Pursuit-evasion dynamics for Bazykin-type predator-prey model with indirect predator taxis

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Abstract

We study a pursuit-evasion diffusive predator-prey model which combines prey-taxis in predators with evasive defense strategy of prey being capable to move in the opposite direction to the gradient of a chemical signal secreted by the predators (indirect predator taxis). The kinetic part of the model extends the Rosenzweig MacArthur predator-prey model by assuming an intraspecific competition among predators, as in the classical Bazykin model. The prey-taxis takes into account density-dependent velocity suppression of predators while chasing the prey. The assumptions enable us to prove the existence of global-in-time classical solutions for space dimension $n \leq 3$ which are not expected to exist for the Rosenzweig MacArthur model according to numerical simulations which depict a finite time blow-up of solutions for $n = 2$.
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1. Introduction

We consider a diffusive predator-prey model formulated in (1.3) which accounts for two taxis mechanisms. One of them is the prey taxis ([20,22]) which concerns a directional movement of predators towards the gradient of prey density (pursuit). Such a strategy is typical for visually foraging predators. The second one is the repulsive chemotaxis as the evasive anti-predator strat-

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egy of prey being the reaction to a chemical signal (odor of predator) secreted by predator. It manifests as moving in the opposite direction with respect to the gradient of the chemical. The latter is referred to as the indirect taxis (see e.g. [40] and [2]) in contrast to direct one implemented by predators considered in the model. Many chemicals (e.g. pheromones, kairomones) released by animals are used as means of inter and intraspecific communication and olfaction is a primary means by which prey animals detect predators [28] and trigger anti-predator responses. Many types of anti-predator responses to chemical cues are described in the literature [6,14,15,31,13,18,21,46], among which in our model evasion (escape) is considered.

From the mathematical viewpoint the structure of the model investigated in the present paper is in a sense intermediate between the full cross-diffusion pursuit-evasion model with direct taxis in prey and predators [36] and the corresponding model with indirect taxis in both predator and prey which has been recently studied in [45]. The pursuit-evasion model (with direct taxis) reads

$$\begin{cases} P_t = d_P \Delta P - \xi \nabla \cdot P \, \nabla N + f(P, N), \\ N_t = d_N \Delta N + \chi \nabla \cdot N \, \nabla P + g(P, N), \end{cases} \tag{1.1}$$

where P and N denote predator and prey densities, the functions f and g describe local predator-prey interactions, $d_P, d_N > 0$ are diffusion constants, $\xi, \eta > 0$ are taxis sensitivity parameters. The model appeared for the first time in [38] and [39] where the formal analysis of soliton solutions and stability analysis of the constant steady state where carried out.

For comparison let us consider the following model in which the direct repulsive predator-taxis is replaced by indirect repulsive predator-taxis which amounts to assume that the prey senses not the presence of predator itself but rather its odor, a diffusive chemical with density W, and use evasive strategy moving in the opposite direction with respect to the gradient of W. Similarly, the movement of predators is oriented towards the gradient of chemical with density U secreted by prey and not the prey itself;

$$\begin{cases} P_t = d_P \Delta P - \xi \nabla \cdot P \, \nabla U + f(P, N), \\ N_t = d_N \Delta N + \chi \nabla \cdot N \, \nabla W + g(N, P), \\ \tau W_t = d_W \Delta W + \gamma P - \mu W, \\ \tau U_t = d_U \Delta U + \alpha N - \beta U \end{cases} \tag{1.2}$$

where $\gamma, \alpha > 0$ and $\mu, \beta > 0$ are rate coefficients related to the production and degradation of the chemicals secreted by predator and prey respectively. Systems (1.1) and (1.2) are supplemented by initial conditions and no-flux boundary conditions describing the lack of migration through the boundary of a region where the species under consideration are distributed.

In [41] only local in time existence of solutions to (1.1) was proved, so far, provided some smallness condition on the taxis coefficients holds. The recent papers [35,36] show how difficult is the problem of global existence of solutions to (1.1) even in the case of one space dimension if there are no restrictions made on the size of the chemotaxis sensitivity coefficients and initial data.

The existence of global weak solutions to (1.2) was shown in [3] for space dimension $n \leq 2$ in a parabolic-elliptic case when the distribution of chemicals is governed by the system of elliptic equations ($\tau = 0$ in (1.2) which amounts to assume that the diffusion of the chemical happens in a much faster time scale than the movement of individuals). This result was improved in [23] where

the existence of classical solutions and its long time behavior were proved for $n \leq 3$. Recently it was proved in [45] that global classical solutions to (1.2) (with $\tau = 1$) exist under additional assumption on the chemotactic sensitivity coefficients and kinetic coefficients responsible for the dissipativity of the system which appear in the functions f and g.

On the other hand it was recently proved in [37] that global boundedness of solutions also holds for the predator-prey system with pursuit-evasion and chemical signaling under additional assumptions on highly nonlinear diffusion of species which turns out to preclude blow-up formation in a finite time.

Notice that contrary to system (1.1) the main elliptic part of the system (1.2) (with $\tau = 1$) is triangular which has crucial consequence on the prolongation of local solutions as in the case of reaction diffusion systems with a triangular elliptic part an *a priori* bound in the space $L^\infty(\Omega)$ is sufficient to prolong a local solution with initial data in the Sobolev space $W^{1,r}(\Omega)$ with $r > n$ where n is the space dimension (see [4] for details).

In the following model investigated in this paper the direct repulsive predator taxis is replaced by the indirect predator-taxis while foraging strategy of predator is based on direct taxis. More precisely, denoting the densities of the predator, prey and the chemical by $P, N, W : \Omega \times (0, \infty) \mapsto I\!R$, respectively, the model reads

$$\begin{cases} P_t = d_P \Delta P - \xi \nabla \cdot P \Psi(P) \nabla N + b F(N, P) P - \delta P - \delta_1 P^2\,, \\ N_t = d_N \Delta N + \chi \nabla \cdot N \nabla W - F(N, P) P + r N - r_1 N^2\,, \\ W_t = d_W \Delta W + \gamma P - \mu W\,, \end{cases} \tag{1.3}$$

defined in a bounded domain $\Omega \subset I\!R^n$ with smooth boundary and outer normal ν, supplemented with initial conditions

$$N(\cdot, 0) = N_0, \ \ P(\cdot, 0) = P_0, \ W(\cdot, 0) = W_0 \tag{1.4}$$

and homogeneous Neumann boundary conditions

$$\langle \nabla N, \nu \rangle = \langle \nabla P, \nu \rangle = \langle \nabla W, \nu \rangle = 0, \quad \text{on} \quad \partial\Omega, \ t > 0\,. \tag{1.5}$$

The chemotaxis term in the P-equation contains density dependent predator velocity $V = \Psi(P)\nabla N$ with $\Psi(P)$ assumed to be a decreasing function accounting for the mutual interference among predators chasing the prey. Such a modification was earlier proposed in the literature in different contexts, namely in modeling the transport of macrophages in the early stages of multiple sclerosis where

$$\Psi(P) = (1 + \sigma P)^{-1} \quad \text{for} \quad \sigma \geq 0 \tag{1.6}$$

(see [24]) or in the modeling of feather morphogenesis with $\Psi(P) = \exp\{-P\}$ (see [30]). It is convenient to rewrite the chemotaxis term in the following way

$$P V = P \Psi(P) \nabla N := G(P) \nabla N\,. \tag{1.7}$$

The function F in ecology is referred to as functional response which describes the consumption rate per predator. Typical examples of functional responses are covered by Holling type functions [16]:

$$F = F_H(N) = \frac{aN^d}{1 + \beta N^d} \quad a, \beta > 0, \quad d = 1 \text{ (Holling type II) or } d = 2 \text{ (Holling type III)},$$

(1.8)

as well as Beddington-DeAngelis functional response (cf. [11])

$$F = F_H(N) = \frac{aN}{1 + \beta N + cP}, \quad a, \beta, c > 0,$$

(1.9)

or that of Crowley–Martin (cf. [12])

$$F(N, P) = \frac{aN}{1 + \beta N + cP + dNP}, \quad a, \beta, c, d > 0.$$

(1.10)

The last two functions incorporate mutual interference of predators while consuming the prey (see e.g. [33] for a survey and comparison with experimental data). The boundedness of F is a consequence of its interpretation as the consumption rate which meets obvious physiological limitation. The coefficient b in the P-equation corresponds to the efficiency of conversion of food into offspring. The interpretation of remaining coefficients is following; the coefficients r and r_1 correspond to the birth rate and intraspecific competition in prey, respectively, while δ and δ_1 are the death rate and intra-specific competition in predator. We refer the reader to most recent survey papers which contain mathematical methods and modeling perspectives for chemotaxis systems [8,9,29] as well as to the introduction in [36] which contains an extensive overview of cross-diffusion systems in science with updated references to the literature.

It is worth underlining that the predator-prey model with bounded functional response and intraspecific competition in prey ($r_1 > 0$) and in predator ($\delta_1 > 0$) is known in the literature as the Bazykin model [7] which turns out to have much more complex dynamics than the Rosenzweig-MacArthur model [32] of prey-predator interactions in which $\delta_1 = 0$ and only $r_1 > 0$. In particular in the case of reaction-diffusion system with Bazykin kinetic part not only Hopf bifurcation is possible for the coexistence constant steady state but also Turing bifurcation as reported in the recent paper [25].

The model (1.3) extends model A from [26] where the intraspecific competition was not taken into account (i.e. $\delta_1 = 0$) and there was no additional suppression of predator velocity assumed i.e. $\Psi \equiv 1$ or $\sigma = 0$ in (1.6). For the aforementioned model A the existence of solutions was proved only for space dimension $n = 1$ and numerical simulation indicated that the solution may blow-up in finite time for $n = 2$. The main goal of the present paper is to find a possibly minimal modification of model A from [26] which warrants prevention of blow-up formation in finite time. Indeed it turns out that under the assumption of sufficiently strong intra-specific competition in predators and prey (δ_1 and r_1 big enough) in conjunction with density-dependent predator velocity suppression the effect of rapid sharp aggregation of predators is precluded and in consequence there is a global classical solution to model (1.3) shown in Theorem 2.1 whose proof is inspired by the article [45].

It is worth highlighting that for the space dimension $n = 2$ the formation of blowing-up solutions is precluded for both the predator-prey model with prey taxis ($\chi = 0$ and $\xi > 0$) with

$\Psi \equiv 1$ as proved in [19] and for predator prey model without prey taxis term ($\xi = 0$, $\chi > 0$) and $\delta_1 = 0$ proved in [26], so neither of taxis mechanisms alone can lead to the blow-up of solution and it is expected to happen only when there is a cumulative effect of both taxis mechanisms in conjunction with the lack of suppressive terms i.e. $\delta_1 = 0$, $\Psi \equiv 1$ and suitable choice of the initial densities.

We assume the following hypotheses:

(H0) The function Ψ in (1.7) satisfies

$$\Psi \in C^2(I\!R_+) \quad \text{and} \quad G(r) = r\Psi(r) \leq \bar{G} \quad \text{for} \quad r \geq 0 \qquad (1.11)$$

for some $\bar{G} > 0$. Notice that for the case of (1.6) we have $\bar{G} := \sigma^{-1}$.

(H1) The parameters d_P, d_N, d_W, ξ, χ, δ, δ_1, r, r_1, b, γ, μ in (1.3) are positive,

(H2) The function $F : I\!R_+ \mapsto I\!R_+$ is a C^2-function such that for some constants $\bar{F} > 0$,

$$F(N) \leq \bar{F} \quad \text{for any } N \geq 0.$$

(H3) The parameters r_1 and δ_1 satisfy the following restrictions

$$\delta_1 \geq \left(\frac{\gamma^2(16+n)}{d_W} + d_W \right), \qquad (1.12)$$

$$r_1 \geq \left(\frac{\chi^2 A_N}{(d_N)^2} + \frac{2\chi^2}{d_W} + d_W \right), \qquad (1.13)$$

where

$$A_N = \frac{2\left((d_N)^2 + (d_W)^2 + \xi^2 \bar{G}^2\right)}{d_W} + 1. \qquad (1.14)$$

It is worth noticing that conditions (1.12)-(1.14) contain only explicit model parameters and are independent of constants used in the proof of Theorem 2.1.

As has been mentioned before, only a cumulative effect of both taxis mechanisms may lead to the formation of blow-up solutions in finite time provided the conditions (1.12)-(1.14) are not fulfilled. If (1.6) is used to define $G(P) = P\Psi(P)$ then for $P > 0$, $G(P) = P(1+\sigma P)^{-1} \to P$ with $\sigma \to 0$ and system (1.3) may be viewed formally as a regularization of the one with $\sigma = 0$. Therefore in Section 3 the impact of the parameter σ on the formation of blow-up solutions is illustrated by means of numerical simulations (notice that then $\bar{G} = \sigma^{-1}$). Our hypothesis that the product $\xi \bar{G}$ rather and not only sole $\bar{G}$ plays a crucial role in blow-up formation, provided $\chi > 0$, is confirmed by the results of numerical simulations which are not included to the paper. They confirmed that for the occurrence of blow-up matters whether the value of $\xi \sigma^{-1}$ is super-critical. For suitably chosen initial data with very high density concentration and selected values of model parameters such that conditions (1.12)-(1.14) are not satisfied we observe numerical solutions showing a rapid blow-up formation for the range $\sigma \in [0, \sigma_c)$ where σ_c is a critical value depending on the remaining parameters. For $\sigma > \sigma_c$ and other values of parameters kept as before, so that the conditions (1.12)-(1.14) are satisfied the stabilizing role of parameter σ reveals and the blowing-up solutions do not occur.

2. Existence of global-in-time solutions

Let us consider the following auxiliary initial boundary value problem

$$u_t + Au + \eta u = \nabla \cdot Q + \varphi, \quad u(0) = u_0 \in W^{1,r}(\Omega), \ r > n \tag{2.15}$$

where $\eta > 0$, Ω is a regular domain with smooth boundary, $W^{k,r}(\Omega)$, $k \in \{0, 1, 2\}$, $r \geq 1$, is the Sobolev space with the norm denoted by $\| \cdot \|_{k,r}$. For short the norm in the space $L^q(\Omega)$, $\Omega \subset IR^n$, will be denoted by $\| \cdot \|_q$. Let us recall also the Sobolev embedding theorem

$$W^{1,r}(\Omega) \subset L^\infty(\Omega) \quad \text{for } r > n. \tag{2.16}$$

For $D > 0$ the operator

$$Au = -D\Delta u \text{ for } u \in D(A) = \{v \in W^{2,q}(\Omega) : \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\}$$

is a $L^q(\Omega)$-realization, $q \in (1, \infty)$, of the Laplace operator with homogeneous Neumann boundary condition and

$$Q \in X_q := C([0, T) : (W^{1,q}(\Omega))^n) \cap C([0, T) : C(\bar\Omega))^n, \tag{2.17}$$

$$\varphi \in C([0, T) : L^{q_0}(\Omega)) \cap C([0, T) : C(\bar\Omega)). \tag{2.18}$$

In the notation we will sometimes drop the arguments of time and space-dependent functions writing $P(\cdot, t)$ or $P(t)$ or just P instead of $P(x, t)$ etc. depending on the length of formulae in which they appear.

For the convenience of the reader we have compiled below some results from the literature which will be used later on:

(R0) Suppose that in (2.15) for some $\bar\varphi \geq 0$ there holds $\varphi(t, \cdot) \leq \bar\varphi$ for $(x, t) \in \Omega \times [0, T)$ and $q > n$. Then there exists a constant $\tilde C$ such that

$$\|u(\cdot, t)\|_\infty \leq \|u_0(\cdot)\|_\infty + \tilde C \sup_{s \in [0,t)} \|Q(\cdot, s)\|_{(L^q(\Omega))^n} + \bar\varphi \eta^{-1} \quad \text{for} \quad t \in [0, T). \tag{2.19}$$

(R1) Bochner's type inequality (see e.g. [44]): For $v \in C^2(\bar\Omega)$ there holds

$$2\nabla v \nabla \Delta v = \Delta |\nabla v|^2 - 2|D^2 v|^2. \tag{2.20}$$

(R2) Let $u \in C^2(\bar\Omega)$ satisfy $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ and Ω is a bounded domain with regular boundary. Then there holds the following pointwise inequality [27, Lemma 4.2]

$$\frac{\partial |\nabla u|^2}{\partial \nu} \leq K |\nabla u|^2 \quad \text{on} \quad \partial\Omega \tag{2.21}$$

where $K = K(\Omega) > 0$ is un upper bound on the curvature of $\partial\Omega$.

(R3) The following inequality is a well known consequence of the Gagliardo-Nirenberg inequality: for $u \in W^{1,2}(\Omega)$ and any $\eta > 0$ there holds

$$\int_{\partial\Omega} u^2 dS \leq \eta \int_{\Omega} |\nabla u|^2 dx + C_G(\eta) \left(\int_{\Omega} u dx \right)^2. \tag{2.22}$$

(R4) Let $T > 0$ and $\tau \in (0, T)$ and $y : [0, T) \to I\!R_+$ satisfies

$$\frac{dy}{dt} + Ay(t) \leq B(t) \quad \text{for a.e.} \quad t \in (0, T) \tag{2.23}$$

where $A > 0$ and $B \in L^1_{loc}([0, T)$ is a non-negative function satisfying

$$\int_t^{t+\tau_0} B(s) ds \leq B_1 \quad \text{for all } t \in [0, T - \tau_0),$$

with $B_1 > 0$ [34, Lemma 3.4]. Then

$$y(t) \leq \max\left\{ y(0) + B_1, \frac{B_1}{A\tau_0} + 2B_1 \right\} \quad \text{for } t \in (0, T). \tag{2.24}$$

(R5) The following result may be found in [45, Lemma 3.6]. Assume in (2.15) that for some $\eta_1, \eta_2 > 0$,

$$\varphi(u) \leq \eta_1 u - \eta_2 u^2$$

and $Q = uV$ and

$$V \in C([0, T) : (W^{1,\varrho}(\Omega))^n),$$

with $\varrho = 4$ and $n \leq 3$. Then for any $k > 1$ there exists a constant $C_k > 0$ independent of T such that

$$\sup_{t \in [0,T)} \|u(\cdot, t)\|_k \leq C_k.$$

The result (R0) is an adjustment of the well known result from [43] and its extension from [10]. Indeed, by the maximum principle the heat-Neumann semigroup $e^{(D\Delta - \eta I)t}$ corresponding to the linear problem

$$u_t - D\Delta u + \eta u = 0 \quad u(\cdot, 0) = u_0(\cdot)$$

with homogeneous Neumann boundary condition, is order preserving (with the natural order in the space $C(\Omega)$) and

$$\|e^{(D\Delta-\eta)t}w\|_\infty \le e^{-\eta t}\|w\|_\infty \quad \text{for} \quad w \in C(\bar\Omega)$$

Using the above properties in the variation of constant formula

$$u(\cdot,t) = e^{(D\Delta-\eta)t}u_0 + \int_0^t e^{(D\Delta-\eta)(t-s)}\nabla \cdot Q(\cdot,s)ds + \int_0^t e^{(D\Delta-\eta)(t-s)}\varphi(\cdot,s)ds$$

for (2.15) we obtain, using [10, Lemma 2.1 (iv)] in the term with the divergence operator,

$$\|u(\cdot,t)\|_\infty \le \|u_0\|_\infty + \int_0^t \|e^{(D\Delta-\eta)(t-s)}\nabla \cdot Q(\cdot,s)\|_\infty ds + \int_0^t \|e^{(D\Delta-\eta)(t-s)}\varphi(\cdot,s)\|_\infty ds$$

$$\le \|u_0\|_\infty + C\int_0^t e^{-\eta(t-s)}\left(1 + (t-s)^{-\frac{1}{2}\left(1+\frac{n}{q}\right)}\right)\|Q(\cdot,s)\|_q ds$$

$$+ \int_0^t e^{-\eta(t-s)}\bar\varphi ds, \qquad t \in [0,T). \tag{2.25}$$

where C is a positive constant. Next, to obtain (2.19) we assume $q > n$, then let $t \to \infty$ in the last two integrals. Finally we find the value of $\tilde C$ in (2.19)

$$\tilde C = C\left(\eta^{-1} + \left(\eta^{\frac{1}{2}\left(\frac{n}{q}-1\right)}\right)\Gamma\left(\frac{1}{2}\left(1-\frac{n}{q}\right)\right)\right)$$

where $\Gamma(\cdot)$ is Euler's gamma function.

Lemma 2.1.

(i) *Suppose that hypotheses (H1)-(H3) are satisfied and P_0, N_0, $W_0 \in W^{1,r}(\Omega)$, $r > n$ are non-negative functions. Then there exists a unique non-negative maximal solution to system (1.3) satisfying initial and boundary conditions (1.4)-(1.5) such that*

$$(P,N,W) \in (C([0,T_{max}): W^{1,r}(\Omega)) \cap C^{2,1}(\bar\Omega \times (0,T_{max})))^3.$$

Moreover, if $T_{max} < \infty$ then

$$\lim_{t \to T_{max}}(\|P(\cdot,t)\|_\infty + \|N(\cdot,t)\|_\infty + \|W(\cdot,t)\|_\infty) = \infty. \tag{2.26}$$

(ii) *There exist positive constant C_1, C_2, C_3 such that*

$$\sup_{t\in[0,T_{max})}\|P(\cdot,t)\|_1 \le C_1, \quad \sup_{t\in[0,T_{max})}\|N(\cdot,t)\|_1 \le C_2, \quad \sup_{t\in[0,T_{max})}\|W(\cdot,t)\|_1) \le C_3. \tag{2.27}$$

Proof of Lemma 2.1. The local in-time existence of solutions for similar quasilinear parabolic systems have been considered in many papers therefore we present it in an abbreviated form. We first notice that the main part of the quasilinear parabolic system is a normally elliptic operator with upper-triangular structure and the existence and uniqueness of maximal classical solution

$$(N, P, W) \in (C([0, T_{max}) : W^{1,r}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})))^3$$

satisfying boundary and initial conditions (1.5)-(1.4) follows from Amann's theory [4, Theorems 14.4 & 14.6] (see e.g. [1,19,42] for details). The non-negativity of solutions easily follows from the maximum principle. Moreover in this case it is known that a uniform in time L^∞-bound for the solution is sufficient to warrant that in fact $T_{max} = +\infty$. Next we proceed to item (i). We shall use several times the following inequality which is a consequence of the Hölder inequality

$$\int_\Omega v^2(x)dx \geq \frac{1}{|\Omega|} \left(\int_\Omega v(x)dx \right)^2 \quad \text{for} \quad v \in L^2(\Omega).$$

Then integrating separately each of the equations in (1.3) we get

$$\frac{d}{dt} \int_\Omega P \leq -\delta \int_\Omega P + \bar{F} \int_\Omega P - \delta_1 \int_\Omega P^2 \tag{2.28}$$

$$\leq -\delta \int_\Omega P + \bar{F} \int_\Omega P - \frac{\delta_1}{|\Omega|} \left(\int_\Omega P \right)^2$$

$$\leq -\delta \int_\Omega P + \frac{\bar{F}^2}{4\delta_1} |\Omega|$$

and solving the resulting differential inequality we obtain

$$\int_\Omega P(\cdot, t) \leq \max \left\{ \int_\Omega P_0, \frac{\bar{F}^2}{4\delta\delta_1} |\Omega| \right\} := C_1 \quad \text{for} \quad t \in [0, T_{max}). \tag{2.29}$$

Next we have

$$\frac{d}{dt} \int_\Omega N \leq r \int_\Omega N - r_1 \int_\Omega N^2 \tag{2.30}$$

$$\leq -r \int_\Omega N + 2r \int_\Omega N - \frac{r_1}{|\Omega|} \left(\int_\Omega N \right)^2$$

$$\leq -r \int_\Omega N + \frac{r^2}{r_1} |\Omega|,$$

hence,

$$\int_{\Omega} N(\cdot, t) \leq \max \left\{ \int_{\Omega} N_0, \frac{r}{r_1} |\Omega| \right\} := C_2 \quad \text{for} \quad t \in [0, T_{max}) \tag{2.31}$$

and finally

$$\frac{d}{dt} \int_{\Omega} W \leq -\mu \int_{\Omega} W + \gamma \int_{\Omega} P \leq -\mu \int_{\Omega} W + \gamma C_1.$$

Hence,

$$\int_{\Omega} W(\cdot, t) \leq \max \left\{ \int_{\Omega} W_0, \frac{\gamma C_1}{\mu} \right\} := C_3 \quad \text{for} \quad t \in [0, T_{max}). \quad \square \tag{2.32}$$

Lemma 2.2. *Let* (P, N, W) *be a solution to (1.3), $T_{max} < \infty$, and $\tau_0 = \min\{1, \frac{T_{max}}{2}\}$. Then for $t \in (0, T_{max} - \tau_0)$ there exist constants $C_4 \ldots C_7$ such that*

$$\int_t^{t+\tau_0} \int_{\Omega} P^2(x, s) ds dx \leq C_4, \qquad \int_t^{t+\tau_0} \int_{\Omega} N^2(x, s) ds dx \leq C_5, \tag{2.33}$$

$$\sup_{t \in [0, T_{max})} \int_{\Omega} |\nabla W(x, t)|^2 dx \leq C_6. \tag{2.34}$$

Proof of Lemma 2.2. On integrating (2.28) and (2.30) with respect to time from t to $t + \tau_0$ we easily find that

$$\int_t^{t+\tau_0} \int_{\Omega} P^2 ds dx \leq \frac{C_1}{\delta_1}(\tau_0 \bar{F} + 1) = \frac{C_1}{\delta_1}(\bar{F} + 1) := C_4, \qquad \int_t^{t+\tau_0} \int_{\Omega} N^2 ds dx \leq \frac{C_1}{r_1}(r + 1) := C_5. \tag{2.35}$$

Next by multiplying the W-equation by $-\Delta W$ we use the Young inequality to obtain

$$\frac{d}{dt} \int_{\Omega} |\nabla W|^2 + 2\mu \int_{\Omega} |\nabla W|^2 + d_W \int_{\Omega} |\Delta W|^2 \leq \gamma^2 \int_{\Omega} P^2.$$

It leads to the differential inequality of form (2.23) with $y(t) = \int_{\Omega} |\nabla W(\cdot, t)|^2$ and owing to (2.35) we make use of the result (R4) to deduce (2.34). $\square$

Lemma 2.3. *Let (P, N, W) be a solution to (1.3). Then for $t \in (0, T_{max})$*

$$\frac{d}{dt} \int_\Omega P^2 + d_P \int_\Omega |\nabla P|^2 + \int_\Omega P^2$$

$$\leq \frac{\xi^2 \bar{G}^2}{d_P} \int_\Omega |\nabla N|^2 + 2 \int_\Omega (b\bar{F}P^2 - \delta_1 P^3) + \int_\Omega P^2 \tag{2.36}$$

$$\frac{d}{dt} \int_\Omega N^2 + d_N \int_\Omega |\nabla N|^2 + \int_\Omega N^2$$

$$\leq \frac{\chi^2}{d_N} \int_\Omega N^2 |\nabla W|^2 + 2 \int_\Omega (rN^2 - r_1 N^3) + \int_\Omega N^2. \tag{2.37}$$

Proof of Lemma 2.3. The inequalities are typical energy estimates which come from multiplying the P-equation by P and the N-equation by N. Next the Young inequality was applied and the terms $\int_\Omega P^2$ and $\int_\Omega N^2$ were added to both sides of inequalities for the P-equation and N equation respectively. □

The following Lemma is one of crucial counterparts of the existence proof providing an estimate on $\int_\Omega |\nabla W(x, t)|^4 dx$ for $t \in (0, T_{max})$.

Lemma 2.4. *Let (P, N, W) be a solution to (1.3) and $T_{max} < \infty$. Then there exists a constant $C_7 > 0$ such that for $t \in (0, T_{max})$.*

$$\frac{d}{dt} \int_\Omega |\nabla W|^4 + d_W \int_\Omega \left| \nabla(|\nabla W|^2) \right|^2 + 4\mu \int_\Omega |\nabla W|^4$$

$$\leq \gamma^2 \left(\frac{16 + n}{d_W} \right) \int_\Omega |\nabla W|^2 P^2 + C_7. \tag{2.38}$$

Proof of Lemma 2.4. Using (2.20) we obtain

$$(|\nabla W|^2)_t = d_W \Delta |\nabla W|^2 - 2d_W |D^2 W|^2 + 2\nabla W \cdot \nabla(\gamma P - \mu W) \tag{2.39}$$

and then this formula along with integration by parts is used to compute

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla W|^4 = \int_\Omega |\nabla W|^2 (|\nabla W|^2)_t = d_W \int_\Omega |\nabla W|^2 \Delta |\nabla W|^2$$

$$- 2d_W \int_\Omega |\nabla W|^2 |D^2 W|^2 + 2 \int_\Omega |\nabla W|^2 \nabla W \cdot \nabla(\gamma P - \mu W) \tag{2.40}$$

$$= -d_W \int_\Omega \left| \nabla(|\nabla W|^2) \right|^2 + d_W \int_{\partial\Omega} |\nabla W|^2 \frac{\partial |\nabla W|^2}{\partial \nu} - 2d_W \int_\Omega |\nabla W|^2 |D^2 W|^2$$

$$-2\mu \int_\Omega |\nabla W|^4 - 2\gamma \int_\Omega (\nabla(|\nabla W|^2) \cdot \nabla W) P - 2\gamma \int_\Omega (|\nabla W|^2 \Delta W) P. \qquad (2.41)$$

To estimate the last two terms we use the pointwise inequality (see e.g. [44])

$$|\Delta W|^2 \le n |D^2 W|^2$$

and the Young inequality which yield

$$2\gamma \int_\Omega (\nabla(|\nabla W|^2) \cdot \nabla W) P + 2\gamma \int_\Omega (|\nabla W|^2 \Delta W) P$$

$$\le \varepsilon_0 \int_\Omega \left| \nabla(|\nabla w|^2) \right|^2 + \frac{\gamma^2}{\varepsilon_0} \int_\Omega |\nabla W|^2 P^2 + 2\gamma \sqrt{n} \int_\Omega |\nabla W|^2 |D^2 W| P$$

$$\le \varepsilon_0 \int_\Omega \left| \nabla(|\nabla W|^2) \right|^2 + \gamma^2 \left(\frac{1}{\varepsilon_0} + \frac{n}{2 d_W} \right) \int_\Omega |\nabla W|^2 P^2 + 2 d_W \int_\Omega |\nabla W|^2 |D^2 W|^2.$$

Notice that the last term cancels out with the first term in (2.40) and next we use the results (R2) and (R3) to estimate in (2.41) the term

$$d_W \int_{\partial\Omega} |\nabla W|^2 \frac{\partial |\nabla W|^2}{\partial \nu} \le d_W K \int_{\partial\Omega} \left(|\nabla W|^2 \right)^2$$

$$\le d_W K \eta \int_\Omega \left| \nabla(|\nabla W|^2) \right|^2 + C_G(\eta) d_W K \left(\int_\Omega |\nabla W|^2 \right)^2$$

where $\eta > 0$. Coming back to (2.41) we get

$$\frac{d}{dt} \int_\Omega |\nabla W|^4 + 2 d_W \int_\Omega \left| \nabla(|\nabla w|^2) \right|^2 + 4\mu \int_\Omega |\nabla W|^4$$

$$\le 2(\varepsilon_0 + \eta d_W K) \int_\Omega \left| \nabla(|\nabla w|^2) \right|^2 + 2\gamma^2 \left(\frac{2}{\varepsilon_0} + \frac{n}{2 d_W} \right) \int_\Omega |\nabla W|^2 P^2 + 2 d_W K C_G(\eta)(C_6)^2.$$

and setting $\varepsilon_0 = \frac{d_W}{4}$ and $\eta = \frac{1}{4K}$ we obtain (2.38) with $C_7 = 2 d_W K C_G(\frac{1}{4K})(C_6)^2$ where (2.34) has been used. $\quad\square$

The main step in the proof of global-in-time existence of solutions is to find a differential inequality which enables to find a bound on

$$y(t) = \int_\Omega |\nabla W|^4 + \int_\Omega P |\nabla W|^2 + \int_\Omega N |\nabla W|^2 + A_1 \int_\Omega N^2 + A_2 \int_\Omega P^2.$$

for suitably chosen A_1 and A_2. We are now in the position to formulate our main theorem.

Theorem 2.1. *Suppose that hypotheses (H1) -(H3) are satisfied and P_0, N_0, $W_0 \in W^{1,r}(\Omega)$, $r > n$ are non-negative functions. If $n \leq 3$ there exists the unique global and uniformly bounded classical solution (N, P, W) to (1.3) satisfying boundary and initial conditions (1.5)-(1.4) defined on $\bar{\Omega} \times [0, \infty)$ such that*

$$(P, N, W) \in (C([0, T) : W^{1,r}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T)))^3 \quad \text{for any } T > 0.$$

Proof. We begin with finding differential inequalities which contain $\int_\Omega P|\nabla W|^2$ and separately $\int_\Omega N|\nabla W|^2$. To this end we use (2.39) and integration by parts

$$\frac{d}{dt} \int_\Omega P|\nabla W|^2 + \int_\Omega P|\nabla W|^2 = \int_\Omega P_t|\nabla W|^2 + \int_\Omega P\left(|\nabla W|^2\right)_t + \int_\Omega P|\nabla W|^2 \qquad (2.42)$$

$$= \underbrace{\left\{ d_P \int_\Omega \Delta P|\nabla W|^2 - \xi \int_\Omega |\nabla W|^2 \nabla \cdot G(P)\nabla N + \int_\Omega |\nabla W|^2 \left(bF(N)P - \delta P - \delta_1 P^2\right)\right\}}_{I_1}$$

$$+ \underbrace{\left\{ d_W \int_\Omega P\Delta(|\nabla W|^2) - 2d_W \int_\Omega P|D^2 W|^2 + 2\int_\Omega P\nabla W \cdot \nabla(\gamma P - \mu W)\right\}}_{I_2} + \int_\Omega P|\nabla W|^2$$

Using the Young inequality and differentiation by parts we obtain

$$I_1 + \int_\Omega P|\nabla W|^2 \leq d_P \int_\Omega |\nabla P|\left|\nabla(|\nabla W|^2)\right| + \xi\bar{G}\int_\Omega |\nabla(|\nabla W|^2)||\nabla N|$$

$$+ b\bar{F}\int_\Omega P|\nabla W|^2 + (1-\delta)\int_\Omega P|\nabla W|^2 - \delta_1 \int_\Omega P^2|\nabla W|^2$$

$$\leq \varepsilon_1 \int_\Omega \left|\nabla(|\nabla W|^2)\right|^2 + \frac{(d_P)^2}{4\varepsilon_1}\int_\Omega |\nabla P|^2 + \varepsilon_2 \int_\Omega \left|\nabla(|\nabla W|^2)\right|^2$$

$$+ \frac{\xi^2\bar{G}^2}{4\varepsilon_2}\int_\Omega |\nabla N|^2 + (b\bar{F} - \delta + 1)\int_\Omega P|\nabla W|^2 - \delta_1 \int_\Omega P^2|\nabla W|^2$$

and

$$I_2 \leq d_W \int_\Omega \nabla P \cdot \nabla(|\nabla W|^2) + d_W \int_{\partial\Omega} P\frac{\partial|\nabla W|^2}{\partial\nu} + 2\gamma \int_\Omega P\nabla W \cdot \nabla P - 2\mu \int_\Omega P|\nabla W|^2$$

$$\leq \varepsilon_3 \int_\Omega \left| \nabla |\nabla W|^2 \right|^2 + \frac{(d_W)^2}{4\varepsilon_3} \int_\Omega |\nabla P|^2 + \int_{\partial\Omega} P \frac{\partial |\nabla W|^2}{\partial \nu}$$

$$+ \varepsilon_4 \int_\Omega P^2 |\nabla W|^2 + \frac{\gamma^2}{\varepsilon_4} \int_\Omega |\nabla P|^2 - 2\mu \int_\Omega P|\nabla W|^2 .$$

Finally, we get in (2.42)

$$\frac{d}{dt} \int_\Omega P|\nabla W|^2 + \int_\Omega P|\nabla W|^2 \leq (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \int_\Omega \left| \nabla (|\nabla W|^2) \right|^2$$

$$+ \left(\frac{(d_P)^2}{4\varepsilon_1} + \frac{(d_W)^2}{4\varepsilon_3} + \frac{\gamma^2}{\varepsilon_4} \right) \int_\Omega |\nabla P|^2 + \frac{\xi^2 \bar{G}^2}{4\varepsilon_2} \int_\Omega |\nabla N|^2$$

$$+ (\varepsilon_4 - \delta_1) \int_\Omega P^2 |\nabla W|^2 + (b\bar{F} - 2\mu - \delta + 1) \int_\Omega P|\nabla W|^2$$

$$+ d_W \int_{\partial\Omega} P \frac{\partial |\nabla W|^2}{\partial \nu} . \tag{2.43}$$

Similarly, for the term $\int_\Omega N|\nabla W|^2$ we obtain

$$\frac{d}{dt} \int_\Omega N|\nabla W|^2 + \int_\Omega N|\nabla W|^2 = \int_\Omega N_t |\nabla W|^2 + \int_\Omega N \left(|\nabla W|^2 \right)_t + \int_\Omega N|\nabla W|^2$$

$$= d_N \int_\Omega \Delta N |\nabla W|^2 + \chi \int_\Omega |\nabla W|^2 \nabla \cdot N \nabla W + \int_\Omega |\nabla W|^2 (-F(N)P + rN - r_1 N^2)$$

$$+ d_W \int_\Omega N\Delta(|\nabla W|^2) - 2d_W \int_\Omega N|D^2 W|^2 + 2\int_\Omega N\nabla W \cdot \nabla(\gamma P - \mu W) + (1+r) \int_\Omega N|\nabla W|^2$$

$$\leq \varepsilon_5 \int_\Omega \left| \nabla(|\nabla W|^2) \right|^2 + \frac{d_N^2}{4\varepsilon_5} \int_\Omega |\nabla N|^2 + \varepsilon_6 \int_\Omega \left| \nabla(|\nabla W|^2) \right|^2 + \left(\frac{\chi^2}{4\varepsilon_6} - r_1 \right) \int_\Omega N^2 |\nabla W|^2$$

$$+ \varepsilon_7 \int_\Omega |\nabla|\nabla W|^2|^2 + \frac{(d_W)^2}{4\varepsilon_7} \int_\Omega |\nabla N|^2 + (r + 1 - 2\mu) \int_\Omega N|\nabla W|^2$$

$$+ d_W \int_{\partial\Omega} N \frac{\partial |\nabla W|^2}{\partial \nu} + \varepsilon_8 \int_\Omega N^2 |\nabla W|^2 + \frac{\gamma^2}{\varepsilon_8} \int_\Omega |\nabla P|^2 .$$

After the rearrangement of the terms above we have

$$\frac{d}{dt} \int_\Omega N|\nabla W|^2 + \int_\Omega N|\nabla W|^2 \leq (\varepsilon_5 + \varepsilon_6 + \varepsilon_7) \int_\Omega \left| \nabla(|\nabla W|^2) \right|^2 + \left(\frac{(d_N)^2}{4\varepsilon_5} + \frac{(d_W)^2}{4\varepsilon_7} \right) \int_\Omega |\nabla N|^2$$

$$+ (r + 1 - 2\mu) \int_{\Omega} N |\nabla W|^2 + \left(\frac{\chi^2}{4\varepsilon_6} - r_1 + \varepsilon_8 \right) \int_{\Omega} N^2 |\nabla W|^2$$

$$+ \frac{\gamma^2}{\varepsilon_8} \int_{\Omega} |\nabla P|^2 + d_W \int_{\partial\Omega} N \frac{\partial |\nabla W|^2}{\partial \nu} . \tag{2.44}$$

To estimate the last terms in (2.43) and (2.44) we use the Young inequality then (R2) and next (R3). Then we have the following

$$d_W \int_{\partial\Omega} P \frac{\partial |\nabla W|^2}{\partial \nu} \leq d_W K \int_{\partial\Omega} P |\nabla W|^2 \leq \int_{\partial\Omega} \left| |\nabla W|^2 \right|^2 + \frac{(d_W)^2 K^2}{4} \int_{\partial\Omega} P^2$$

$$\leq \eta_1 \int_{\Omega} \left| \nabla (|\nabla W|^2) \right|^2 + \int_{\Omega} |\nabla P|^2 + C_G(\eta_1) \left(\int_{\Omega} |\nabla W|^2 \right)^2$$

$$+ \frac{(d_W)^2 K^2}{4} C_G \left(\frac{4}{(d_W)^2 K^2} \right) \left(\int_{\Omega} P \right)^2 . \tag{2.45}$$

Similarly, we proceed for the case of (2.44) and finally we obtain that for some $\eta_2 > 0$

$$d_W \int_{\partial\Omega} P \frac{\partial |\nabla W|^2}{\partial \nu} + d_W \int_{\partial\Omega} N \frac{\partial |\nabla W|^2}{\partial \nu}$$

$$\leq \eta_2 \int_{\Omega} \left| \nabla (|\nabla W|^2) \right|^2 + \int_{\Omega} |\nabla P|^2 + \int_{\Omega} |\nabla N|^2 + C_{10} \tag{2.46}$$

where C_{10} is a constant depending on K and constants C_1, C_2 and C_6 from (2.27) and (2.34). Next we add (2.38), (2.43) and (2.44) making use of (2.46)

$$\frac{d}{dt} \left(\int_{\Omega} |\nabla W|^4 + \int_{\Omega} P |\nabla W|^2 + \int_{\Omega} N |\nabla W|^2 \right) + 4\mu \int_{\Omega} |\nabla W|^4 + \int_{\Omega} P |\nabla W|^2$$

$$+ \int_{\Omega} N |\nabla W|^2 + d_W \int_{\Omega} \left| \nabla (|\nabla w|^2) \right|^2 \leq \left(\sum_{i=1, i \neq 4}^{i=7} \varepsilon_i + \eta_2 \right) \int_{\Omega} \left| \nabla (|\nabla W|^2) \right|^2$$

$$+ \left(\gamma^2 \left(\frac{16 + n}{d_W} \right) + \varepsilon_4 - \delta_1 \right) \int_{\Omega} P^2 |\nabla W|^2 + \left(\frac{\chi^2}{4\varepsilon_6} + \varepsilon_8 - r_1 \right) \int_{\Omega} N^2 |\nabla W|^2$$

$$+ (b\bar{F} - 2\mu - \delta + 1) \int_{\Omega} P |\nabla W|^2 + (r + 1 - 2\mu) \int_{\Omega} N |\nabla W|^2$$

$$+ A_P \int_\Omega |\nabla P|^2 + A_N \int_\Omega |\nabla N|^2 + C_{10} \tag{2.47}$$

where

$$A_P = \left(\frac{(d_P)^2}{4\varepsilon_1} + \frac{(d_W)^2}{4\varepsilon_3} + \frac{\gamma^2}{\varepsilon_4} + \frac{\gamma^2}{\varepsilon_8} \right) + 1, \tag{2.48}$$

$$A_N = \left(\frac{(d_N)^2}{4\varepsilon_5} + \frac{(d_W)^2}{4\varepsilon_7} + \frac{\xi^2 \bar{G}^2}{4\varepsilon_2} \right) + 1. \tag{2.49}$$

Setting $\varepsilon_i = \eta_2 = \frac{d_W}{8}$ for $i = 1, \ldots, 7, i \neq 4$ and $\varepsilon_4 = \varepsilon_8 = d_W$ we obtain

$$\left(\sum_{i=1, i \neq 4}^{i=7} \varepsilon_i + \eta_2 \right) = \frac{7}{8} d_W$$

and

$$A_P = \frac{2 \left((d_P)^2 + (d_W)^2 + \gamma^2 \right)}{d_W} + 1, \tag{2.50}$$

$$A_N = \frac{2 \left((d_N)^2 + (d_W)^2 + \xi^2 \bar{G}^2 \right)}{d_W} + 1. \tag{2.51}$$

In order to cancel out the last two integrals in (2.47) we add to (2.47) side by side (2.36) multiplied by $\frac{A_P}{d_P}$ and then (2.37) multiplied by $\frac{A_N}{d_N}$ to get

$$\frac{d}{dt} \left\{ \int_\Omega |\nabla W|^4 + \int_\Omega P |\nabla W|^2 + \int_\Omega N |\nabla W|^2 + \frac{A_P}{d_P} \int_\Omega P^2 + \frac{A_N}{d_N} \int_\Omega N^2 \right\}$$

$$+ \left\{ 4\mu \int_\Omega |\nabla W|^4 + \int_\Omega P |\nabla W|^2 + \int_\Omega N |\nabla W|^2 + \frac{A_P}{d_P} \int_\Omega P^2 + \frac{A_N}{d_N} \int_\Omega N^2 \right\}$$

$$+ A_P \int_\Omega |\nabla P|^2 + A_N \int_\Omega |\nabla N|^2$$

$$\leq \left(\gamma^2 \left(\frac{16 + n}{d_W} \right) + d_W - \delta_1 \right) \int_\Omega P^2 |\nabla W|^2 + \left(\frac{\chi^2 A_N}{(d_N)^2} + \frac{2\chi^2}{d_W} + d_W - r_1 \right) \int_\Omega N^2 |\nabla W|^2$$

$$+ (b\bar{F} - 2\mu - \delta + 1) \int_\Omega P |\nabla W|^2 + (r + 1 - 2\mu) \int_\Omega N |\nabla W|^2$$

$$+ A_P \int_\Omega |\nabla P|^2 + A_N \int_\Omega |\nabla N|^2 + \frac{A_P}{d_P} (2b\bar{F} + 1) \int_\Omega P^2 - \frac{2A_P}{d_P} \delta_1 \int_\Omega P^3$$

$$+ \frac{A_N}{d_N}(2r+1) \int_\Omega N^2 - \frac{2A_N}{d_N} r_1 \int_\Omega N^3 + C_{10}. \tag{2.52}$$

Next we use the Young inequality to estimate

$$(b\bar{F} - 2\mu - \delta + 1) \int_\Omega P|\nabla W|^2 \le \frac{3}{2}\mu \int_\Omega |\nabla W|^4 + \frac{1}{6\mu}(b\bar{F} - 2\mu - \delta + 1)^2 \int_\Omega P^2$$

$$(r + 1 - 2\mu) \int_\Omega N|\nabla W|^2 \le \frac{3}{2}\mu \int_\Omega |\nabla W|^4 + \frac{1}{6\mu}(r + 1 - 2\mu)^2 \int_\Omega N^2$$

and then denoting

$$\alpha_1 := \frac{A_P}{d_P}(2b\bar{F} + 1) + \frac{1}{6\mu}(b\bar{F} - 2\mu - \delta + 1)^2, \qquad \alpha_2 := \frac{2A_P}{d_P}\delta_1$$

$$\sigma_1 := \frac{A_N}{d_N}(2r + 1) + \frac{1}{6\mu}(r - 2\mu + 1)^2, \qquad \sigma_2 := \frac{2A_N}{d_N}r_1$$

we find estimates for the polinomial terms

$$\alpha_1 \int_\Omega P^2 - \alpha_2 \int_\Omega P^3 + \sigma_1 \int_\Omega N^2 - \sigma_2 \int_\Omega P^3 \le \frac{4}{27}\left(\frac{\alpha_1^3}{\alpha_2^2} + \frac{\sigma_1^3}{\sigma_2^2}\right)|\Omega| := C_{11}. \tag{2.53}$$

Denoting

$$y(t) = \left\{\int_\Omega |\nabla W|^4 + \int_\Omega P|\nabla W|^2 + \int_\Omega N|\nabla W|^2 + \frac{A_P}{d_P}\int_\Omega P^2 + \frac{A_N}{d_N}\int_\Omega N^2\right\}$$

and using the assumption (H3):

$$\left(\gamma^2\left(\frac{16 + n}{d_W}\right) + d_W - \delta_1\right) \le 0,$$

$$\left(\frac{\chi^2 A_N}{(d_N)^2} + \frac{2\chi^2}{d_W} + d_W - r_1\right) \le 0,$$

we are in the position to rewrite (2.52) in the form of the following differential inequality

$$\frac{d}{dt}y(t) + y(t) \le C_{10} + C_{11} \quad \text{for} \quad t \in [\tau, T_{max})$$

for some $\tau \in (0, T_{max})$. Notice that we have assumed that $W_0 \in W^{1,r}(\Omega)$ for $r > n$, thus, for $n = 3$ there could be that $r \in (3, 4)$. Whence it follows that y is a bounded function for $t \in [\tau, T_{max})$ and in particular

$$\sup_{s\in[\tau,T_{max})}\int_{\Omega}|\nabla W(x,s)|^4dx < \tilde{C}_W \qquad (2.54)$$

with $\tilde{C}_W$ independent on time. Next, using the result (R5) to the N-equation with $\varrho = 4$ we deduce that there is a time-independent constant C_k

$$\sup_{t\in[0,T_{max})} \|N\|_k < C_k \quad \text{for any} \quad k \geq 1. \qquad (2.55)$$

By the Hölder inequality it follows that for some $\varepsilon \in (0, \frac{1}{2})$

$$\sup_{s\in[\tau,T_{max})} \|N(\cdot,s)\nabla W(\cdot,s)\|_{4-\varepsilon} < C_{NW}$$

with a constant C_{NW} independent of time. Hence, in view of (R0) setting $\eta = 1$ and

$$Q = N\nabla W,$$
$$\varphi = (r+1)N - r_1 N^2 - F(N)P.$$

we find that $\varphi \leq \bar{\varphi} = \frac{(r+1)^2}{4r_1}$ and we infer that for $n \leq 3 < 4 - \varepsilon$ there is a constant C_N independent on time such that

$$\|N(\cdot,t)\|_{\infty} \leq C_N \quad \text{for} \quad t \in [0, T_{max}). \qquad (2.56)$$

Owing this we proceed to find an estimate for ∇N. To this end we look at the N-equation as at the parabolic equation of form (2.15) with $\eta = 1$ and in view of (2.54), (2.56) and boundedness of $\int_{\Omega} P^2(\cdot, t)$ being the component of $y(t)$ we obtain may set $q = 4$, $q_0 = 2$ in (2.17)-(2.18). The variation of constant formula for (2.15) reads

$$u(t) = e^{-(A+I)(t-\tau)}u(\tau) + \int_{\tau}^{t} e^{-(t-s)(A+I)}\nabla \cdot Q(s)ds + \int_{\tau}^{t} e^{-(t-s)(A+I)}\varphi(s)ds \qquad (2.57)$$

for $t \in (\tau, T_{max})$. Notice that for $u \in D(A + I)^\theta, \theta \in (0, 1)$, and $m \in \{0, 1\}$ there exists $\tilde{C}_1$

$$\|u\|_{m,p} \leq \tilde{C}_1 \|(A + I)^\theta u\|_q \qquad (2.58)$$

where $p \in [1, \infty]$ and

$$\frac{m}{2} - \frac{n}{2p} < \theta - \frac{n}{2q}. \qquad (2.59)$$

On the other hand using properties of analytic semigroup we get (see e.g. [17] or [42]) for $\varphi \in L^{q_0}(\Omega)$, $q \geq q_0$, there exist $\tilde{C}_2 > 0$ and μ_0 such that

$$\|(A + I)^\theta e^{-t(A+I)}\varphi\|_q \leq \tilde{C}_2 t^{-\theta - \frac{n}{2}(1/q_0 - 1/q)}e^{-\mu_0 t}\|\varphi\|_{q_0}. \qquad (2.60)$$

Moreover, using [42, Lemma 2.3] for the taxis term we have that for some $\tilde{C}_3 > 0$ and $\varepsilon \in (0, \frac{1}{2})$

$$\|(A + I)^\theta e^{-t(A+I)} \nabla \cdot Q\|_q \le C_3 t^{-\theta - \frac{1}{2} - \varepsilon} e^{-\mu_0 t} \|Q\|_q. \tag{2.61}$$

Thus, making use of (2.57), (2.58), (2.60) and (2.61) we obtain for $t > \tau$ and $\tilde{C}_0 = \max\{\tilde{C}_1 \tilde{C}_2, \tilde{C}_1 \tilde{C}_3\}$

$$\|u(t)\|_{m,p} \le \tilde{C}_1 \|(A + I)u(t)\|_q \le \tilde{C}_1 (\|(A + I)^\theta e^{-(t-\tau)(A+I)} u(\tau)\|_q$$

$$+ \tilde{C}_2 \int_\tau^t (t - s)^{-\alpha} e^{-\mu_0(t-s)} \|Q(s)\|_{X_q} ds + \tilde{C}_3 \int_\tau^t (t - s)^{-\beta} e^{-\mu_0(t-s)} \|\varphi(s)\|_q ds$$

$$\le \tilde{C}_0 \{(t - \tau)^{-\theta} \|u_0\|_q + \int_0^\infty \sigma^{-\alpha} e^{-\mu_0 \sigma} d\sigma (\sup_{t \in [\tau, T_{max})} \|Q(t)\|_{X_q})$$

$$+ \int_0^\infty \sigma^{-\beta} e^{-\mu_0 \sigma} d\sigma (\sup_{t \in [\tau, T_{max})} \|\varphi(t)\|_{q_0})\}$$

$$\le \tilde{C}_0 \{(t - \tau)^{-\theta} \|u(\tau)\|_q + \Gamma(1 - \alpha) \mu_0^{1-\alpha} \sup_{\tau \in [0, T)} \|Q(t)\|_{X_q} + \Gamma(1 - \beta) \mu_0^{1-\beta} \sup_{t \in [\tau, T)} \|\varphi(t)\|_{q_0}\}$$

where $\Gamma(\cdot)$ is Euler's gamma function

$$\Gamma(a) = \frac{1}{x^{-a}} \int_0^\infty \sigma^{a-1} e^{-x\sigma} d\sigma \quad \text{for } a > 0, x > 0$$

and α and β satisfy

$$\alpha = \frac{1}{2} + \theta + \varepsilon < 1, \quad \beta = \theta + \frac{n}{2}\left(\frac{1}{q_0} - \frac{1}{q}\right) < 1 \tag{2.62}$$

(see e.g. [17]). Next, setting in (2.59) $q = 4$, $q_0 = 2$ and $m = 1$ we get

$$\frac{1}{2} - \frac{n}{2p} + \frac{n}{8} < \theta. \tag{2.63}$$

On the other hand the condition $\beta < 1$ implies

$$\theta < 1 - \frac{n}{8}$$

and for $n \le 3$ there is such a θ which satisfies condition $\theta < \frac{1}{2} < 1 - \frac{3}{8}$ and combining it with (2.63) we get $p < 4$. Thus, applying it to the N-equation we infer that there exists a constant $\tilde{C}_N$

$$\|\nabla N(\cdot, t)\|_p \le \tilde{C}_N \quad \text{for } t \in [\tau_1, T_{max}) \quad \text{for } p < 4.$$

where $\tau_1 \in (\tau, T_{max})$. It follows from (H0) that $\|G(P)\nabla N\|_q$ is bounded if we take for instance $q = 3.5$. Then of course $q > n = 3$ and applying the result (R0) to the P-equation we infer that there is a constant C_P such that

$$\|P(\cdot, t)\|_\infty \leq C_P \quad \text{for} \quad t \in [0, T_{max}).$$

Finally, from the W- equation and parabolic regularity we immediately obtain that

$$\|W(\cdot, t)\|_\infty \leq C_W \quad \text{for} \quad t \in [0, T_{max})$$

where C_W is a positive constant and we conclude that $T_{max} = \infty$, which completes the proof. □

3. Numerical results

This section is devoted to the numerical illustration of theoretical results discussed in the previous section. Our main goal is to show how the parameter σ prevents blow-up of solutions to model (1.3) with (1.6). To this end, we use open source FreeFem++ software to run numerical simulations in 2D for the model

$$
\begin{cases}
P_t = d_P \Delta P - \xi \nabla \cdot \dfrac{P}{1 + \sigma P} \nabla N + \dfrac{abNP}{1 + \beta N} - \delta P - \delta_1 P^2, \\[2mm]
N_t = d_N \Delta N + \chi \nabla \cdot N \, \nabla W - \dfrac{aNP}{1 + \beta N} + rN - r_1 N^2, \\[2mm]
W_t = d_W \Delta W + \gamma P - \mu W,
\end{cases}
\tag{3.64}
$$

which has the kinetic part as in the Bazykin model [7] with Holling type II functional response (1.8). A discretization of the problem is based on the finite element discretization scheme with $\Delta x = \Delta y = 0.125$ and time step $\Delta t = 0.001$. We applied fractional step method and used semi-implicit scheme for the weak formulation of the system. A fractional step method allows users to successively deal with each term of the PDE system.

It can be checked numerically that model (3.64) posses a constant steady state solution $E^\star = (P^\star, N^\star, W^\star) = (0.741, 1.016, 0.74)$ for the following set of parameter values:

$$
\begin{aligned}
& r = 2, \ r_1 = 1.8, \ a = 0.7, \ b = 0.9, \ \beta = 2, \ \mu = 0.01, \ \delta = 0.1, \ \delta_1 = 0.15, \\
& \gamma = 0.015, \ d_n = 1, \ d_p = 0.1, \ d_w = 0.05.
\end{aligned}
\tag{3.65}
$$

It turns out that for the parameter values (3.65) along with $\xi = 30$ and $\chi = 0.1$. the conditions (1.12)-(1.14) hold if and only if $\sigma > \sigma_c := 19.7$. Therefore in the forthcoming simulations, we intend to show that for $\sigma < \sigma_c$, predator-prey model (3.64) exhibits finite-time blow-up of solution and that assuming $\sigma > \sigma_c$ prevents blow-up formation. In order to investigate impact of σ on the solutions, we run simulations subject to suitably chosen initial functions with bell-shaped graphs, each resembling a spike with a smooth peak
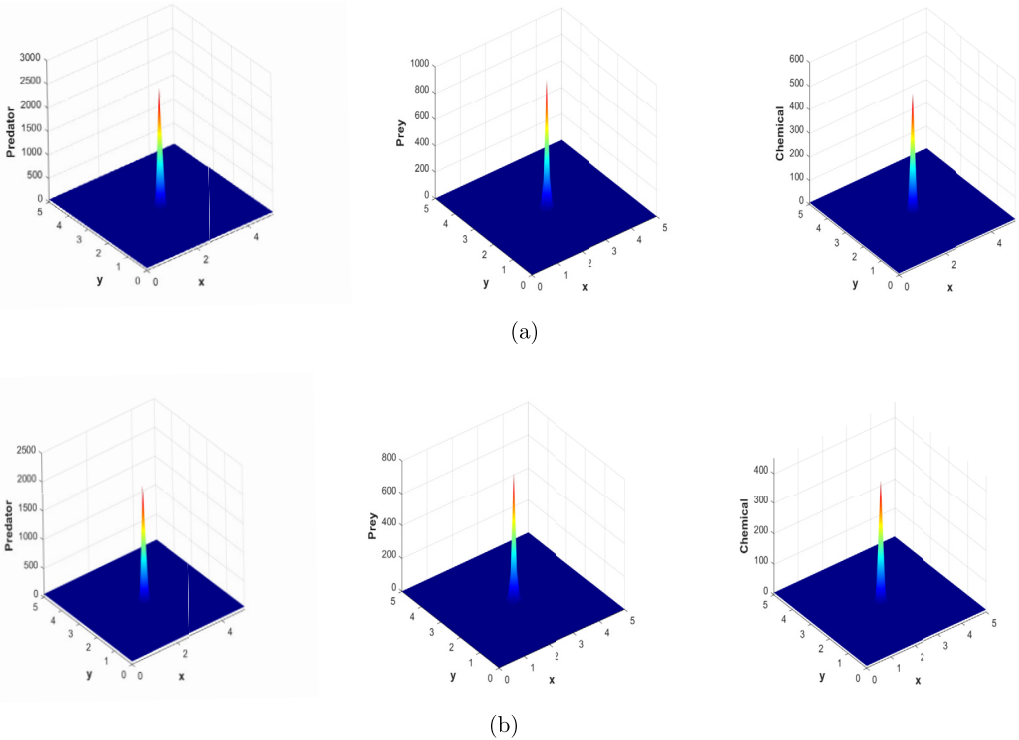
(a)

(b)

Fig. 1. (a) Approximated blowup solution at time $t = 1.5 \times 10^{-4}$ for $\sigma = 0.0$. (b) Approximated blowup solution at time $t = 2.3 \times 10^{-4}$ for $\sigma = 5.0$ subject to initial condition (3.66). The parameter values are as in (3.65) with $\chi = 0.1$, $\xi = 30$.

$$P(x, y, 0) = P^\star + 500e^{-100((x-2.5)^2+(y-2.5)^2)},$$
$$N(x, y, 0) = N^\star + 800e^{-100((x-2.5)^2+(y-2.5)^2)},$$
$$W(x, y, 0) = W^\star + 100e^{-100((x-2.5)^2+(y-2.5)^2)}$$

(3.66)

where $(x, y) \in \Omega = (0, 5) \times (0, 5)$. It is worth pointing out that the solution starting from an initial data of similar shape but less concentrated and smaller in L^∞-norm stabilizes after some time and in this case there is no blow-up formation. We assume that the numerical solution evaluated at some time moment t is an approximation of the blow-up solution if the numerical solver stops at time $t + \Delta t$, because the solution's gradient exceeds some suitably chosen high threshold. For given values of parameters the blow-up time was confirmed by the mesh refinement while the position in space of the singular solution is forced by the choice of the initial functions.

The Fig. 1a depicts results of simulations for the initial data (3.66) at the time moment preceding the stop of the numerical solver. The left column in figures shows spatial distribution of predator, the right one that of the chemical and in the middle the prey distribution is represented. During numerical simulations, it has been observed that predator population increases over time very rapidly and blow-up occurs within first 15 time steps (see Fig. 1a). A similar rapid increase has been also observed in the chemical concentration while the spike in prey density is also formed in the center of the domain that is close in height to the initial data. Here the prey appears to be passive and the indirect predator taxis does not help the prey to move out of the chemical

dominant zone. Similarly, in Fig. 1b, we observe that a relatively small $\sigma = 5$ does not have a significant impact on the solutions behavior and blow up occurs at time about $t = 2.3 \times 10^{-4}$. Further numerical simulations suggest that the blow-up time increases along with the increase of the parameter σ. It is worth noticing that the coefficient of direct prey taxis ξ is multiple times higher than predator taxis coefficient χ. This indicates that the predator avoidance by prey is very weak while the predator velocity is relatively high which helps the predators to reach the prey very quickly. Here the initial prey population is dominant in the center and before the prey leaves the center predator moves faster and becomes more packed, causing the explosion of predator density in the very center of the domain. As it was mentioned in the Introduction the occurrence of blow-up is a cumulative effect of both taxis mechanisms as in the cases of models with $\chi = 0$ and $\xi > 0$ or $\chi > 0$ and $\xi = 0$ it does not appear. Noticing that for this particular initial data the gradient of the chemical is directed toward the center of the domain we infer that even with small predator taxis coefficient χ they prey moves outward in such a way that the norm of the prey density gradient, oriented toward the center, increases significantly. This in turn amplifies the strength of prey taxis being oriented toward the center leading in consequence to the blow-up of predator density.

The Fig. 2 illustrates the situation when the control parameter σ is greater than the threshold σ_c. Keeping all other parameters and initial data the same as before and running simulations we observe that indeed selecting $\sigma > \sigma_c$ prevents the blowup formation and the numerical solutions starting from the bell shaped initial data rapidly converges to the steady state solution. In order to make clear the impact of σ on the solution development in time we chose $\sigma = 25$ and capture snapshots of numerical solutions at different time steps. In Fig. 2a, we observe that prey are quickly able to leave the center and form four small aggregations which manifest as density spikes at time step $t = 13$, however predator and chemical still remain dominant in the center (see Fig. 2b). In this figure, the predators seem to be passive and cannot effectively "chase" the prey. Snapshots captured at time step $t = 50$ suggest that prey leaves the center and begins to form a depletion region in the middle so that the graph of prey-density forms a volcano-like structure. Though the average density of prey and predator decreases gradually a weak predator avoidance can be noticed here so that during farther solution development the prey become spread around the central spot occupied by the predator as depicted in Figs. 2c–2d. We observe that prey density first drops significantly ($N \in (0.0365, 0.03675)$) at time $t = 100$ and in the long run at $t = 500$ it is restored such that $N \in (1.0095, 1.0105)$ which is very close to the prey component of the steady state solution. Similarly, the predator density also approaches its steady state value, and only the chemical density attains relatively high values $W \in (6.9, 7.5)$ at time moment $t = 500$ being stabilized only after a longer time which is not represented in the selected figures. This suggests that the steady state $E^\star$ to the model becomes locally stable and no singularity formation takes place in the long run. This observation suggests that the conditions (1.12)-(1.14) not only prevent blowup, but also promote the stabilization at the space-homogeneous steady state. Finally, we may conclude that along with the increase of σ the role of prey taxis is waning and solutions are expected to share some properties of predator prey model with indirect predator taxis studied in [26, Model B].

4. Final remarks and open questions

This paper contributes to the classical problem in the theory of reaction diffusion systems which amounts to finding a relation between the main diffusive part of the system and the reaction part such that solutions to the system are global-in-time. The main part of the system contains
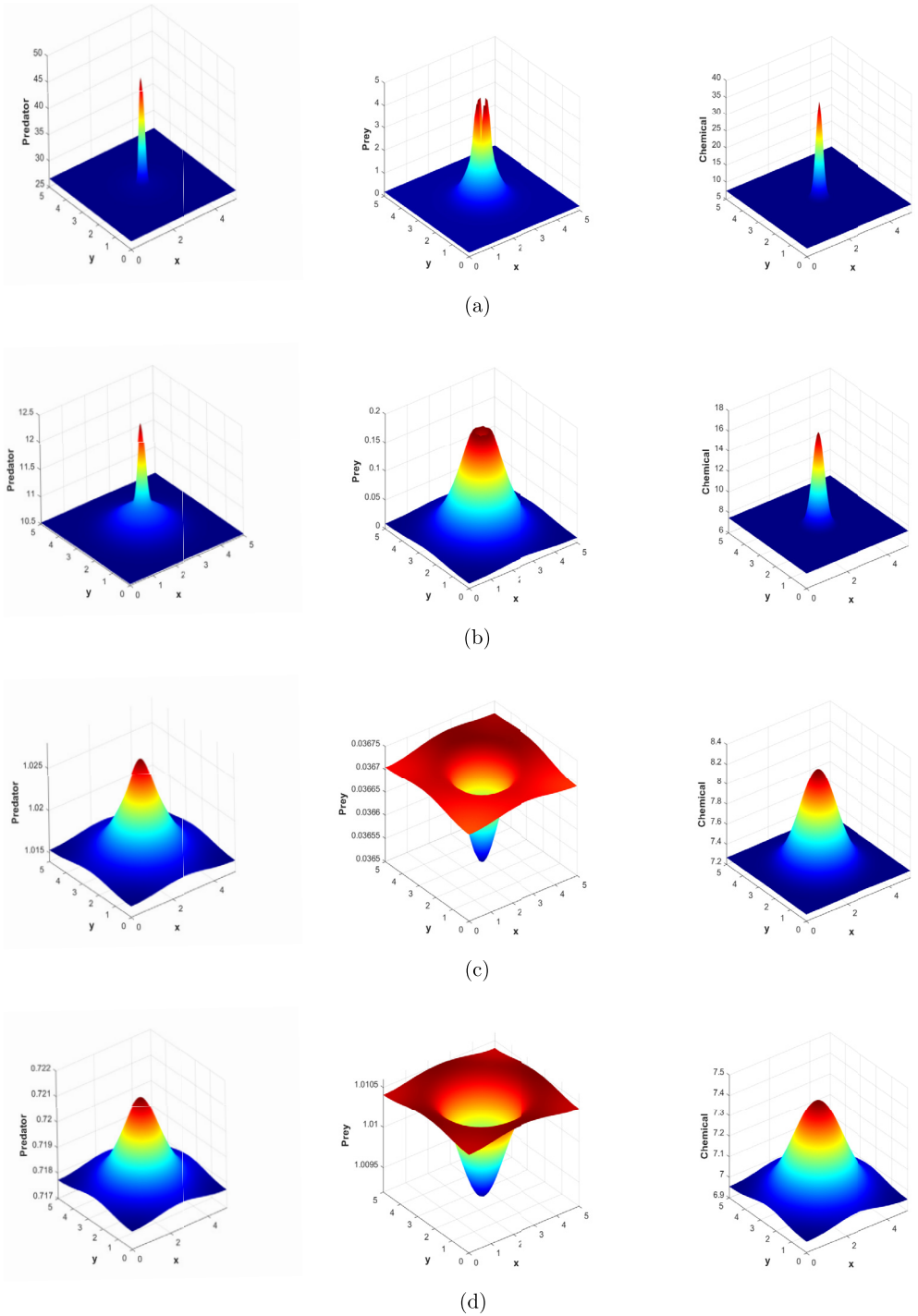
Fig. 2. Snapshots for $\sigma = 25$ at different time steps. (a) $t = 13$, (b) $t = 50$, (c) $t = 100$, (d) $t = 500$. All other parameter values and initial condition is same as in figure (1).

two taxis terms and the analysis of the problem led to introducing a kind of regularization in one of them which along with quadratic terms in the reaction part enable to find conditions precluding the blow-up of solutions. The regularization has clear interpretation in the context of the model being a density dependent suppression of the predators velocity.

The numerical simulations provide strong evidence that in 2D case the two taxis mechanisms jointly, without the aforementioned regularization, for some range of model parameters have significant impact on the finite time blow-up of solutions. Moreover, numerical experiments suggest that formation of singularity in finite time depends also on the size of the initial data and similarly to the Keller-Segel model of chemotaxis some initial density threshold is expected to exist below which there is no blow-up formation. The study of radial solutions may bring some insight into how the two taxis forces together induce the blow-up though none of them can do it alone. From the mathematical viewpoint it would be interesting also to study the existence of weak solutions being "weak enough" to grasp the singularities. To this and some estimates based on an entropy-like function are needed, so far unknown.

A finite-time blow-up in a population model suggests on one hand the appearance of processes leading to the formation of aggregations and on the other it indicates model limitations. Therefore finding realistic modifications of such a model which ensure prevention of overcrowding is important from the modeling viewpoint. There are many ways to attain it, one of them is to build in the model a density threshold that was already studied in the context of predator-prey model in [5]. Another way is to consider nonlinear diffusion operators as in [37]. In this article, we propose yet another approach that takes into account intraspecies competition in both predators and prey, as well as density-dependent velocity suppression in predators. This leads to finding a range of parameters such that the global solutions of the model exist without any restrictions on the "size" of the initial data. On the other hand, the range of parameters limits the strength of both taxis mechanisms. Thanks to the numerical simulations, we also noticed that the high value of the parameter σ related to the damping of the density-dependent velocity not only prevents the formation of blow-up, but also forces the stability of a constant steady state.

Similarly to the case when $\delta_1 = 0$, studied in our earlier paper [26, Model A], for model (1.3) studied in this paper periodic patterns in space and time (or quasi-periodic), triggered by the Hopf bifurcation, are expected to appear for some range of parameters. The quasi-periodic patterns essentially correspond to predators' pursuit of prey and their evasion from predators in space which manifests itself by occurrence of aggregations and low-density spots varying in time and space. However, when $\delta_1 > 0$ also Turing stationary patterns are expected similarly to the case of reaction -diffusion system with Bazykin's kinetic part without taxis [25]. The linear stability analysis of constant steady states, bifurcation analysis as well as numerical study of pattern formation for system (1.3) deserve a separate paper.

Data availability

No data was used for the research described in the article.

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