $\mathcal{F}$ is a $\sigma$-algebra of all events on $\Omega$, $(\mathcal{F})_{t\geq 0}$ is a right continuous family of $\sigma$-subalgebras of $\mathcal{F}$, $P$ is a probability measure on $\mathcal{F}$; all the above $\sigma$-algebras are assumed to be complete with respect to (w.r.t. in what follows) the measure $P$, i.e. containing all subsets of zero measure; the symbol $E$ stands below for the expectation related to the probability measure $P$.

Let $Z(t) := (z_1(t), ..., z_m(t))^T$, $t \geq 0$, be an $m$-dimensional semimartingale defined on this stochastic basis (see e.g. [27]). A simimartingale is a stochastic process, not necessarily continuous, with respect to which one can define a stochastic integral satisfying natural properties. An example of a continuous semimartingale is given by $(t, \mathcal{B}_2(t), ..., B_m(t))^T$, where $\mathcal{B}_i$ are independent standard Brownian motions (Wiener processes).

The following spaces of stochastic processes will be used in the sequel:

- The space $k^n$ consists of all $n$-dimensional, $\mathcal{F}_0$-measurable random variables, and $k = k^1$ is a commutative ring of all scalar $\mathcal{F}_0$-measurable random variables.
- The space
$$k_p^n = \{\alpha : \alpha \in k^n, E|\alpha|^p < \infty\} \quad (p \geq 1)$$
is a linear subspace of $k^n$.
- $\mathcal{L}^n$ consists of all predictable $n \times m$-matrix stochastic processes on $[0, \infty)$, the rows of which are locally integrable w.r.t. the semimartingale $Z$ (see e.g. [27] for the details).

- $\mathcal{D}^n$ consists of all $n$-dimensional stochastic processes on $[0,\infty)$, which can be represented as

$$x(t) = x(0) + \int_0^t H(s)dZ(s),$$

where $x(0) \in k^n$, $H \in \mathcal{L}^n$.

- For any positive continuous function $\gamma(t)$, $t \geq 0$, we put

$$M_p^\gamma = \{x : x \in \mathcal{D}^n, \sup_{t\geq 0} E|\gamma(t)x(t)|^p < \infty\}, \quad M_p^1 \equiv M_p.$$

For any $p \geq 1$ the linear spaces $k_p^n$ and $M_p^\gamma$ are normed spaces with the norms $\|\alpha\|_{k_p^n} = (E|\alpha|^p)^{1/p}$ and $\|x\|_{M_p^\gamma} = \sup_{t\geq 0}(E|\gamma(t)\ x(t)|^p)^{1/p}$, respectively. In the sequel, $k^n$ and $k_p^n$ will serve as "the spaces of initial values", while $\mathcal{D}^n$ and $M_p^\gamma$ will be interpreted as "the spaces of solutions".

In what follows, we consider $k$-linear Volterra operators in the spaces of stochastic processes. Let $\mathcal{A}$ and $\mathcal{B}$ be two $k$-linear spaces of stochastic processes defined on some interval $I$. Recall that $V : \mathcal{A} \to \mathcal{B}$ is

- *$k$-linear* if $V(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 V x_1 + \alpha_2 V x_2$ for all $x_1, x_2 \in \mathcal{A}$ and $\alpha_1, \alpha_2 \in k$;
- *Volterra* if for any stopping time $\tau : \Omega \to I$ a.s. and for any stochastic processes $x, y \in \mathcal{A}$ the equality $x(t) = y(t)$ $(t \in [0,\tau)$ a.s.) implies the equality $(Vx)(t) = (Vy)(t)$ $(t \in [0,\tau]$ a.s.). Note the asymmetry in this definition: it is needed at the discontinuity points of the stochastic processes involved.

We are dealing with the following linear stochastic functional differential equation w.r.t. a semimartingale $Z$ in this paper (abbr. SFDE):

eqno_2

$$dx(t) = [(Vx)(t) + f(t)]dZ(t)\ (t > 0), \tag{1}$$

where $f \in \mathcal{L}^n$ and $V : \mathcal{D}^n \to \mathcal{L}^n$ is a $k$-linear Volterra operator.

The initial condition for SFDE (1) reads as

eqno_2a

$$x(0) = x_0, \quad \text{where} \quad x_0 \in k^n. \tag{2}$$

By a solution of the problem (1)-(2) we understand a stochastic process $x \in \mathcal{D}^n$ satisfying the equation

$$x(t) = x_0 + (Fx)(t) \quad (t \geq 0),$$

where $(Fx)(t) = \int_0^t [(Vx)(s) + f(s)]dZ(s)$ is a Volterra operator in the space $\mathcal{D}^n$, and the integral is understood as a stochastic one w.r.t. the semimartingale $Z$ [27]. Under natural assumptions the initial value problem (1)-(2) has a unique (up to the natural equivalence of stochastic processes) solution, see e. g. [17].

Particular cases of SFDE (1) are: deterministic functional differential equations, linear ordinary and delayed stochastic differential equations, integro-differential equations driven by an arbitrary semimartingale, some classes of stochastic neutral equations, stochastic difference equations, differential equations driven by random Borel measures and other classes of hereditary and non-hereditary stochastic equations [18]. If $Z(t) = (t, \mathcal{B}_2(t), ..., \mathcal{B}_m(t))^T$, then we obtain *a linear functional differential Itô equation*, which is, of course, a particular case of SFDE (1).

Below we describe an algorithm explaining how a general stochastic hereditary equation can be represented as SFDE (1). This algorithm is an adaptation of the deterministic scheme, presented in the monograph [5], to the stochastic case.

Consider the equation

$$dx(t) = (\Gamma x)(t)dZ(t) \quad (t > 0) \tag{3}$$

coupled with the prehistory condition

$$x(s) = \varphi(s) \quad (s < 0), \tag{4}$$

where $\varphi$ belongs to some $k$-linear space $\mathcal{N}^n$ of $n$-dimensional $\mathcal{F}_0$-measurable stochastic processes defined on $(-\infty, 0)$. For $s = 0$ one also needs an initial condition for (3)-(4), which simply coincides with (2). The operator $\Gamma$ in (3) is a $k$-linear Volterra operator defined on a $k$-linear space of stochastic processes on $(-\infty, \infty)$, the restrictions of which to the subsets $(-\infty, 0)$ and $[0, \infty)$ belong to the spaces $\mathcal{N}^n$ and $\mathcal{D}^n$, respectively. The values of this operator belong to the space $\mathcal{L}^n$. Eq. (3) coupled with (4) will be addressed as *a stochastic hereditary equation.* Sometimes by stochastic functional differential equation one means Eq. (3), but we follow the approach and the terminology from the monograph [5], according to which functional differential equations are defined by (1), and in order to represent Eq. (3) as Eq. (1), the aforementioned equation should be coupled with the prehistory condition (4).

A solution of (3)-(4) is a stochastic process $x(t)$ coinciding with $\varphi(t)$ if $t < 0$ and obeying the integral equation $x(t) = x(0) + \int\limits_0^t (\Gamma x)(s)dZ(s)$ if $t \geq 0$. The existence and uniqueness of the solutions of this problem will follow from the representation of (3)-(4) as SFDE (1), see below. This representation will also justify the following terminology: the hereditary equation (3)-(4) is *homogeneous* if $\varphi = 0$.

**Remark 1.** *In most applications, the space $\mathcal{N}^n$ consists of all $n$-dimensional $\mathcal{F}_0$-measurable stochastic processes defined on $(-h, 0)$, where $0 \le h \le \infty$, and satisfied the condition $\operatorname{ess\,sup}_{s<0} E|\varphi(s)|^p < \infty$. In this case, $\mathcal{N}^n$ becomes a normed space with the natural norm.*

To represent (3)-(4) as SFDE (1) we need some additional notation. Given stochastic processes $x(t)$ $(t \ge 0)$ and $\varphi(t)$ $(t < 0)$ we put

$$x_+(t) = \begin{cases} x(t) & (t \ge 0) \\ 0 & (t < 0) \end{cases} \quad \text{and} \quad \varphi_-(t) = \begin{cases} 0 & (t \ge 0) \\ \varphi(t) & (t < 0) \end{cases}$$

and define $(Vx)(t) \equiv (\Gamma x_+)(t)$ and $f(t) := (\Gamma\varphi_-)(t)$ for $t > 0$ (i.e. we restrict the processes in both formulas from $(-\infty, \infty)$ to $(0, \infty)$). It is then easy to see that $x_+(t) + \varphi_-(t)$, defined for $t \in (-\infty, \infty)$ will be a solution of (3)-(4) if $x(t)$ $(t \in [0, \infty))$ satisfies SFDE (1). Indeed, by linearity $\Gamma(x_+ + \varphi_-) = \Gamma(x_+) + \Gamma(\varphi_-) = Vx + f$, which gives (1) with $V$ and $f$ just defined. Note that $f$ is uniquely defined by the prehistory function $\varphi$ and that $\varphi(s) = 0$ $(s < 0)$ implies $f(t) = 0$ $(t \ge 0)$, so that homogeneity of (3)-(4) is simply a consequence of homogeneity of (1).

For instance, a linear stochastic differential equation with distributed delay

$$dx(t) = (\Delta x)(t)dZ(t) \quad (t > 0),$$

where

$$(\Delta x)(t) = \left( \int_{(-\infty,t)} d_s\mathcal{R}_1(t,s)x(s), ..., \int_{(-\infty,t)} d_s\mathcal{R}_m(t,s)x(s) \right),$$

$\mathcal{R}_i$ are vector functions defined on $\{(t,s) : t \in [0,\infty),\ -\infty < s \le t\}$ for $i = 1, ..., m$, coupled with the prehistory condition

$$x(s) = \varphi(s) \quad (s < 0),$$

can be, under natural assumptions on $\mathcal{R}_i$ and $\varphi$ (see e.g. [18]), rewritten as SFDE (1), where

$$(Vx)(t) = (\textstyle\int_{[0,t)} d_s\mathcal{R}_1(t,s)x(s), ..., \int_{[0,t)} d_s\mathcal{R}_m(t,s)x(s)),$$

$$f(t) = (\textstyle\int_{(-\infty,0)} d_s\mathcal{R}_1(t,s)\varphi(s), ..., \int_{(-\infty,0)} d_s\mathcal{R}_m(t,s)\varphi(s)).$$

def2 **Definition 1.** *SFDE (1) obtained from the stochastic hereditary equation (3)-(4) in the way described above will be addressed as the canonical representation of (3)-(4).*

sec2

## 3. Two types of stability

The key idea of the $W$-method is a parallelism between the Lyapunov stability and the input-output stability. Below we look at the stochastic versions of these concepts. In the next definition it is tacitly assumed that the initial value problem for the hereditary equation (3)-(4), (2) has a unique solution $x(t, x_0, \varphi)$ for all $x_0 \in k^n$ and $\varphi \in N^n$. Then the stochastic Lyapunov stability may be defined as follows:

def1 **Definition 2.** *Given $p \geq 1$ we call the zero solution of the homogeneous hereditary equation (3)-(4) ($\Leftrightarrow \varphi(s) = 0,\ s < 0$)*

- *p-stable (w.r.t. the initial data, i.e. w.r.t. $x_0$ and the prehistory function $\varphi$) if for any $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that $E|x_0|^p + \operatorname{ess\,sup}_{s<0} E|\varphi(s)|^p < \delta$ implies $E|x(t, x_0, \varphi)|^p \leq \epsilon$ for all $t \geq 0$ and all (admissible) $\varphi$, $x_0$.*
- *Asymptotically p-stable (w.r.t. the initial data) if it is p-stable and, in addition, any $\varphi$, $x_0$ such that $E|x_0|^p + \operatorname{ess\,sup}_{s<0} E|\varphi(s)|^p < \delta$ satisfies $\lim_{t\to\infty} E|x(t, x_0, \varphi|^p = 0$;*
- *Exponentially p-stable (w.r.t. the initial data) if there exist positive constants $\bar{c}$, $\beta$ such that the inequality $E|x(t, x_0, \varphi)|^p \leq \bar{c}(E|x_0|^p + \operatorname{ess\,sup}_{s<0} E|\varphi(s)|^p) \exp\{-\beta s\}$ holds true for all $t \geq 0$ and all $\varphi$, $x_0$.*



For SFDE (1) we define input-output stability w.r.t. the input data $x_0 \in k_p^n$ and $f$ belonging to a certain linear space $B^\gamma$, which can be described as follows: given a linear subspace $B$ of the space $\mathcal{L}^n$ equipped with some norm $\|.\|_B$ and a positive and continuous function $\gamma(t)$ ($t \in [0, \infty)$) we define $B^\gamma = \{f : f \in B, \gamma f \in B\}$. This is a normed space as well with the norm $\|f\|_{B^\gamma} \equiv \|\gamma f\|_B$. In the next definition we assume again that the initial value problem (1)-(2) has a unique solution $x_f(\cdot, x_0)$ if $f \in \mathcal{L}^n$. Then we can introduce

Def4 **Definition 3.** *Let $p \geq 1$. We say that SFDE (1) is $(M_p^\gamma, k_p^n \times B^\gamma)$-stable if*

(1) *$x_f(\cdot, x_0) \in M_p^\gamma$ for any $x_0 \in k_p^n$, $f \in B^\gamma$ and*
(2) *there exists $\bar{c} > 0$ such that*

$$||x_f(\cdot, x_0)||_{M_p^\gamma} \leq \bar{c}(||x_0||_{k_p^n} + ||f||_{B^\gamma}).$$

This definition says that the solutions belong to $M_p^\gamma$ whenever $f \in B^\gamma$ and $x_0 \in k_p^n$ and that they continuously depend on $f$ and $x_0$ in the appropriate topologies.

**Remark 2.** *In the deterministic case the space $B^\gamma$ is usually Banach, so that the second condition on Definition 3 follows from the first one due to the Banach theorem on inverse* *linear operators. In the stochastic case, the space $B^\gamma$ may not be Banach, so that we do need both conditions.*

The parallelism between Definitions 2 and 3 is described in the main result of this section, which is Theorem 1 proved in [18]. The theorem explains how to choose $B^\gamma$ to get different kinds of stochastic Lyapunov stability.

Th1

**Theorem 1.** *Let $\gamma(t)$ $(t \geq 0)$ be a positive continuous function and $p \geq 1$. Assume that the canonical representation (1) of the hereditary equation (3)-(4) has the property $f(\cdot) \equiv (\Gamma\varphi_-)(\cdot) \in B^\gamma$ for all $\varphi$ satisfying*

$$\operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p < \infty \quad \text{and} \quad \|f\|_{B^\gamma} \leq K \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p$$

*, where $K > 0$ is some constant. Then*

- *if $\gamma(t) = 1$ $(t \geq 0)$ and SFDE (1) is $(M_p^\gamma, k_p^n \times B^\gamma)$-stable, then the zero solution of the homogeneous hereditary equation (3)-(4) is p-stable;*
- *if $\lim_{t\to\infty} \gamma(t) = \infty$, $\gamma(t) \geq \delta > 0$, $t \in [0,\infty)$ $(t \geq 0)$ for some $\delta$, and SFDE (1) is $(M_p^\gamma, k_p^n \times B^\gamma)$-stable, then the zero solution of the homogeneous hereditary equation (3)-(4) is asymptotically p-stable;*
- *if $\gamma(t) = \exp\{\beta t\}$ $(t \geq 0)$ for some $\beta > 0$ and SFDE (1) is $(M_p^\gamma, k_p^n \times B^\gamma)$-stable, then the zero solution of the homogeneous hereditary equation (3)-(4) is exponentially p-stable.*

The proof is close to that for the deterministic case [6].

sec3

## 4. Two versions of the stochastic $W$-method

The input-output stability of SFDE (1), implying by Theorem 1 the Lyapunov stability of the stochastic hereditary equation (3)-(4), can be studied using the $W$-method, i.e. by transforming SFDE (1) to an integral equation. A key step in this transformation is a choice of an auxiliary equation

eqno_4

$$dx(t) = [(Qx)(t) + g(t)]dZ(t) \quad (t > 0), \tag{5}$$

where $Q: D^n \to \mathcal{L}^n$ is a $k$-linear Volterra operator, and $g \in \mathcal{L}^n$. Assuming the existence and uniqueness property for this equation for any initial condition $x(0) = x_0 \in k^n$ and using the $k$-linearity of the operator $Q$ we get the following representation of the solutions of the auxiliary equation (5):

$$x(t) = U(t)x(0) + (Wg)(t) \quad (t \geq 0) \tag{6}$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, which is an $n \times n$-matrix whose columns satisfy this homogeneous equation and $U(0) = I_n$, and $W : \mathcal{L}^n \to D^n$ is the Cauchy operator, for which $(Wf)(0) = 0$ and $Wf$ is a solution of the equation (5) for any $g \in \mathcal{L}^n$.

Using the auxiliary equation let us rewrite SFDE (1) as

$$dx(t) = [(Qx)(t) + ((V - Q)x)(t) + f(t)]dZ(t) \quad (t > 0),$$

or, using the representation (6), as

$$x(t) = U(t)x(0) + (W(V - Q)x)(t) + (Wf)(t) \quad (t \geq 0).$$

Thus, we obtain the operator equation

$$x(t) = (\Theta x)(t) + U(t)x(0) + (Wf)(t) \quad (t \geq 0), \quad \text{where} \quad \Theta = W(V - Q). \tag{7}$$



**Theorem 2.** *Let $\gamma(t)$ $(t \geq 0)$ be a positive continuous function and $p \geq 1$ and assume that*

1. *$V$, $Q$ are continuous operators from $M_p^\gamma$ to $B^\gamma$;*
2. *the auxiliary equation (5) is $(M_p^\gamma, k_p^n \times B^\gamma)$-stable;*
3. *the operator $I - \Theta : M_p^\gamma \to M_p^\gamma$, defined in (7), has a bounded inverse in this space.*

*Then SFDE (1) is $(M_p^\gamma, k_p^n \times B^\gamma)$-stable as well.*

The proof follows the lines of the deterministic proof presented in [6] and can be found in [19].

Theorem 2 makes it possible to prove input-output stability of SFDE (1) using the same stability property of the auxiliary equation (5), which in practice is chosen to be simpler. Applying after that Theorem 1 gives the Lyapunov stability of the associated hereditary equation (3). This short description uncovers the very essence of the $W$-method.

Estimating the norms in (7) we obtain

$$\|x\|_{M_p^\gamma} \leq \|\Theta\|_{M_p^\gamma} \|x\|_{M_p^\gamma} + \bar{c}\|x_0\|_{k_p^n} + \hat{c}\|f\|_{B^\gamma}.$$

Thus, if $\|\Theta\|_{M_p^\gamma} < 1$, then SFDE (1) becomes $(M_p^\gamma, k_p^n \times B^\gamma)$-stable. This simple observation has been applied to various classes of stochastic hereditary equations and produced verifiable stability tests in terms of the parameters of the equations. The list of the relevant publications before 2016 can be found in the review article [19]. In the remaining part of the section we will concentrate therefore on the recent developments of the stochastic $W$-method, which adopts the idea of positivity. To this end, we need

f_M-matrix **Definition 4.** *An invertible matrix $B = (b_{ij})_{i,j=1}^m$ is called positive invertible if all entries of the matrix $B^{-1}$ are positive.*

According to [3, p. 830], $B$ is positive invertible if $b_{ij} \leq 0$ $(i, j = 1, ..., m,\ i \neq j)$, and one of the following conditions is satisfied:

(1) the leading principal minors of the matrix $B$ are positive;

(2) there exist numbers $\xi_i > 0$, $i = 1, ..., m$ such that $\xi_i b_{ii} > \sum_{j=1, i \neq j}^{m} \xi_j |b_{ij}|$, $i = 1, ..., m$;

(3) there exist numbers $\xi_i > 0$, $i = 1, ..., m$ such that $\xi_j b_{jj} > \sum_{i=1, i \neq j}^{m} \xi_i |b_{ij}|$, $j = 1, ..., m$.

In particular, if $\xi_i = 1$, $i = 1, ..., m$, then we obtain the class of matrices with strict diagonal dominance and non-positive off-diagonal entries.

Suppose that a componentwise estimation in (7) gave the following matrix inequality:

eqno_7 (8)
$$\bar{x} \leq C\bar{x} + \bar{c}||x_0||_{k_{2p}^n} e_n + \hat{c}\|f\|_{B^\gamma} e_n,$$

where $C$ is some $n \times n$-matrix and $\bar{c} > 0, \hat{c} > 0$. Then we have the following

th3 **Theorem 3.** *If $I_n - C$ is positive invertible, then SFDE (1) is $(M_p^\gamma, k_p^n \times B^\gamma)$-stable.*

*Proof.* Introducing the notation $X_i = \sup_{t \geq 0} (E|\gamma(t)x_i(t)|^p)^{1/p}$, $X = col(X_1, ..., X_n)$ we can rewrite the matrix inequality (8) as

$$X \leq (I_n - C)^{-1}(\bar{c}||x_0||_{k_{2p}^n} e_n + \hat{c}\|f\|_{B^\gamma} e_n),$$

where the matrix $(I_n - C)^{-1}$ has positive entries. Then

eqno_8 (9)
$$|X| \leq K(\|x_0\|_{k_{2p}^n} + \|f\|_{B^\gamma}),$$

where $K = ||(I_n - C)^{-1}|| \max\{\bar{c}, \hat{c}\}$. Replacing $x(t)$ with $x_f(t, x_0)$ (see the Definition 3), using the inequality $\|x_f(\cdot, x_0)\|_{M_p^\gamma} \leq |X|$ and the estimate (9) yield

$$x(\cdot, x_0) \in M_p^\gamma \quad \text{and} \quad \|x(\cdot, x_0)\|_{M_p^\gamma} \leq c(\|x_0\|_{k_p^n} + \|f\|_{B^\gamma})$$

for all $x_0 \in k_p^n$ and $f \in B^\gamma$, where $c > 0$ is some constant. This implies $(M_p^\gamma, k_p^n \times B^\gamma)$-stability of SFDE (1). □

In the next section we provide some stability tests obtained by this modification of the $W$-method. More examples can be found in [20]-[25].

## 5. Some examples

The stability tests below are all formulated in terms of positive invertibility of certain matrices. The specific conditions ensuring positive invertibility are listed in Section 4.

**Example 1.** *Consider the deterministic hereditary system*

$$\dot{x}(t) = -\sum_{j=1}^{m} A_j x(t-h_j) \quad (t \geq 0), \tag{10}$$

*coupled with the prehistory condition*

$$x(s) = \varphi(s) \quad (s < 0) \tag{11}$$

*and equipped with the initial condition*

$$x(0) = x_0 \in R^n, \tag{12}$$

*where* $A_j = (a_{sl}^j)_{s,l=1}^n, j = 1, ..., m$ *are real* $n \times n$*-matrices,* $h_j \geq 0$, $j = 1, ..., m$ *are real numbers, and* $\varphi$ *is a Borel measurable and essentially bounded function.*

*Putting* $\sum\limits_{j=1}^{m} a_{ss}^j = a_s > 0, s = 1, ..., n$ *we denote by* $C$ *the* $n \times n$*-matrix, the entries of which are defined as*

$$c_{ss} = 1 - \frac{1}{a_s} \sum_{k=1}^{m} \sum_{j=1}^{m} |a_{ss}^k| h_k |a_{ss}^j| \quad (s = 1, ..., n),$$

$$c_{sl} = -\frac{1}{a_s} \left[ \sum_{k=1}^{m} \sum_{j=1}^{m} |a_{ss}^k| h_k |a_{sl}^j| + \sum_{j=1}^{m} |a_{sl}^j| \right] \quad (s, l = 1, ..., n, s \neq l).$$

*If* $I_n - C$ *is positive invertible, then the zero solution of the homogeneous hereditary system (10)-(11) is exponentially stable with respect to the initial data.*

**Example 2.** *Consider the system of linear hereditary Itô equations with constant delays*

$$dx(t) = -\sum_{j=1}^{m_1} A_{1j} x(t - h_{1j}) dt + \sum_{i=2}^{m} \sum_{j=1}^{m_i} A_{ij} x(t - h_{ij}) d\mathcal{B}_i(t) \quad (t \geq 0), \tag{13}$$

*coupled with the prehistory*

$$x(s) = \varphi(s) \quad (s < 0) \tag{14}$$

*and equipped with the initial condition*

$$x(0) = x_0 \in k_p^n, \quad (15)$$

*where* $A_{ij} = (a_{sl}^{ij})_{s,l=1}^n, i = 1, ..., m, j = 1, ..., m_i$ *are real* $n \times n$*-matrices,* $h_{ij} \geq 0$, $i = 1, ..., m, j = 1, ..., m_i$*, are real numbers and* $\varphi$ *is a* $\mathcal{F}_0$*-measurable stochastic process such that* $\operatorname{ess\,sup}_{s<0} E|\varphi(s)|^p < \infty$.

*Putting* $\sum_{j=1}^{m_1} a_{ss}^{1j} = a_s > 0, s = 1, ..., n$*, we denote by* $C$ *the* $n \times n$*-matrix, the entries of which are defined as*

$$c_{ss} = 1 - \frac{1}{a_s} \sum_{k=1}^{m_1} \sum_{j=1}^{m_1} |a_{ss}^{1k}| h_{1k} |a_{ss}^{1j}| - \frac{c_p}{\sqrt{2a_s}} \left[ \sum_{k=1}^{m_1} \sum_{i=2}^{m} \sum_{j=1}^{m_i} |a_{ss}^{1k}| \sqrt{h_{1k}} |a_{ss}^{ij}| + \sum_{i=2}^{m} \sum_{j=1}^{m_i} |a_{ss}^{ij}| \right] \quad (s = 1, ..., n),$$

$$c_{sl} = -\frac{1}{a_s} \left[ \sum_{k=1}^{m_1} \sum_{j=1}^{m_1} |a_{ss}^{1k}| h_{1k} |a_{sl}^{1j}| + \sum_{j=1}^{m_1} |a_{sl}^{1j}| \right] - \frac{c_p}{\sqrt{2a_s}} \left[ \sum_{k=1}^{m_1} \sum_{i=2}^{m} \sum_{j=1}^{m_i} |a_{ss}^{1k}| \sqrt{h_{1k}} |a_{sl}^{ij}| + \sum_{i=2}^{m} \sum_{j=1}^{m_i} |a_{sl}^{ij}| \right] \quad (s, l = 1, ..., n, s \neq l),$$

*where* $c_p$ *is a constant from the inequality 4 on page 65 in the monograph* [27].

*If* $I_n - C$ *is positive invertible, then the zero solution of the homogeneous stochastic hereditary system (13)-(14) is exponetially 2p-stable with respect to initial data.*

To demonstrate the power of the $W$-method we consider a less classical system, which is known as semidiscrete or hybrid system. The system can be regarded as the one drive by a discontinuous semimartingale, so that the theory presented in the previous sections is fully applicable.

**Example 3.** *Let*

$$d\hat{x}(t) = -\sum_{j=1}^{m_1} A_{1j} x(t - h_{1j}) dt + \sum_{i=2}^{m} \sum_{j=1}^{m_i} A_{ij} x(t - h_{ij}) d\mathcal{B}_i(t) \quad (t \geq 0),$$

$$\tilde{x}(s+1) = \tilde{x}(s) - A_1 \sum_{j=s-d_1}^{s} x(j) h + \sum_{i=2}^{m} A_i \sum_{j=s-d_i}^{s} x(j) (\mathcal{B}_i((s+1)h) - \mathcal{B}_i(sh)) \quad (s = 0, 1, 2, ...) \quad (16)$$

*where* $x(t) = (x_1(t), ..., x_l(t), x_{l+1}([t]), ..., x_n([t]))^T$ $(t \geq 0)$ *is an* $n$*-dimensional stochastic process, which contains the* $l$*-dimensional continuous component and* $n - l$*-dimensional discrete component (*$[t]$ *is the integer part of* $x$*),* $A_{ij} = (a_{kr}^{ij})_{k,r=1}^{l,n}$, $i = 1, ..., m, j = 1, ..., m_i$

*are real* $l \times n$*-matrices,* $A_i = (a^i_{kr})^n_{k=l+1,r=1}$, $i = 1, ..., m$ *are real* $(n-l) \times n$*-matrices,* $h$ *and* $h_{ij} \geq 0, i = 1, ..., m, j = 1, ..., m_i$ *are real numbers.*

*The hereditary system (16) is equipped with the prehistory*

$$x(\varsigma) = \varphi(\varsigma) \quad (\varsigma < 0), \tag{17}$$

*and the initial condition*

$$x(0) = x_0 \in k^n_p, \tag{18}$$

*where* $\varphi(\varsigma) = col(\varphi_1(\varsigma), ..., \varphi_l(\varsigma), \varphi_{l+1}([\varsigma]), ..., \varphi_n([\varsigma]))$ $(\varsigma < 0)$ *is a* $\mathfrak{F}_0$*-measurable stochastic process such that* $\operatorname{ess\,sup}_{\varsigma<0} E|\varphi(\varsigma)|^p < \infty$.

*Setting* $\sum_{j=1}^{m_1} a^{1j}_{kk} = a_k, k = 1, ..., l$ *we define the entries* $c_{ij}$ *of the* $n \times n$*-matrix* $C$ *as*

$$c_{kk} = \frac{1}{a_k} \sum_{j=1}^{m_1} |a^{1j}_{kk}| \left( \sum_{\nu=1}^{m_1} |a^{1\nu}_{kk}| h_{1j} + c_p \sum_{i=2}^{m} \sum_{\nu=1}^{m_i} |a^{i\nu}_{kr}| \sqrt{h_{1j}} \right) + \frac{c_p}{\sqrt{2 a_k}} \sum_{i=2}^{m} \sum_{j=1}^{m_i} |a^{ij}_{kk}|, \quad k = 1, ..., l,$$

$$c_{kr} = \frac{1}{a_k} \left( \sum_{j=1}^{m_1} |a^{1j}_{kr}| \left( \sum_{\nu=1}^{m_1} |a^{1\nu}_{kr}| h_{1j} + c_p \sum_{i=2}^{m} \sum_{\nu=1}^{m_i} |a^{i\nu}_{kr}| \sqrt{h_{1j}} \right) + \sum_{j=1}^{m_1} |a^{1j}_{kr}| \right) +$$

$$\frac{c_p}{\sqrt{2 a_k}} \sum_{i=2}^{m} \sum_{j=1}^{m_i} |a^{ij}_{kr}|, \quad k = 1, ..., l, \; r = 1, ..., n, \; k \neq r,$$

$$c_{kk} = \frac{c_p(d_i+1)}{a^1_{kk}\sqrt{h}} \sum_{i=2}^{m} |a^i_{kk}|, \quad k = 1+1, ..., l,$$

$$c_{kr} = \frac{(d_1+1)|a^1_{kr}|}{a^1_{kk}} + \frac{c_p(d_i+1)}{a^1_{kk}\sqrt{h}} \sum_{i=2}^{m} |a^i_{kr}|, \quad k = 1+1, ..., l, \; r = 1, ..., n, \; k \neq r.$$

*If the matrix* $I_n - C$ *is positive invertible and, in addition,* $a_k > 0$, $k = 1, ..., l$, $a^1_{kk} > 0$, $k = l+1, ..., n$, *then the zero solution of the homogeneous stochastic hereditary system (16)-(17) is exponetially* $2p$*-stable with respect to initial data.*

Some particular cases of the above examples are considered in the papers [20] and [25] in more details.

**Remark 3.** *As it was shown in the papers* [10] *and* [23]*, the theory of positive invertible matrices incorporated into the* $W$*-method can be used to study global stability of deterministic and stochastic systems. A comprehensive nonlinear theory of the* $W$*-method suffers, however, a necessary level of generality, which is achieved in the case of the linear theory, and remains, therefore, a challenge.*

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