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En klasseromsstudie: Elevers kreative matematiske resonnement og lærerhandlinger. Designprinsipper for å lære ved og gjennom CCMR kompetanse

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“Isn't it splendid to think of all the things there are to find out about? It just makes me feel glad to be alive—it's such an interesting world. It wouldn't be half so interesting if we know all about everything, would it? There'd be no scope for imagination then, would there?”

—Lucy Maude Montgomery, *Anne of Green Gables*, 1908

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Abstract

This design-based research study presents results based on observations and analysis of student–student and student–teacher interactions in a Norwegian upper secondary school. The aim of this study was to examine student interactions during collaborative mathematical reasoning tasks about functions to identify insights to support collaborative problem-solving competency. The study also sought to investigate teacher actions in productive interactions and how students' potential learning outcomes are affected by interactions. Analysis of student–student interactions and related teacher interactions revealed strategies for facilitating productive problem solving among student dyads. The productive interaction pattern—a bi-directional interaction—presents inherent learning opportunities. This study adds to the field of mathematics education by suggesting an extension of the concept of collaborative problem-solving competency (CPS) by connecting the competencies of collaboration, reasoning, and problem solving in a new model for facilitating productive interaction in mathematics classrooms. The suggested competency model has potential as an analytical tool for teacher educators and researchers to utilize in classroom studies focusing on interactional patterns in students' mathematical problem solving.

Sammendrag

Denne design-baserte forskningsstudien presenterer resultater basert på observasjoner og analyser av elev–elev og elev–lærer interaksjoner fra en norsk videregående skole. Målet med denne studien var å undersøke elevinteraksjoner når de samarbeidet og resonnerte matematisk om funksjoner for å identifisere og gi innsikt omkring samarbeidskompetanse i problemløsing. Forskningsprosjektet undersøkte lærerhandlinger tilknyttet elevinteraksjoner og hvordan elevenes potensielle læringsutbytte påvirkes av interaksjonen. Analyse av elev–elev interaksjoner og relaterte lærerinteraksjoner avdekket strategier for å legge til rette for produktiv problemløsing blant elevparene. Det produktive interaksjonsmønsteret—en toveisinteraksjon—gir iboende læringsmuligheter. Dette forskningsprosjektet bidrar til matematikkdidaktikkfeltet ved å foreslå en utvidelse av konseptet «problemløsing ved samarbeid» gjennom å koble kompetansene samarbeid, resonnement og problemløsing til en ny modell for produktive interaksjoner i matematikklasserommet. Den foreslåtte kompetansemodellen har et potensial som et analytisk verktøy for lærerutdannere og forskere i klasseromsstudier med fokus på interaksjonsmønstre i elevenes matematiske problemløsing.

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As a student in elementary school, who found mathematics to be difficult to grasp, I could not have imagined myself doing research on the very subject that provoked tears in the classroom. Many years later, with the joy of being engaged with mathematics, I was educated as a mathematics teacher in upper secondary school. My interests were in students' learning of mathematics trough mathematical discussions, and students' difficulties with algebra. I am grateful for the opportunity of a doctoral scholarship in Mathematics Education at the University of Life Sciences in Ås, Norway, that allowed me to deepen and study my interests.

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Moss, June 2022

List of papers

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III. Hansen, E. K. S., & Naalsund, M. (2022). An analytical model for analysing interactional patterns in creative collaborative mathematical reasoning. Submitted

1. Introduction

This design-based research project focused on productive student–student interactions in student-centered classrooms, related teacher actions, and assessments of these interactions. A student interaction can be defined as "a complex social phenomenon which is composed of non-verbal and social properties in addition to its verbal characteristics" (Kumpulainen & Mutanen, 1999, p. 455). There are inherent learning opportunities involved in sharing ideas and experiencing mathematics as meaningful to oneself and together with peers (Krummheuer, 2007; Sidenvall, 2019). If given the opportunity, students can construct their own solution procedures to mathematical tasks and problems, which is important to their mathematical understanding (cf., Lithner, 2017; Mueller et al., 2012; Stockero et al., 2019). In such situations, they may propose ideas and answers, defend and justify their ideas, and become producers of mathematics (Schoenfeld, 2013). Thus, how well students interact in pairs or in small groups is central to their progress in becoming a producer of mathematics rather than an imitator who reproduces mathematics without understanding the conceptual parts (Lithner, 2017).

Student-centered environments that focus on collaborative interactions have been central to mathematics education reforms and in research on classroom practices for decades (Mueller et al., 2012; NCTM, 2014; Webb, 1982). However, there is still a need for knowledge on how to facilitate productive student interactions (Langer-Osuna et al., 2020), further exploration of the processes in students' collaborations (Seidouvy & Schindler, 2019; van de Pol et al., 2018; Varhol et al., 2020), and specifically on students' interactions when reasoning mathematically (Erath et al., 2021). Therefore, investigating of students' participation and their participation patterns with their inherent interaction aspects could provide further insights into quality student interactions and the dynamics of the processes involved. These insights may subsequently strengthen the knowledge of instructional design for teachers, the design of tasks, and curricula (Erath et al., 2021).

Collaborative problem-solving competency (CPS) involves learning mathematics together in a problem-solving setting and is, therefore, a key competency connected to the project's focus. CPS can be defined as a "coordinated attempt between two or more people to share their skills and knowledge for the purpose of constructing and maintaining a unified solution to a problem" (c.f., OECD, 2017, Roschelle and Teasley, 1995, as cited in Sun et al., 2020, p. 2). In recent years, CPS has become an important domain-independent skill for students to succeed in group-based activities (Sun et al., 2020). Moreover, CPS is recognized as a so-called 21st-century skill; that is, a skill that is key to successfully connecting and working together globally and locally in schools, workplaces, and communities (Child & Shaw, 2016). *Mathematical competency* can be defined as "a clearly recognizable and distinct, major constituent of mathematical competence" (Niss, 2003, p. 7). Competency-related activities involve interpreting, doing and using, and judging mathematics (Boesen et al., 2014). Thus, mathematical competencies can be understood as specific ways of understanding and doing mathematics involving the utilization of different skills. CPS can be developed in classroom settings when students get to attend to others' ideas and actions, merging different thoughts and knowledge into a unified solution (Cobb, 1995; Graesser et al., 2018; Mueller, 2009). To do this, one must evaluate others' input, possibly negotiate different points of view, and come to an agreement. Such actions require knowledge about the domain, the ability to work effectively in and with diverse groups, and the sharing of responsibility (Lai et al., 2017). Therefore, CPS is recognized as a skill that may enhance collaborative learning in schools and further contribute to personal success in workplaces (Lai et al., 2017; Sun et al., 2020).

A recent review of research literature on CPS focuses specifically on the CPS concept, rather than related terms, such as "teamwork," "cooperation," and "problem-solving" (Sun et al., 2020). The work of Roschelle and Teasley (1995), Nelson (1999), the Assessment of Teaching of Twenty-first Century Skills (ATC21S) (Griffin et al., 2012), and the PISA framework for summative assessment of CPS skills (OECD, 2017) are considered influential frameworks that emphasize the CPS concept (Sun et al., 2020). A unifying feature of the four frameworks is the underlying constructs of CPS skills: establishing shared knowledge, resolving divergence and misunderstanding, monitoring progress and results, and maintaining a functional team (Sun et al., 2020).

In this research project, three related interaction aspects are emphasized: mathematical reasoning, collaborative processes, and exercised agency. Moreover, the four foundational constructs of CPS skills serve as a foundation in this research project as well. By identifying the interplay of interactional components tied to the CPS construct, another approach is suggested for understanding students' CPS competency and related learning outcomes. This approach emphasizes students' interaction patterns and related teacher actions. The three interactional aspects and teacher actions are presented below.

Mathematical reasoning is defined as "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2017, p. 939). When suggesting an idea, one can collectively or individually create "lines of thought," such as building upon an argument, justifying an idea, or explaining why something is true or not. The definition of mathematical reasoning is connected to an empirically developed framework known as *creative mathematically founded reasoning*[1] (CMR), which defines different paths of reasoning in order to reach a conclusion in task solving (Lithner, 2008, 2015, 2017). *Collaborative process* is defined as "a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (Roschelle & Teasley, 1995, p. 70). This definition views students' collaboration through different activities that produce unified solutions. Furthermore, Roschelle and Teasley (1995) suggest that the co-construction of a shared understanding happens through processes of building, monitoring, and repairing meaning or a strategy for solving a problem. These processes are characterized by an interplay of ideas, which are interwoven into shared actions and shared explanations (Martin & Towers, 2015). The project views students' *exercised agency* as "the way in which he or she acts, or refrains from acting, and the way in which her or his action contributes to the joint action of the group in which he or she is participating" (Gresalfi et al., 2009, p. 53).

Facilitating reasoning and argumentation in mathematics classrooms is challenging, and there is a need for a better understanding of how to facilitate these aspects (e.g., Ayalon & Hershkowitz, 2018; Maher et al., 2018; van de Pol et al., 2018). Teachers require knowledge about students' interactions and how to encourage students to share their thinking with one another, which may include their reasoning processes (Lithner, 2008) and how they act or refrain from acting in the mathematical conversations–their agency (Gresalfi et al., 2009; Mueller et al., 2012). Drageset (2014) developed a detailed framework of teacher actions for the purpose of studying in-depth specific teacher actions and how they influence

[1] In line with the work of Lithner and colleagues (2008, 2017) on creative, mathematically founded reasoning, the present project uses the wording "creative reasoning" or the acronym "CMR" for linguistically simplicity.

students' interactional patterns in situations of collaboration, reasoning, and exercised agency (though this was not specifically connected to the interactional patterns focus on in this project). This *redirecting, progressing, and focusing actions* framework (Drageset, 2014) made it possible for me, as the researcher, to investigate teacher actions related to student interactions. The three interaction aspects and teacher actions are further elaborated on in chapter 2.

The first three constructs of the CPS competency (establishing shared knowledge, resolving divergence and misunderstanding, monitoring progress and results) are similar to the three collaborative processes in this project—building, monitoring, and repairing—for establishing and maintaining a shared understanding (Roschelle & Teasley, 1995). The last area, maintaining a functional team, concerns processes of upholding the collaborative dynamics. These processes could be informed by the collaborative processes or an interplay of different interactional aspects, such as students' exercised agency and collaborative processes. Therefore, an in-depth study of interwoven interactional aspects facilitating quality student interaction may extend the CPS competency. By extending means emphasizing how details of student interactions and related teacher actions provides a foundation to operationalize productive interactions through insights of interaction patterns and inherent learning opportunities. Consequently, aspects of productive interactions may be more recognizable to students, teachers, teacher educators, and researchers. This may prove a useful tool for each party. For instance, students can recognize their interaction as productive and learn through and of the competency, and a teacher can recognize characteristics of students' interaction patterns and further support and guide productive interactions.

Aim

This research project comprises three studies: two case studies (Article 1 and Article 2) and one conceptual contribution (Article 3). The aim of the doctoral thesis is to tie the three studies together and build on previous research to extend the knowledge on productive student interactions related to collaborative problem-solving competency. Considering competency aspects in productive student interaction patterns might improve the *learning of* and *learning through* a competency. *Learning of* a competency is the learning goal of specific skills comprising a competency; in comparison, *learning through* involves using the skills of a competency as a means of learning (Sidenvall, 2019).

The theoretical points that are made in synthesizing the findings and building on related research literature are guided by the following research question:

> *How can a design-based research process focusing on aspects of students' interactions and related teacher actions contribute to design principles that support productive student interactions in mathematics?*

The aims of the three articles were as follows:

- Article 1: To investigate the patterns of interplay between creative reasoning, collaboration, and exercised agency in a mathematical problem-solving session involving more or less productive student interactions.
- Article 2: To explore interactional conditions in light of the role of teacher actions with their limitations and opportunities to influence the productivity of students' interactional patterns in the collaborative problem solving of linear functions.
- Article 3: To explore how the interactional aspects were connected through students' ways of participation and how their participation in dyads serves as an indicator for assessing their interaction as productive or unproductive. This resulted in an analytical model that was informed by the results of Article 1 and Article 2 and a further data analysis.

The context of the study was a Norwegian upper secondary school. The participants included three teachers and their mathematics classes. The students were 15–16 years old and enrolled in their first year of a chosen theoretical mathematics program. During the research period, the teachers and I planned lessons together that focused on functions. The function concept is integrated into many areas of school mathematics and has a central role in organizing and connecting many mathematical ideas (Michelsen, 2006). Thus, it is important to connect different function representations, to prevent a fragmented view on functions (Best & Bikner-Ahsbahs, 2017). With its diverse representations and with the need of algebraic knowledge (Leinhardt et al., 1990; Lepak et al., 2018), it is often an area of difficulty for many students. There is substantial research on how to develop a rich understanding of the function concept, but less attention has been paid to how teachers should design instruction and curricula to help students overcome difficulties with the function concept (Dubinsky & Wilson, 2013). To summarize the motivation for choosing functions as the area of study while studying students' interactions: the function topic was suitable to the students' curriculum, it engaged the teachers, and it reflected a topic that was difficult for the students to understand yet important for connecting many mathematical ideas.

2. Theoretical foundations

How students participate when working collaboratively influences their productivity, their progression, and their learning opportunities. For instance, encouraging a peer to make suggestions, listening to a peer, justifying an idea, exploring different ideas, and sometimes just being nice to a peer could make all the difference to a group's dynamics and progress. However, in a collaborative mathematical situation, some interactional aspects are more prominent than others. For students to learn mathematics together, it is crucial that they reason mathematically together. Creating a reasoning sequence anchored in mathematical properties, which is not solved by familiar imitative reasoning, ensures that a student 1) must be responsible for the reasoning, 2) must verify a suggested solution or strategy choice, and 3) must explain why the mathematical concepts used are relevant, thus involving mathematical properties (Lithner, 2015). Therefore, when students are engaged in "the explicit act of justifying choices and conclusions by mathematical arguments" (Boesen et al., 2014, p. 75), they are presented with learning opportunities related to the involved mathematics (Lithner, 2017).

When engaging in collaborative mathematical reasoning, students can establish a shared understanding of a mathematical problem. Martin and Towers (2015) describe a shared understanding as a collective mathematical understanding consisting of an "ever-changing interactive process, where shared understandings exist and emerge in the discourse of a group working together" (p. 6). Coming to a unified understanding involves a shared learning process in which students share ideas through verbal expressions of suggestions, explanations, and disagreement. Thus, collaborative processes (Roschelle & Teasley, 1995) are a central interaction aspect of collaborative mathematical reasoning.

Although students might collaborate well, they sometimes withdraw from the conversation but eventually continue their mutual work where they left off. Other times, they might consistently work toward a unified solution. Thus, students may participate in different roles, which can quickly change during problem solving (Child & Shaw, 2019). Therefore, a third central aspect of productive student interactions is the consideration of students' exercised agency (Mueller et al., 2012) in collaborative reasoning situations. Students' agency is situational and can be seen in how students participate and act in a group (Gresalfi et al., 2009). Thus, an individual's method of participation is not a fixed pattern of actions but is dependent on the collaborative processes and mathematical reasoning involved in a group or a community (Gresalfi et al., 2009; Langer-Osuna, 2018; Mueller et al., 2012).

2.1 Characteristics of the three interactional frameworks

The frameworks related to the three interactional aspects defined in chapter 1 and further elaborated on below have different theoretical underpinnings. Although four central frameworks were used in this project, three of them were used to study student–student interactions. Therefore, teacher actions are not included in the discussion below but are elaborated on in section 2.3.

This research project adopted collaborative processes based on Rochelle and Teasley's (1995) definition of collaboration. The goal of the processes is for students to make mutual utterances and engage actively in suggesting, questioning, and listening. Thus, coordination (Baker, 2015; Sarmiento & Stahl, 2008) and turn-taking (Sidnell, 2010) are important aspects of developing a shared understanding through reasoning (Barron, 2000). Collaborative work may begin or progress through the suggestion of ideas for how to solve a problem or the exploration of ideas, which indicates the process of *building* a shared conception of a problem (Roschelle & Teasley, 1995). This is an inquiry process involving the introduction of new ideas (Alrø & Skovsmose, 2004; Child & Shaw, 2019). If students continue to build, they may "zoom in" on an idea (Alrø & Skovsmose, 2004) to further investigate a proposal. Students may also address givens and constraints about the proposal (Sun et al., 2020). When explaining and exploring the details of an idea, actions such as drawing figures, calculating, or using digital software facilitate the building of a shared understanding. Thus, building a shared understanding is important in initiating the process of collaboration, as well as continuing or finishing a problem-solving process.

If a suggested idea is not making sense, a peer can ask questions about the suggestion, which can then be explained. Having different perspectives, asking questions, and providing explanations are important parts of *monitoring* the collaborative work (Roschelle & Teasley, 1995). In monitoring the collaborative problem-solving process, students can experience their ideas being built upon. Thus, a peer is able to influence the thoughts and actions of the group. Such social interdependence is positive and implies a degree of synchronicity (Child & Shaw, 2019). If students believe they can succeed by themselves only if others fail and not together as a group, their interdependence is negative (Child & Shaw, 2019). Experiencing a negative interdependency might lead to asynchronous activity (not working together at the same time) or no activity at all. Thus, acknowledging others' contributions support the problem-solving progress (Sun et al., 2020).

Sometimes, students experience divergences in their understanding or opinion of which strategies to use, which indicate conflicting ideas or a lack of understanding of one another. Experiencing divergences in opinions means that students need to *repair* their shared understanding (Roschelle & Teasley, 1995). Resolving conflicting ideas can involve reformulating ideas, such as paraphrasing or repeating utterances in one's own words (Alrø & Skovsmose, 2004). Clarifying misunderstandings provides learning opportunities by making thoughts easier to understand (Sun et al., 2020). Thus, this presents opportunities to justify mathematical arguments. The utilized framework on collaborative processes views learning as a social activity, and the theory is grounded in *socio-cognitivism* (Roschelle & Teasley, 1995).

Students' mathematical discussions can be viewed through the CMR framework (Lithner, 2017). Mathematical reasoning is an important interactional aspect of learning mathematics through the processes of argumentation (Krummheuer, 2007; Yackel, 2001). Argumentation is a method of reasoning in which one justifies thoughts and ideas, aiming to convince oneself or someone else that the reasoning is appropriate and correct (Bergqvist et al., 2007). Thus, mathematical reasoning can be seen as a thinking process that produces claims and conclusions that may involve more or less correct assumptions (Lithner, 2015). Moreover, students' mathematical reasoning may provide opportunities for students to learn "how to find solution methods by themselves or how to engage in other mathematical processes" (Lithner, 2017, p. 937). This contrasts with students' use of task-solution templates without considering conceptual parts, which can lead to rote learning. Although rote learning can "free up cognitive resources to be used for more advanced problem solving," there is "little or no transfer" from imitation of solution templates to developing central mathematical competencies (Lithner, 2017, pp. 937-938). In the research project, students' mathematical reasoning was viewed as an interactional accomplishment in which participants accepted and suggested arguments individually and collectively; this did not consider whether the arguments were formal or logically correct (Lithner, 2017; Yackel, 2001). Through the CMR framework and related principles to facilitate CMR, students' creative reasoning might enhance task fluency and mathematical understanding (Lithner, 2017), and thus students might be prevented from engaging in mainly rote learning.

The CMR framework identifies two main types of reasoning: creative reasoning and imitative reasoning (Lithner, 2008). Three criteria must be fulfilled to call a reasoning sequence "creative" (Lithner, 2017): it must be creative, plausible, and anchored. *Creativity* refers to creating a reasoning sequence not experienced previously or re-creating a forgotten one. *Plausibility* involves arguments supporting the strategy choice or strategy implementation that explain why the conclusions are true or plausible. *Anchoring* means that arguments are anchored in the intrinsic mathematical properties of the components of the reasoning. Arguments are considered to be intrinsic if they are based on mathematical concepts or relations; they are considered superficial if based on appearance and not on underlying mathematics.

Imitative reasoning is recognized as memorized reasoning or algorithmic reasoning. An example of memorized reasoning is remembering that each of the angles of an equilateral triangle measure 60°. Algorithmic reasoning is when the students use a given or recalled algorithm to solve a task. The strength of using an algorithm in school mathematics is the speed and the high reliability it provides when solving a task (Lithner, 2015). However, if the algorithm is used without its conceptual component, such as consideration of its meaning, it may lead to rote learning (Lithner, 2017). The CMR framework concerns individuals' mathematical reasoning and is grounded in *social constructivism* (Lithner, 2017).

Participating in a dyad or in a group means that a student can engage in mathematical problem solving, which is important to developing a mathematical voice and becoming recognized as a producer of mathematics (Schoenfeld, 2013). Moreover, a sense of agency can be produced through progress in problem solving while engaged in a meaningful task (Schoenfeld et al., 2019). Thus, attempting to collaborate with peers can influence students' experience of having agency in a problem-solving situation. When students interact during a mathematical activity, the interaction functions as an opportunity to communicate ideas and position students in relation to one another (Langer-Osuna, 2018). This reflects a "distribution of authority" and can fundamentally affect the possibilities of collaborative mathematical reasoning (Langer-Osuna, 2018). In line with this perspective, Mueller et al. (2012) present a framework for students' argumentation connected to exercised agency in different discursive practices when collaboratively solving mathematics problems. Students can exercise *shared agency* when co-constructing arguments (Mueller et al., 2012). In those situations, students simultaneously build arguments from the ground up, and "without one of the participants, the argument would not exist" (Mueller et al., 2012, p. 378). Moreover, students may exercise individual forms of agency, called *primary agency* or *secondary agency.* A student is a primary agent in situations in which he or she produces the final argument based on corrections from a peer or by making sense of a peer's faulty or flawed idea. A secondary agent provides input that influences the original argument. These forms of input include corrections, extensions, or flawed arguments, which are then further formed into a final argument by the primary agent (Mueller et al., 2012). Both a primary agent and a secondary agent can be responsible for the original idea.

In this project, agency is defined as a student's participation through acting or resisting acting in a mathematics conversation (Gresalfi et al., 2009) in which agency is exercised individually or shared through attempted collaboration. The theoretical concepts used to analyze exercised agency (Gresalfi et al., 2009; Mueller et al., 2012; Pickering, 1995) originate from work in the *sociocultural* tradition.

2.2 Compatibility and connecting the three interactional frameworks

The theoretical foundations of the three frameworks are anchored in different perspectives: cognitivism, constructivism, and socioculturalism. Thus, there are different perspectives of how students' knowledge acquisition works and whether the knowledge is "already there" and, to some extent, shaped by the student (cognitivism), constructed internally by the student (constructivism), or shaped by the student's social interactions (socioculturalism) (Cobb, 1994; Lerman, 1996). However, the social aspect of the frameworks does connect them through the perspective that it is in a social context of interaction that a student's learning develops. The compatibility and connection between the three frameworks are elaborated on below.

Research on collaborative problem solving can have different focuses and approaches. One approach in classroom studies is, for instance, to examine students' conversations or learning outcomes. Another approach is to look at the conditions for organizing collaboration, such as the types of questions used in group discussions, the designated roles of the participants, and the variety of problem types to be solved. These different focuses are commonly divided into a focus on the *collaborative process* or a focus on the *collaborative outcome* (Dillenbourg et al., 1996; Lai et al., 2017; Seidouvy & Schindler, 2019). This division is found in educational studies (Child & Shaw, 2019), as well as in studies of mathematics education studies (Mueller et al., 2012). When research is focused on the use of language and interaction in *collaborative processes*, such as in this research project, the underpinning theories are mainly cognitive theories and sociocultural theories (Seidouvy & Schindler, 2019). If the focus is on the *collaborative outcome*, the theories represented are primarily cognitive theories (Seidouvy & Schindler, 2019).

When referring to outcomes, one can talk about students' learning observed during pre- and post-tests (Child & Shaw, 2019) or students' solution procedures in solving a mathematical task. The outcome of students' individual reasoning or co-constructed reasoning is another focus of this project. To summarize: the process view comprises social aspects, such as students' interactions in collaborative work, and the outcome view concerns individuals' learning.

A theoretical framework for research on collaborative problem solving should view the intertwined nature of social and individual aspects to prevent a social-individual dichotomy (Noorloos et al., 2017). To overcome the tension between theories on social versus individual, Noorloos et al. (2017) suggest using a theory that "can describe the learning activity of the student simultaneously and essentially in both cognitive and social terms" (p. 441). In this project's research studies the three interactional frameworks comprised the coding framework (Appendix A). The frameworks were utilized for studying the interplay of the interactional aspects, which is a multi-faceted phenomenon. Such a phenomenon "…cannot be described, understood or explained by one monolithic theory alone, a variety of theories is necessary to do justice to the complexity of the field" (Bikner-Ahsbahs & Prediger, 2010, p. 484). Thus, a fruitful starting point for operating with different theories and theoretical approaches is to connect them for further development in mathematics education (Bikner-Ahsbahs & Prediger, 2010). This supports the suggested approach to Noorloos et al. (2017) above.

Connecting theories may contribute to "gaining a more applicable network of theories to improve teaching and learning in mathematics education" (Bikner-Ahsbahs & Prediger, 2010, p. 491). A fundamental criterion for compatibility is a consistency between frameworks, according to Jungwirth (2010). Meaning that there is a shared ground where concepts from the theories are not conflicting but provide a starting point for data analysis and theory development. Regarding the theoretical underpinnings of the chosen frameworks: the CMR framework concerns individuals' mathematical reasoning and is grounded in social constructivism; collaborative processes view learning as a social activity, and the theory is grounded in socio-cognitivism and exercised agency, which originates from the sociocultural tradition. Although these three frameworks are differing in their theoretical underpinnings, they are logically compatible when focusing on describing learning activities in cognitive and social terms (Noorloos et al., 2017). Thus, it is possible to connect the frameworks for the purpose of studying both individual aspects of interactions, such as a student's contribution through suggestion or disagreement, as well as a group's outcome in their shared solution procedure.

2.3 Teacher actions

Asking students to work collaboratively on mathematical tasks is not enough to provoke agency (Mueller et al., 2012). Students are afforded agency when teachers attempt to make every student in a group or dyad accountable for their mathematical ideas (Langer-Osuna, 2018). Thus, shared authority is distributed between students and teachers (Langer-Osuna et al., 2020). Teachers can share authority, allowing students to exercise their own agency, by offering students opportunities to address mathematics problems and holding students accountable for their strategies, solutions, and ideas (Bell & Pape, 2012; Hamm & Perry, 2002, as cited in Langer-Osuna, 2018). When students are mutually engaged in problem solving, they take turns suggesting, explaining, and resolving misunderstandings, thereby exercising shared agency while co-constructing arguments (Mueller et al., 2012).

Classroom situations where students are afforded agency have the potential to foster conceptual agency (Cobb et al., 2009), meaning students have the opportunity to construct their own meaning and methods (Mueller et al., 2012). Moreover, if students choose problem solving paths and connect mathematical ideas, a teacher is more likely to support students' mathematical learning through shared agency (Cobb et al., 2009). Thus, students are held accountable for the co-constructed mathematical arguments in their reasoning. However, students often do not justify a reasoning sequence because they feel it is not necessary to convince anyone since a textbook or a teacher should be responsible for that (Bergqvist & Lithner, 2012).

In classroom situations where teachers exercise authority, students are only permitted to exercise disciplinary agency (Mueller et al., 2012). Disciplinary agency is a concept posed by Pickering (1995), a complementary concept to conceptual agency described as "utilizing established procedures" (Mueller et al., 2012, p. 374). Consequently, in teacher–student interactions with disciplinary agency, a teacher is responsible for determining the validity of student responses (Cobb et al., 2009; Lithner, 2015). Therefore, it is important that teacher actions facilitate students' co-constructed arguments in a shared agency and that teachers are knowledgeable about which actions to take (Maher et al., 2018).

The framework for teacher actions, called *redirecting, progressing, and focusing actions*, is both empirically and theoretically grounded (Drageset, 2014). The three main categories (redirecting, progressing, and focusing) elucidate tools and techniques teachers use to make students' thoughts and strategies visible, help students progress in their problem solving, and redirect students in an alternative direction (Drageset, 2014). The teacher interactions may facilitate different types of student responses (Drageset, 2015, 2019). The main teacher-action categories entail 13 subcategories (Appendix B) built on concepts drawn from theories about mathematical discourse grounded in perspectives on student-centered versus teacher-centered classrooms.

If the teacher actions components are detailed, they can provide greater insight into how different teacher actions influence students' interactions. Different teacher actions influence students' interactions when they collaborate, discuss, reason mathematically, and take ownership of a problem. Although teacher actions influence students' collaborative work, they are also shaped through interactions with students and by their methods of participation (Staples, 2007). This complex relation needs to be addressed by a fine-grained analytical model to provide detailed insights into teachers' role in promoting students' collaborative interactions. It is possible to investigate this using the framework provided by Drageset (2014) since it separates teachers' actions from students' talk and actions. Paying attention to students' specific interactional aspects, as emphasized above, and specific teacher actions provides opportunities to explore how teacher actions are related to students' interactional patterns.

Two overarching categories, *funneling* and *focusing* (Wood, 1998), organize the areas of teacher actions in Drageset's (2014) framework. If a teacher is *funneling* students' thinking, it means that "the student's thinking is focused on trying to figure out the response the teacher wants instead of thinking mathematically himself" (Wood, 1998, p. 172). Thus, mainly the teacher is doing the intellectual work. Redirecting and progressing actions are primarily categories of funneling actions in which the teacher is the intellectual authority. Drageset (2014) explains *redirecting actions* as corrections exhibited implicitly or explicitly (Alrø & Skovsmose, 2002). Moreover, redirecting actions are categorized as a teacher's attempt to challenge the students (Drageset, 2014), which means "questioning already established knowledge" (Alrø & Skovsmose, 2004, p. 55). An alternative to *funneling* is what Wood (1998) calls *focusing*, where the intellectual responsibility falls on the students. Hence, a teacher's focusing actions promoting productive interactions, as previously reviewed, concern facilitating reasoning (Ayalon & Even, 2016; Lithner, 2017; Maher et al., 2018), collaboration (Howe et al., 2007; Staples, 2007; van de Pol et al., 2018), and agency (Langer-Osuna, 2018; Mueller et al., 2012).

Thus, Drageset's (2014) three main categories—redirecting, progressing, and focusing— provided a useful approach to investigate opportunities and limitations of teacher actions for the productivity of students' interactional patterns. Using this framework, with its 13 subcategories (see Appendix B), interpreted in light of theories on teacher actions for the emphasized interactional aspects, allowed for greater exploration of teachers' guidance of student interactions.

3. Methodology

The project term comprised the iterative design steps from beginning to end through collaboration with the teachers, classroom interventions, analysis, and redesign. Each study represented part of the overall project. The studies are called Study 1, Study 2, and Study 3 (see 4. Summary of studies).

3.1 Design-based research project

The research project was a classroom study of student interactions during collaborative problem solving of function problems and their teachers' actions related to promoting mathematical reasoning and collaborative problem solving. DBR aims to "improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researchers and practitioners in real-world settings, and leading to contextually sensitive design principles and theories" (Wang & Hannafin, 2005, pp. 6-7). The key elements of the DBR methodology guided the project in the following way: 1) the basis of the project was a collaboration with teachers for a classroom study in a naturalistic setting; 2) the processes involved iterative cycles of planning, observing, and reflecting upon lessons; 3) the outcome of the project grew out of the iterative processes of analyzing student–student and teacher–student interactions and translating findings into design principles for classroom observations of students' creative, collaborative problem solving (see chapter 5). The three areas of DBR are addressed in greater detail in the following paragraphs.

A collaboration between teachers and researchers is a useful partnership when conducting research in the complexity of a regular classroom (Anderson & Shattuck, 2012). The researcher–teacher interaction provides contextual sensitivity to the research process and findings (Bungum & Sanne, 2021). A teacher possesses knowledge of school politics, the classroom culture, and his or her students. Thus, teacher involvement and role are therefore important in accounting for contextual factors in design procedures or principles that are likely to be relevant in a similar setting (Wang & Hannafin, 2005). A researcher contributes their theoretical knowledge of the object of study and frames for how to conduct the study. Thus, teachers and researchers often have different roles in a research study. The researcher must attend to balancing the theoretical and practical aspects and involving teachers and students in the process (Wang & Hannafin, 2005). Moreover, the role of the researcher is to be a facilitator and to adapt to the participants' perspectives, beliefs, and strategies. However, the researcher does not adopt the participants' values nor impose their own but considers others' input while aligning and extending the design process (Wang & Hannafin, 2005). Iversen and Jonsdottir (2018) suggest that a study's research design should be flexible in terms of teacher inclusion and in which phases they are included. Therefore, with the practical and theoretical perspectives being the researcher's responsibility, it is useful to consider how and when to involve teachers in the different aspects of the DBR process. However, such decisions should be intentional to avoid limiting the value of the research outcomes by exhibiting a possible lack of contextual sensitivity (Bungum & Sanne, 2021).

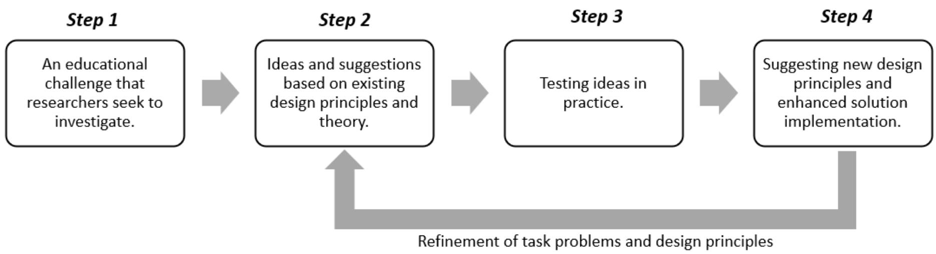


Fig. 1. The iterative DBR process (based on Amiel and Reeves (2008, p. 34))

The DBR process often starts with an educational challenge the researcher seeks to explore (see step 1 in Fig. 1, illustrating the DBR process). The process proceeds to testing possible solution designs through a tool, curricular, or pedagogical approach to intervene with the identified challenge (Vakil et al., 2016). The initial design, suggested in step 2, then evolves over time in an iterative cycle of analysis, design, implementation, and redesign in step 3 (Wang & Hannafin, 2005). This may result in design principles and theorization after reflections and discussions between researchers and teachers in Step 4 (Anderson & Shattuck, 2012). The identified challenge in the present research project started by emphasizing that teaching and learning mathematics often involve imitating solution methods from a textbook, a teacher, or a peer, resulting in rote learning (Lithner, 2017). Students' engagement in mathematical reasoning, CMR (Lithner, 2008), presented a potential solution to this problem. At the time this project was conducted, limited research on students' CMR had been studied in a naturalistic setting (Lithner, 2017). Thus, the starting point of the project was to explore students' CMR related to other aspects of mathematical engagement for learning and understanding.

DBR involves collaboration and context-specific iterative cycles and is theory driven through a "dual commitment to theory refinement and local impact" (Ryu, 2020, p. 234), visualized as step 4. Central in theory refinement is the process of generating design principles to achieve practice goals (Euler, 2017) (see Fig. 2 for a visualization of the process). The process begins with theory-based assumptions, which form principles to be tested and further developed. The design principles are theoretically anchored and further developed in light of iterative analysis processes. Thus, design principles function as a bridge between scientific knowledge production, in this case, the different interactional aspects and teacher actions, and a defined practice goal (Euler, 2017). The practice goal in the present project was for students to become competent mathematical producers through collaborative creative reasoning.

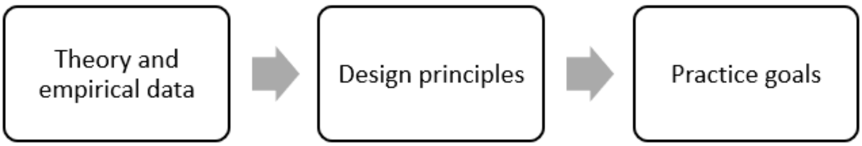


Fig. 2 A simplified example of design principles as a bridge between the theoretical foundation and a defined practice goal.

Theory generation from DBR studies is locally relevant to the specific context. However, design principles generated are also relevant outside a particular classroom study if properly anchored in theoretical claims that go beyond the local context (Barab & Squire, 2004). The design principles are prescriptive statements that should be formulated as explicitly as possible to achieve given practice goals (Euler, 2017). However, the design principles should not only describe the activities but also "transcend the immediate problem setting and context to guide designers in both evolving relevant theory and generating new findings" (Wang & Hannafin, 2005, p. 11).

This methodological perspective provided the structure of the process of investigating student–student interactions and student–teacher interactions, which resulted in design principles that are context-sensitive and possibly relevant beyond the given classroom context (see chapter 5).

3.2 Participants and data collection

The data collection took place in a Norwegian upper secondary school in 2017, where three teachers and their mathematics classes participated for five consecutive months (August–December). The participating teachers and students were informed of the research purpose and gave written informed consent according to the ethical requirements of the Norwegian Research Council (The Research Council of Norway, 2018). There were 69 students in total enrolled in the first year of a chosen theoretical mathematics program. Students were 15–16 years old.

In 2017, when the research project was conducted, Norwegian students were showing weak performances in the most basic areas of mathematics: numbers in primary school (age 6–12) and algebra in later school years (Grønmo & Hole, 2017). At the time, the concepts of variables and functions were first introduced to Norwegian students in grade 8–10 (age 13–15) mathematics.[2] It is now included in earlier years and explicitly as a competence after grade 6 in the new national curriculum: "use variables, loops, conditions and functions in programming to explore geometric figures and patterns" (Utdanningsdirektoratet, 2019). However, the students in the project had recently transitioned from grade 10 to grade 11 and had had no recent teaching on linear functions, except for those in lower secondary school. Learning about functions is important for connecting different mathematical ideas (Michelsen, 2006). Functions are also a useful tool not only in school mathematics but also in daily life for understanding models of change and development present in various subjects and in media. Thus, the topic of functions was of interest to the teachers, suitable for the students' curriculum, and could be considered an important mathematical area (Dubinsky & Wilson, 2013), yet one that is challenging for students to understand (Best & Bikner-Ahsbahs, 2017).

The three teachers with the pseudonymized names, Jacob, Lucas, and Sophie, had all worked for several years at the same upper secondary school. All the students were organized in dyads, in which they worked and sat in the classroom for the entire project period, including times when the researcher was not present.

[2] http://timssandpirls.bc.edu/timss2015/encyclopedia/countries/norway/the-mathematics-curriculum-in-primary-and-lower-secondary-grades/ (retrieved 29.08.2018)

In each of the three classes, two of the student dyads were followed closely in some of the problem-solving situations (see section 3.2.2). The student dyads were as follows: Emma and Hannah, and Sarah and Ella (teacher Jacob); Philip and Noah, and Leah and Isaac (teacher Lucas); Olivia and Oscar, and William and Maja (teacher Sophie). All student pairs worked on the same tasks in the classroom during the project period.3.2.1 Conversations with the teachers and implementation in the classrooms.

3.2.1 Conversations with the teachers and implementation in the classrooms

At the beginning of the project design process, the teachers and I held our first of 12 planned conversations (see Fig. 3). In the first meeting, I presented the research projects, their focus, and ideas for implementation. We also discussed practicalities for our future conversations, consent forms, and how to execute the video recordings. We agreed that I would make a draft outline for lesson planning sessions, informational sessions with the classes, and classroom observation sessions with and without cameras. Based on the draft, the teachers suggested dates and times. The appointments were flexible. There were 12 meetings during which we planned lessons together and evaluated planned function problems and the teachers' engagement. We ended with individual conversations by addressing the project process and the teachers' experiences with it. I visited the classrooms 24 times in total: six times to video record the entire classrooms, nine times to record specific student dyads during problem solving, three times for informational sessions with the classes, three times to observe the classes without cameras, and three times to attend their mathematical test and observing (see Fig. 3 for an overview of the timeline).

All the talks were audio recorded. In the orientation session about the project, I posed some questions to focus the teachers' thoughts on student-centered classrooms. For instance: What are your experiences with engaging students to talk about mathematics? What would you say are good guidelines for promoting collaborative work and mathematical talk? After the teachers shared their thoughts, I asked if they had any questions about the project and their expectations regarding their participation.

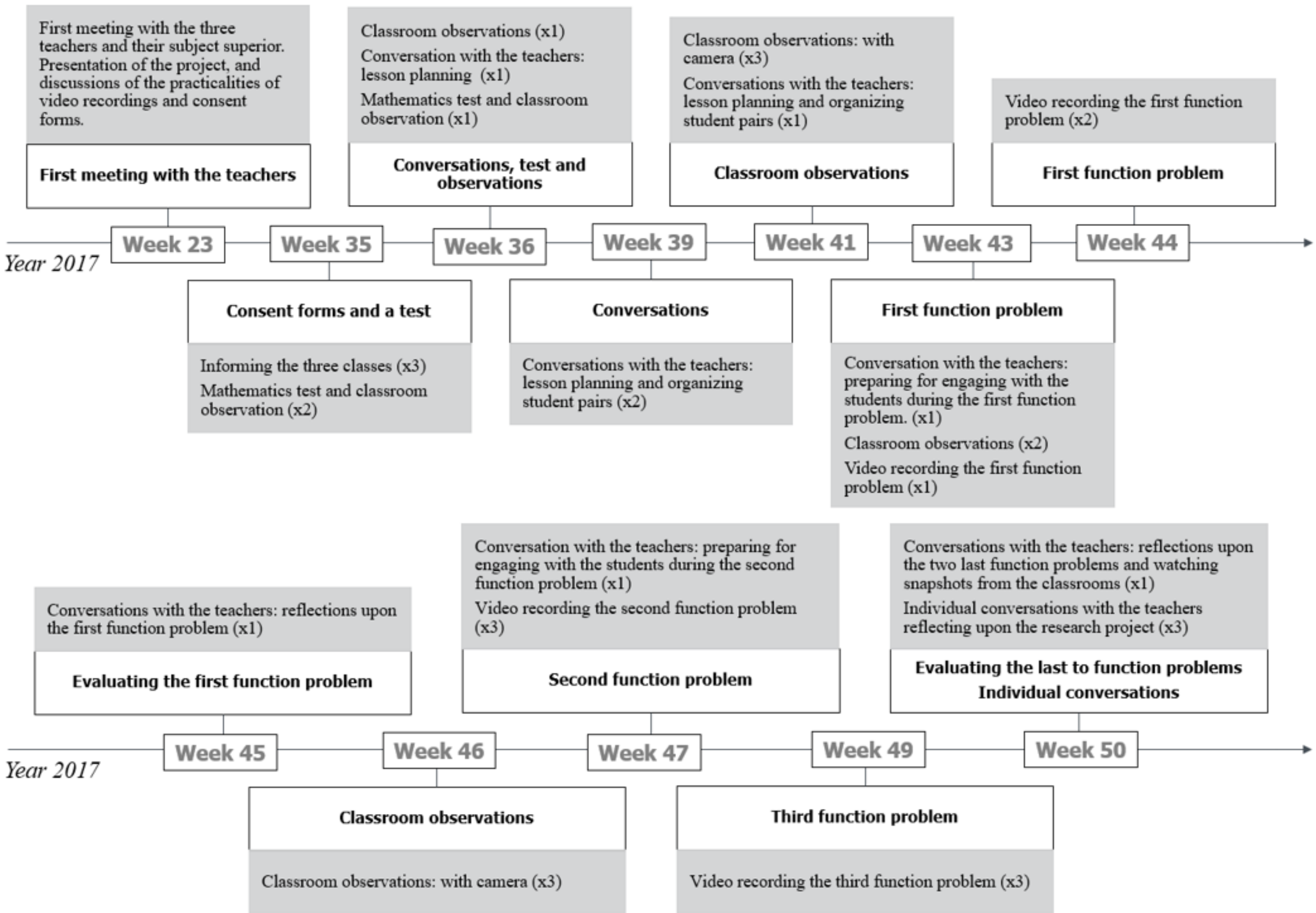


Fig. 3. Timeline of the DBR process of data collection.

I deliberately used the word “mathematical talk” in the conversation due to a previous conversation with Lucas. I had wanted to hear his opinions on the CMR framework (Lithner, 2008) and his reflections on students’ mathematical reasoning. From the conversation, we agreed on addressing the issue as “mathematical talk,” since he found the CMR framework difficult to discuss. “Mathematical talk” implied that students being observed were discussing a given task, a mathematical concept, or properties of a solution procedure.

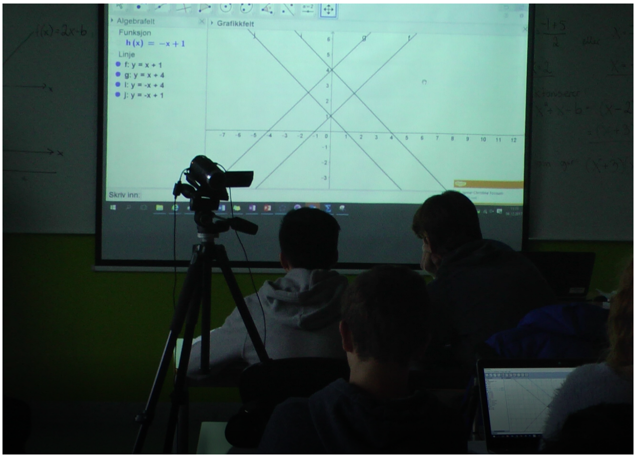
In subsequent conversations, we continued to discuss questions similar to the initial questions mentioned above. At the same time, I started observing the mathematics classrooms. My observations of student interactions, teacher–student interactions, and whole classroom talks and instructions, combined with the teachers’ thoughts conveyed in our conversations, inspired me to formulate conversational questions related to the practice of how to promote mathematical discussions in classrooms. The questions focused on the mathematical problems the students were going to discuss in their classrooms as they related to teacher actions in those situations. The five practices of initiating mathematical discussions, suggested by Stein et al. (2008), informed the formulation of the questions. These practices were used to guide the questions because in observing the teachers, I had noticed that, in advance of their lessons, they rarely reflected on the students’ answers, how students were proceeding in their problem solving, or how students’ solutions or ideas could be connected to a whole classroom summary. I prepared the following questions: How do you think students are going to solve this problem? How many ways of solving the problem can we come up with? When students are going to work together on a problem, how can you support them in the best way possible? What kinds of mathematical concepts do you think the students are going to use? How can you facilitate an expedient summary of students’ solutions?

I experienced this particular conversation as worthwhile because the teachers actively engaged in discussing mathematical concepts and expressed how they would define and explain them to their students, how students often interpreted them, and why students often struggled with the variable concept. Consequently, in the following lessons, the teachers attempted to ask open-ended questions with minimum guiding intervention. Moreover, the teacher–students interaction was organized after four principles of orchestrating mathematical discussions (Stein et al., 2008): first, having the students discuss the problem in pairs; second, allowing them to develop their own solutions; and third and fourth, teachers' attempts to connect the students' solutions and strategies in a plenary conversation with the class.

In the project design process, the three teachers and I planned and evaluated lessons to introduce the function concept. Both prior to and after the planned lessons, we discussed how to assist and interact with the students in order to promote mathematical reasoning and collaborative work. Moreover, I emphasized the importance of a teacher's role in the classroom in encouraging pairwise and collective mathematical discussions. The role of the teacher and students' difficulty understanding functions were central conversation topics in the iterative cycle of reflection between the teachers. In these reflective conversations, the teachers reflected upon their role as initiators of fruitful collaborations who could refrain from guiding students through a solution method and instead let their students take responsibility for the solution method. They expressed that this was challenging.

I observed the mathematics classrooms both with and without video cameras, as seen in Pic. 1. In the last conversation, after two video-recorded lessons of students' problem solving, I showed the teachers some clips of the teachers' interactions with student pairs and plenary discussions. The video clips demonstrated a great variation in the student interactions and teacher actions related to the student pairs' attempts to solve the function problem collaboratively. Moreover, the principles suggested by Stein et al. (2008), which had guided the questions in previous conversations, were also presented to the teachers and discussed. These principles made sense to the teachers, and they expressed a desire to use the principles in their lessons.

The four principles (Stein et al., 2008) informed the different focuses used in watching the video clips together. The first focus was a mathematical discussion related to five chosen video clips, where we discussed the following question: What do you think is an important contribution to maintain or initiate a conversation between students? The second focus was a whole classroom mathematical discussion related to three clips, where we discussed the following questions: When you feel your plenary discussion is going well, what do you think characterizes it? If you think something should have been done differently, what could that be and why?



Pic. 1. A plenary discussion.

In the conversations with the three teachers, we planned, reflected upon lessons, adjusted plans, and developed tasks and mathematical problems to discuss (see section 3.3). The twelfth and final conversation was conducted individually with each of the teachers. They were each invited to reflect on the project, including their experiences of the process of facilitating students' mathematical talk, pairwise collaborations, and teacher actions for such situations. They had not been used to organizing their classroom for this extensive focus on students' collaborative work and mathematical reasoning. They found it difficult at times to pose "good" and open-ended questions to the dyads but enjoyed seeing their students engaged in mathematical conversations and, therefore, expressed that they would continue to facilitate more classroom situations involving collaborative mathematics discussions. Their overall feedback on the project was a unified expression of the usefulness and value for them to discuss mathematical concepts and the challenges related to teaching and learning mathematics with a structured focus guiding their conversations.

3.2.2 Formation of student pairs

From the 69 students in the three classes studied, 33 pairs were formed. Two pairs did not consent to be video recorded, and two groups had three participants, resulting in 31 groups that agreed to be video recorded. The student pairs were organized according to the following criteria: 1) reasoning competence, 2) understanding of functions, and 3) likeliness to engage in "math-talk" with one another. The two first criteria were based on the students' scores on a mathematics test. The last criterion was based on conversations with the teachers in consideration of criteria 1 and 2. The written test had three main tasks. All tasks entailed the translation of one function representation onto another, which required the students to justify their solution or solution strategy choice. Thus, the test provided insights into students' mathematical reasoning, specifically focusing on the use of variables, which is important for understanding functions (Best & Bikner-Ahsbahs, 2017). Students' translation between function representations and their use of variables, including reasoning competence, was categorized according to descriptions of the different levels of function understanding, based on Gjone (1997) and Leinhardt et al. (1990). Students were organized according to the levels of their test results as low-achievers, average achievers, or average-high achievers. Since creating homogenous pairs was likely to increase their collaborative activity (Dillenbourg, 1999), student dyads were organized within the same level. For instance, a student categorized as an average achiever was paired with a student of the same level. Students' test results were discussed with their teachers' thoughts on their performance on previous mathematical tests. However, the teachers' opinions on the pair constructions focused mainly on social aspects, primarily how likely it was for student pairs to be comfortable talking and discussing with one another.

Pic. 2 A student pair collaborating on one laptop.

Due to the practicality of the time allotted to analyze the classroom video recordings before conversations with the teachers and the limited time of the research project overall, two pairs in each of the three classrooms were set as a condition. For an in-depth analysis of the interactional aspects of students' collaborative problem solving, six pairs were chosen. The six pairs were chosen based on the three criteria above. Two aspects particularly stood out: 1) students should express a high level of reasoning competence, which means that they attempt to explain their thinking and anchor it in mathematics; and 2) the likeliness of the student pairs to be verbal and share thoughts with one another. These aspects were discussed with their teachers when making the pairs.

The student pairs sat side-by-side behind two desks (see Pic. 2). They sometimes collaborated by using a shared laptop or having a laptop each, making suggestions through pen and paper, or simply discussing verbally without any physical tools. All student pairs were video recorded from a whole classroom perspective, but only six pairs were recorded for in-depth observations. A video camera was placed beside each of the six pairs, capturing their upper body. The students' laptop screens faced toward them and were therefore not visible to the camera (see Pic. 2). All student pairs worked on the same tasks in the classroom during the time of the study. Furthermore, students were kept in the same pairs during the five-month project period.

Two out of six student pairs did not engage in sharing their thoughts with one another, and therefore their conversations were not productive in terms of collaborative problem solving and mathematical reasoning. The other four student pairs interacted with one another through collaborative processes with mathematical reasoning. Thus, the final sample consisted of four student pairs, totaling eight students, who helped to shed light on how different collaborative interactions are intertwined in students' building of a shared understanding when attempting to solve linear function problems together.

3.3 The linear function problems

3.3.1 Linear functions in school mathematics

The function concept can be referred to in different ways. It is a dynamic mechanism that performs transformation through input and output, it reflects the relationship between two variables, and it can also function as the rule of correspondence between two sets (Malik, 1980). Thus, the function concept may be explained in various ways according to different representations.

Students struggle to understand what is similar and different about these various representations and how they are connected (Akkoç & Tall, 2005; Clement, 2001; Dubinsky & Wilson, 2013; Leinhardt et al., 1990; Thompson, 1994). If students view the representations only as separate entities, their view of functions becomes fragmented (Best & Bikner-Ahsbahs, 2017). To overcome fragmented views and to become flexible in their understanding and usage of the function concept and its representations, students need underlying algebraic knowledge of variables (Leinhardt et al., 1990; Lepak et al., 2018).

Variables in school mathematics are often seen as letters to operate on in the derivation of an expression, for example. Variables have different meanings for different uses (Usiskin, 1988), such as in a formula (e.g., $D = S \cdot T$), where the variables are quantities reflect distance, speed, and time; or in an equation (e.g., $36 = 6x$), where the goal is to find the unknown value x. Yet another example could be the function $y = mx + b$. The algebraic expression of a linear function is a pattern of variables and a formula, which can be challenging for students (Usiskin, 1988).

In order to substitute numbers for slope number and y-intercept (constant), students must understand several elements in the algebraic function representation. First, even though y and x are often used as unknowns, not all variables require you to search for unknowns. Second, it is important to understand which of the letters, m, x, or c, is the argument. Third, y and x can be used as unknowns when finding m by using one pair of numbers but return to not being unknowns when finding b.

Many students struggle to map the construct of one representation onto another representation (Adu-Gyamfi et al., 2012). Being able to see the function as a representation of an algebraic expression transformed, for instance, into a graph, is a manner of translation (Leinhardt et al., 1990). Therefore, considering different aspects of variables, distinguishing between different types of functions (Best & Bikner-Ahsbahs, 2017), and connecting function representation by translation (Akkoç & Tall, 2005) are all important pathways to learning about the function concept. Further, students' exploration of function problems together can contribute to the development and demonstration of their knowledge about functions, connections between quantities, and different representations of these relationships (Lepak et al., 2018).

3.3.2 Three function problems

The planned function problems were designed to emphasize mathematical reasoning and non-routine solving of tasks, where the problem-solving struggle was intended to be more like a challenge rather than an obstacle (Hiebert & Grouws, 2007; Lithner, 2017; Stein et al., 2008). This is what Hiebert and Grouws (2007) identify as a *productive struggle*. A productive struggle is a problem-solving process beneficial for students when learning new concepts (Granberg, 2016). When students are presented with new concepts, mathematical ideas, or problems to be solved, it should be a task that is simultaneously "within reach" and challenging, meaning that there is something new to figure out (Hiebert & Grouws, 2007). During the project, three main function problems were given to the students. However, only two (see Fig. 4 and Fig. 6) of the three problems were presented in the case studies because they facilitated the richest conversations. In these conversations, students more often anchored their arguments in mathematical properties (Lithner, 2017), such as the slope number being a varying parameter. Students also more frequently engaged in all of the collaborative processes of building, monitoring, and repairing their shared understanding (Roschelle & Teasley, 1995). The second problem did not present a productive struggle for the students and thus presented limited opportunities to study the interplay of interactional aspects in students' problem solving.

Another design aspect of the function problems was the use of the dynamic software program GeoGebra. This was presented to the students as a tool to help them in their explorations of ideas. To facilitate a collaborative setting while using GeoGebra, students worked in dyads on one laptop and were encouraged to employ GeoGebra in their problem solving.

GeoGebra provides tools to create, manipulate, and control mathematical content, which allows students to investigate mathematical relations (Granberg & Olsson, 2015; Hall & Chamblee, 2013). Thus, linear function problems present an opportunity to investigate varying parameters of the slope number and the constant. Changing an algebraic expression may cause GeoGebra to change the related graphical representation dynamically (Preiner, 2008). Students thus receive rapid feedback on performed actions, inputs, and changes in GeoGebra. However, GeoGebra does not interpret the generated information. Therefore, students have to make sense of dynamic changes between different linear representations. Olsson (2018) found that students who successfully solved a task with GeoGebra used the given feedback extensively and engaged in reasoning.

During the project period, students were presented with several "math-talk tasks" during their regular lessons. However, to allow for a detailed study of students' interactional patterns, three linear function problems were given particular attention. Students received suggestions or explicit directions to use GeoGebra as a tool in their problem solving. The three function problems that were video recorded are presented below. They are ordered according to the timeline of the study.

The first presented problem (Fig. 4) had previously and successfully been tested for a similar purpose at another school (Olsson, 2018). The focus of the first function problem was translation between algebraic and graphical representations, and it emphasized the importance of variables for understanding the function concept. In order to formulate a rule for a pair of perpendicular lines, students had to engage in a generalization process in which they used patterns identified in their findings to determine the general relationship between two linear functions. The first function problem generated the most fruitful conversations, including the sharing of ideas, negotiations for how to progress, and actions for testing ideas out in GeoGebra.

- Create a straight line $y = mx + c$
- Create another straight line in a way that the corresponding graphs are perpendicular.
- Formulate a rule for when two straight lines are perpendicular.
- Test the rule for other straight lines.

Fig. 4. The first function problem (reformulated from Olsson (2018))

The second problem (Fig. 5) was designed by the teachers to be a function problem that promoted CMR (Lithner, 2017) and a productive struggle (Hiebert & Grouws, 2007). The tasks invited students to explore properties with quadratic functions through algebraic, graphical, and possibly table representations. Students engaged in solving the tasks, but there was little discussion or sharing of ideas. Students primarily guessed and checked their answers with the use of GeoGebra. Thus, this problem was not used in the case studies exploring the interplay of different interactional aspects. However, the absence of mathematical reasoning and collaborative processes used in this problem highlights the importance of task design for promoting CMR.

Task 1

The graph of the function $f(x) = ax^2 + bx + 4$ has the roots $x = -1$ and $x = 4$.

Decide a and b graphically with help of GeoGebra.

Task 2

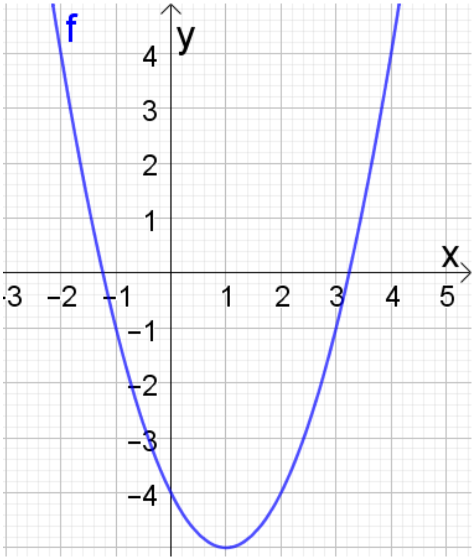
The graph of the function $g(x) = ax^2 + bx + 1$ goes through the point $(-1, -2)$ and $(4, -7)$.

Decide a and b graphically with help of GeoGebra.

Decide a and b by calculations.

Task 3

The graph of the function $h(x) = ax^2 + bx + c$ is shown under:



Decide a, b and c by calculations.

Fig. 5. The second function problem.

In a follow-up conversation with the teachers about the second function problem, they highlighted a few issues which were problematized and reflected on. First, the students did not have terminology for a, b, and c (see Fig. 5), which may have made it difficult to discuss the task. In the first function problem, students knew the mathematical concepts m and c, which may have made it easier to discuss a “tangible concept,” as suggested by Lucas. Second, students did not see that “graphs and equations are two sides of a coin,” as Jacob explained it. Thus, they found it hard to translate between the algebraic and graphical representation and see the connection. Third, students had not yet connected function values and coordinates. This last issue initiated a conversation on variables in which Lucas highlighted that this was a difficult concept for students to understand. The teachers reported that these issues provided insights into students' difficulties with the function concept, which were valuable to their future planning and reflecting on students' thinking.

The third function problem (Fig. 6) had previously and successfully been tested for a similar purpose at another school (Granberg & Olsson, 2015). This function problem was a continuation of the first function problem. In the last problem students were given, students had to find (or remember) the connection between perpendicular lines and then form a square by finding a constant number to the different linear functions. Thus, this task invited students to connect algebraic representation and graphical representation.

The video recordings of the first two problems were executed successfully. However, one teacher did not have time for the final round of video observations of the third function problem due to a test, and I experienced technical problems with the video recording memory cards. Consequentially, there was minimal recording resulting in less data on the third function problem compared to recordings of the other two function problems.

Solve the following problem using GeoGebra.

- Construct four linear functions in such way that they will form a square.
- One of the lines must be parallel with the y-axis.
- Adjust the formulas in GeoGebra's algebra field in such way that you will angel the square less than 90^o clockwise or counterclockwise.

Fig. 6. The third function problem based on Granberg and Olsson (2015)

3.4 Data analysis

The focus areas of student interactions were collaborative processes, mathematical reasoning, and exercised agency connected to teacher actions when solving function problems. These foci were developed through conversations with the teachers, combined with literature on students' collaborative work, difficulties with functions, and mathematical reasoning, and review and analysis of the classroom video recordings.

The data analysis was an iterative process that started with preliminary findings seen in a particular student pair, Philip and Noah, whose method of interaction was particularly strong compared to the other pairs. The two following sections explain the data analysis procedure used to analyze the student–student interactions and teacher–student interactions.

3.4.1 Data analysis of students' creative collaborative interactions

The video recordings of students' problem solving *showed* students' dialogue, gestures, and computer interactions. These were approached using a deductive analytical strategy (Yin, 2014) in which video excerpts were identified and selected for further systematic sampling in light of the research questions (Derry et al., 2010). With a strong orienting theoretical foundation and research aim (i.e., studying student interactional patterns and related teacher actions), the deductive approach was useful in choosing suitable video excerpts to analyze. Appendix A presents the coding framework used to analyze the three interactional aspects: CMR, collaborative processes, and exercised agency.

The video recordings were viewed multiple times. The first step of the analysis was denoting sequences in which students often made justifications and explanations anchored in mathematical properties as creative reasoning (Lithner, 2017). These sequences were considered "critical events" to be further examined (Powell et al., 2003) due to the inherent learning opportunities presented by reasoning creatively about functions (e.g., Granberg & Olsson, 2015; Lithner, 2017; Olsson, 2018). Students' reasoning was coded after Lithner's (2017) framework of *creative* and *imitative reasoning*.

To categorize students' CMR, three criteria in the framework had to be fulfilled (Lithner, 2017). First, *creativity*: the students created a reasoning sequence not experienced previously or re-creating a forgotten one. Second, *plausibility*: the students presented arguments that supported the strategy choice or strategy implementation that explained why the conclusions were true or plausible. Third, *anchoring*: the students' arguments were anchored in the intrinsic mathematical properties of the components of the reasoning. Arguments were considered *intrinsic* if they were based on mathematical concepts or relations and superficial if based on appearance and not on underlying mathematics.

Students' CMR sequences were further transcribed, and each turn-taking utterance was written down. The second step of the analysis was coding students' utterances and actions with respect to collaborative processes (Roschelle & Teasley, 1995). Development of a shared understanding started with the collaborative process of *building*: suggesting, accepting, and agreeing upon an idea to solve a problem. The collaborative process of *monitoring* was categorized as asking questions or explaining an idea when observing and trying to understand each other's interpretations of a problem. If the suggestions or explanations conflicted with a peer's understanding of the shared understanding, a state of collaborative *repairing* was categorized as negotiating and correcting conflicting interpretations using justifications and counter-suggestions.

The third step in the analytical process was to provide rich descriptions (Powell et al., 2003) of the pairs' collaboration and reasoning within their CMR sequences. The descriptions of students' CMR, collaborative processes, and the ways in which they participated in their dyads made it possible to characterize students' agency in their conversations as primary, secondary, or shared (Mueller et al., 2012). First, a description of how the students interacted when engaged in reasoning, concerning how they attempted to engage with one another, was developed. If students constructed a solution sequence where they connected ideas and thoughts to make a shared understanding of the current situation, it was recognized as *shared agency*. If students engaged individually, clearly having different roles when suggesting ideas or explaining thoughts, their agency was recognized as *primary* or *secondary*. After characterizing students' interactions, it was possible to describe students' typical roles in their engagement connected to how they reasoned about linear functions and the ways they collaborated to solve the given task.

3.4.2 Data analysis of teachers' actions

After several rounds of watching and coding students' interactions, the focus was broadened to include how teacher actions promoted productive student interactions. Thus, "critical events" were identified in the recordings based on the following criteria: 1) the dyads had some form of interaction, characterized by the three emphasized aspects (see section 3.4.1), and 2) teacher interactions were tied to these sequences. Four dyads fulfilled these criteria, and each pair's work with the given problem and the interactions they had with the teacher in the process were transcribed and coded. The teachers' interactions with the dyads were coded using the coding scheme presented in Table 2 (see Appendix B). These codes were based on Drageset's (2014, 2019) framework for redirecting, progressing, and focusing teacher actions but were slightly revised through an iterative coding procedure between the data set and the theoretical framework. Table 2 shows which codes were based on the original framework and which ones were added or revised. The following paragraphs outline Drageset's framework, starting with the funneling actions of *redirecting* and *progressing* and then the focusing actions (*focusing*).

A typical redirecting teacher comment is to discard a student's suggestion or comment; this is referred to as *putting aside* in the coding. With such an action, a teacher is not providing help with a presumably pressing question or challenge. The second redirecting action, *advising a new strategy*, means that a teacher's comment suggests an alternative approach or way of thinking to solve a problem. The last redirecting action is *correcting questions*, in which a teacher's question aims to move a student's focus to another approach. In summary, *redirecting* actions are a teacher's strategy for shifting attention to something else.

In line with the funneling of actions, a teacher may aim to move a problem-solving process forward. Drageset (2014) explains four actions for attempting to guide students' progress. *Open progress details* reflect teachers' open questions with possibilities for several answers concerning students' progress toward solving a problem. This action includes questions on how to do, how to think, how to solve, and how to generalize patterns. Thus, an open progress action aims to "[move] the process forward, but without pointing out the direction" (Drageset, 2014, p. 16). In contrast, *closed progress details* concern "how" (How many? How large? How much? How big? How should we do it?) and "what" (What will it become? What shall we write? What is it? What should we do?) questions. Questions typically request details needed to move the process forward, connected to steps in a procedure (Lithner, 2008). These details can be process answers (one step at a time) or details about how the process should proceed to reach the answer.

Another aspect of a teacher action that attempts to move the process forward is a *simplification* of the task at hand. To simplify a task, a teacher may change or add information, tell students how to solve it, or give hints to make the task easier (Wood, 1998). Teachers typically pull students toward the solution (the Topaze effect, cited in Brousseau, 2006): "It often seems that this involvement is meant to ensure the progress of the class and sometimes these comments appear to come as a consequence of a halted progress. Many of the simplification comments could also be characterized as hints" (Drageset, 2014, p. 15).

The last progress action is *demonstration*. A teacher typically demonstrates how to solve the problem or shows students every step in a procedure. It is primarily a monologue given by a teacher that is occasionally broken if students ask questions or if the teacher asks students whether they understand or agree.

Further, Drageset (2014) divides focusing teacher actions (Wood, 1998) into two categories: *request for student's input* and *pointing out*. A teacher may ask for a student's input by asking them to *enlighten details*, *provide justification*, *apply to similar problems*, or *request assessment from other students*. These concepts are related to what Franke et al. (2007) expressed as access to student thinking. If teachers ask students to *enlighten details*, they want them to explain what something means or how something happens. Typically, details are brought into focus. If a teacher asks a student to *provide justification* (Cengiz et al., 2011), it is a request for a more thorough explanation, often to validate why the answer found, or the method used, is right.

Another approach focusing on students' thinking is when a teacher *points out* something. Subcategories include *recapping* and *noticing*. The purpose of *recapping* is to merge information to clarify important elements in a student's explanations. Moreover, a teacher can repeat a student's answer with the purpose of confirming or ending dialogue or sometimes adding information to an answer. A *noticing* action (Cengiz et al., 2011) is when a teacher highlights particular aspects, concepts, or details he wants to make a student aware of. Other aspects of *noticing* include reminding students of new or previous information and adding information.

3.5 The quality of the research project

Data in qualitative research is context-dependent, and interpretations are subjective, although theoretically anchored. Presmeg and Kilpatrick (2019) say that criteria for judging the quality of mathematics education research, which were formulated in the early era of the field (e.g., Hart 1993, Lester & Cooney, 1994, Kilpatrick & Sierpinska, 1993), comprising timeless elements for scientific endeavor. However, there are many different terms and strategies used by researchers to evaluate the "accuracy" of the chosen data, utilized methods, and interpretation of the data (Cresswell, 2007). A validation strategy suggested by Sierpinska (1993) comprises eight strategies: relevance, validity, objectivity, originality, rigor and precision, predictability, reproducibility, and relatedness. Other identified elements, which are similar to these, include worthwhileness, goodness of fit; competence, openness, and credibility (of the researcher); and lucidity, conciseness, and originality (called intangible qualities) (Presmeg & Kilpatrick, 2019).

The quality of the present research project can be evaluated by considering several of the above-mentioned terms. First, a study must have "value" for informing and improving educational practices (Howe & Eisenhart, 1990). Hence, a "so what?" question should be asked about a research project's aim and findings. In this way, a research project's *relevance* or *worthwhileness* can be made clear (Presmeg & Kilpatrick, 2019; Sierpinska, 1993). There is a need for knowledge on how to facilitate productive student interactions (Langer-Osuna et al., 2020) in collaborative mathematical reasoning (Erath et al., 2021). A particular focus that has been previously highlighted as requiring further exploration is students' dynamics in mathematical collaborations (Seidouvy & Schindler, 2019; van de Pol et al., 2018; Varhol et al., 2020). The three studies contained in this project emphasize that the following interactional aspects comprise important processes for collaborative mathematical problem solving: 1) CMR facilitates in-depth learning in contrast to rote learning; 2) engagement in different collaborative processes means that students get to suggest, explain, and defend mathematical ideas, according to their methods of participation (their exercised agency). The interwovenness of these aspects emphasize the complexity of student interactions. A mathematical competency comprises connections of skills, knowledge about the competency, and the ability to use the skills (Gresalfi et al., 2009; Niss, 2003; Sun et al., 2020). Therefore, mathematical competency is suitable and useful to encompass, explore, and the complexities with productive student interactions.

The present research project was design-based and focused on students' interactions, teacher–student interactions, and students' potential learning outcomes when interacting. The process of the project comprised iterative design steps of analysis and interpretation, collaboration with the teachers, and classroom interventions. The outcome of the iterative processes of analyzing student–student and teacher–student interactions were design principles for classroom observations of students' creative, collaborative problem solving (see chapter 5). Design principles in DBR can be tools or conceptual models which guide and inform practices and research (Wang & Hannafin, 2005). The suggested design principles in this research project may impact our understanding of productive student interactions in mathematical problem solving and potentially influence teaching practices through awareness of indicators for productive mathematical interactions. Thus, this research project is anchored in *cognitively relevance*, as it deepens our understanding of a learning and teaching phenomenon, and *pragmatically relevance*, as it may impact teaching practices (Sierpinska, 1993).

Second, another quality criterion is the validity of the research project. The *validity* often refers to the "thing" that is to be investigated, whereas *validation* refers to the investigation of a process (Newton & Shaw, 2014). Another suitable term for validation of the research project is *goodness of fit*, which concerns "the suitability of choices regarding theory, the broad umbrella of methodology, and specific methods of data collection and analysis within this methodology" (Presmeg

& Kilpatrick, 2019, p. 350). The suitability of the theoretical choices was addressed above and in section 2.2.

In summary, three frameworks were chosen to study student interactions. The reasons for selecting these particular frameworks included the research questions and aims, interpretations of the video data, and the comprehensive iterative processes of the project. Regarding these areas, choices were made based on what I observed in the video recordings and my conversations with the teachers. Thus, subjectivity needs to be considered in this qualitative research. If I had chosen different frameworks to study student interactions and teacher actions, something else might have been observed in the first iterative process, and thus, the project could have taken other iteration paths. Moreover, including other interactional aspects could have strengthened the findings further, or it could have emphasized other interactional aspects, resulting in other design principles. This does not suggest contradictory design principles but principles concerning something else. Connecting the frameworks to gain insights and produce new knowledge reflects a research project's *originality* because when "we say something in different words we no longer say exactly the same thing: a new focus may be brought in, a new aspect shown" (Sierpinska, 1993, p. 63). Thus the intangible qualities, *lucidity, conciseness,* and *originality* (Presmeg & Kilpatrick, 2019) concern issues for a project's validation. These qualities concern the trustworthiness of literature, chosen data, and the building of logical arguments (Presmeg & Kilpatrick, 2019).

Throughout this doctoral thesis, I have strived to provide a logical rationale for the importance of the chosen focus of this study, making a clear and coherent theoretical foundation, and being transparent about methodological choices, including the research context. One choice was to make rich descriptions of the recorded video observations (Cresswell, 2007; Derry et al., 2010) and to provide relatively comprehensive transcripts in the published articles and submitted manuscript, thus, enabling readers to evaluate the soundness of the findings and the conclusions, and evaluation of the applicability to similar settings. These efforts were intended to make the research process and outcome as clear as possible to a reader, thereby contributing to the trustworthiness and credibility of the research project. This doctoral thesis also contains two published, peer-reviewed articles with the same theoretical foundation and data. Thus, peer assessment and the revision processes contributed to strengthening and validating the trustworthiness of the posed research questions, literature, methodology, and findings.

The last aspect that reflects the project's quality is the *role of the researcher*. In discussing the researcher, Sierpinska (1993, p. 58) says, "absolute objectivity is impossible to achieve, but we must make an attempt to avoid blind subjectivity." My interest in mathematical reasoning in collaborative work has been a driving force during the research process. Thus, it has been important not to overinterpret data for instances concerning my interest. My supervisor has held me accountable through many conversations and written rationales over the years. In our conversations, I have willingly shared my past experiences as a teacher in upper secondary school, my experience with mathematics, and how I experienced mathematics by engaging in mutual work with peers, thus, expressing my beliefs and values. Therefore, having a co-researcher, or as in this case, a supervisor, has contributed to keeping me honest, responsible, and trustworthy. Moreover, the handling of the video data through the selection of CMR sequences and the transcript analysis served as an iterative process between my supervisor and me. We started with rich descriptions of the instances of CMR in the different student dyads. The descriptions were discussed. Furthermore, I transcribed and coded the CMR sequences. My supervisor coded a few of the CMR sequences. We sat together, compared our codes, and discussed. We attended to the instances that were similar, as well as the instances that were differing. Both of us explained our coding, which was another instance that held me accountable to how I justified the codes and connected it to research literature. This process contributed to form the coding framework further. Thereafter, we had frequent conversations about my interpretations of the coded transcripts. Thus, the close relationship with my supervisor held me accountable and contributed to intercoder reliability (Cresswell, 2007), which further contributed to the trustworthiness of the research process.

During the project process, I also have to build trust with the three teachers and their students. Mutual respect and trust are important for working well together (Bungum & Sanne, 2021), as emphasized in section 3.1. The teachers provided thoughts about their own and each other's interactions with students after watching selected video recordings. Moreover, they expressed their expectations of their teaching, reflected upon lessons, and looked ahead to the coming situations of student problem solving. Cresswell (2007) emphasizes that a project's participants should be asked to give feedback on data, analyses, interpretations, and conclusions. The participants' task is thus to judge the accuracy and credibility of the report. Such an approach to support the process and results could have been utilized in this research project but was not. Working with the data and the process of interpretation took longer than anticipated. I felt I could not return to ask for more of the teachers' time. In retrospect, feedback from the teachers should, therefore, have been a deliberate part of the original project plan.

3.6 Ethical considerations

The participants were informed of the research purpose and gave written informed consent (see Appendix C) according to the ethical requirements of the Norwegian Research Council (The Research Council of Norway, 2018). Video recordings of classroom situations and audio recordings of conversations with the teachers allowed me to revisit the material as needed. Another advantage of this approach was the replay flexibility, such as playing clips slower to accurately observe gestures and conversations or playing them faster to scan for particular moments. Thus, the video and audio recordings provided a richness of detail about the individuals and groups, which also introduced a variety of ethical issues. A written informed consent "does not necessarily protect consenters against a number of problematic situations" (Powell et al., 2003, p. 409). A problematic situation, in this instance, might involve video sharing. Some researchers give other researchers access to their data to provide them with the opportunity to form their own judgment of the analysis and results. This approach can strengthen study validity, but making data more publicly available raises ethical concerns about how others might reuse data. Data could be used out of context and in unanticipated ways. To address this issue, Derry et al. (2010) suggest that the video and audio data be made available to other qualified researchers "with the provision that they agree to abide by legal and ethical guidelines governing use, reuse, and attribution" (p. 32). Derry et al. (2010) further share ideas on how to assist researchers in sharing and reusing video data.

However, researchers also commonly provide readers with transcripts and sufficient contextual information. This was also the case for the present research project. The National Committees for Research Ethics in Norway (NESH, 2006) says, "Personally identifiable information (e.g., lists of names, field notes, interview material) shall be stored responsibly for a limited period of time, and then be deleted once it has served its original purpose" (p. 19). Data in this research project was stored safely and legally, accordingly to the guidelines of the Norwegian Centre for Research Data. Only my supervisor and I have access to the data. Students and teachers could withdraw at any time from the project without further explanation. Data will be deleted in 2027. Furthermore, to protect the participants' privacy, all names have been pseudonymized.

Another ethical consideration was my role as a researcher. Although emphasizing a study in a naturalistic setting, my presence, thoughts, and suggestions, along with cameras and microphones, influenced the participants and their dynamics. This aspect, of affecting the teachers' response and thoughts, was something I was particularly aware of before, during, and after the project. I wanted the teachers to be engaged and share their opinions about teaching and learning of mathematics. However, I was particularly concerned with my enthusiasm and whether it potentially could prevent them for answering honest, in fear of letting me down. In the conversations it was important for me to build trust with the teachers, so that they would openly share their thoughts. Thus, my approach was to be genuine, to build trust, and balancing the questioning approach with enough openness and yet sufficiently directness. As elaborated in the previous chapter, internal biases have an impact on what we see in a situation, and conclusions that might come from listening in a conversation. However, my experience is that the teachers were sincere in their response about the different issues we discussed and planned, as well as their engagements in their classrooms. The issues concerning the interaction with the teachers, can apply to interacting with the students. In my role as a qualitative researcher my cultural attributes might have affected the video data. The students may have acted differently because a researcher was studying them. Moreover, their awareness of that the teachers could watch the video recordings and possibly form another opinion about their mathematical understanding could potentially have affected their interactions.

To summarize, the ethical measures taken included obtaining written informed consent, ensuring responsible storage for a limited period, using fictional names ensure participant confidentiality, and awareness of my role as a qualitative researcher.

4. Summary of studies

The present research project comprises two case studies of student interactions in mathematical conversations about linear function problems (Fig. 4 and 6, section 3.3.2) and one conceptual contribution with an analytical model departing from the case studies. Each case is an instrumental case study that provides in-depth insights: the first, on interactional aspects of collaborating student pairs, and the second, on student interactions and related teacher actions. The following sections describe the three studies, including their research questions, aims, main findings, and how the study informed the design principles for classroom observations of students' creative collaborative problem solving (see chapter 5).

4.1 Study 1: Students' agency, creative reasoning, and collaboration in mathematical problem solving

In a preliminary analysis conducted in advance of Study 1, one of six student pairs, Noah and Philip, stood out in comparison to the other pairs because of how well they collaborated and reasoned mathematically, despite a lack of encouragement from their teacher. Noah and Philip engaged in all aspects of the collaborative processes, discussed, and negotiated to agree upon a unifying solution. Their engagement in collaboration and mathematical reasoning supported previous research on how these interactional aspects are interwoven (Granberg & Olsson, 2015). This observation led to Study 1, an instrumental case study of four student pairs, with the following research question:

> *What are the patterns of interaction for creating a shared understanding through the interplay between students' creative reasoning, collaboration, and exercised agency in a mathematical problem-solving session?*

The aim of Study 1 was to investigate aspects of the dyads' interactional patterns to gain further insights into why some dyads were more or less productive while attempting and wanting to collaborate on a mathematical problem (Fig. 4, section 3.3.2). Three aspects were used as lenses to analyze the dyads' interactions: collaborative processes, mathematical reasoning, and exercised agency. The four student pairs frequently engaged in different collaborative processes (Roschelle &

Teasley, 1995) and exhibited building, monitoring, and repairing while using CMR (Lithner, 2017). Students' exercised agency (Gresalfi et al., 2009; Mueller et al., 2012) was characterized as shared agency or as individual agency (either primary or secondary agency) (Mueller et al., 2012). Connecting students' exercised agency to the interplay of collaborative processes and CMR revealed different roles in students' interaction patterns.

The main outcome of Study 1 was the identification of interaction patterns characterized as *one-directional interaction* and as *bi-directional interaction*. The two interaction patterns provided insights into students' opportunities and limitations connected to their role in the interaction pattern for constructing their own solution procedures, which is important for their mathematical understanding (cf., Lithner, 2017; Mueller et al., 2012; Stockero et al., 2019). In one-directional interactions, students engaged with different agencies either as a primary agent, leading conversations, making suggestions and explanations sometimes anchored in mathematical properties, or as a secondary agent, listening and attempting to understand ideas expressed by a peer. Secondary agents rarely reasoned mathematically. Both students attempted to collaborate but rarely or never disagreed. The interactional pattern in bi-directional interactions highlighted a mutual attempt to collaborate where both students were the driving forces of the problem-solving process. Students acted in similar roles when both were exercising a shared agency, building the final argument together by suggesting, accepting, listening, and negotiating mathematical properties. A critical variable for such a successful interaction was the collaborative process of repairing their shared understanding and reasoning anchored in the mathematical properties of linear functions.

Thus, Study 1 provided insights into students' interactional patterns through three frameworks focusing on students' creative reasoning, collaborative processes, and exercised agency. The three frameworks together made up the analytical design for studying student interactions. The pragmatic use of the analytical design in a naturalistic classroom setting allowed for observations of two distinct methods of interacting and their characteristics. This became a starting point for developing the design principles for understanding and assessing students' competency for productive interaction patterns focusing on the three interaction aspects involved.

4.2 Study 2: The role of teacher actions for students' productive interaction solving a linear function problem

Eight students, the same dyads as in Study 1, and their three teachers constituted the case participants in Study 2, which sought to explore how teachers can facilitate productive student interactions in mathematics learning. The research question guiding the second case study was as follows:

> *What are the opportunities and limitations of teacher actions for the productivity of students' interactional patterns?*

The second study aimed to identify the characteristics of teacher actions in teacher–student communication connected to student interactions, focusing on collaborative processes, mathematical reasoning, and the students' agency related to linear functions. Teacher actions were described as suggested by Drageset (2014). This fine-grained framework was used to better understand teacher actions in teacher–student conversations, to provide detailed insights into how teachers can facilitate students' reasoning and argumentation, and to highlight their collaboration and agency in those situations.

The findings of Study 2 revealed that students who established an interactional pattern, such as a shared agency in a bi-directional interaction or a primary/secondary agency in a one-directional interaction, maintain and progress in the same interactional pattern before, during, and after a teacher interaction. This observation was found in the student–teacher interaction patterns of two pairs. Jacob and Lucas used *progressing actions*, which resulted in less reasoning anchored in mathematical properties and more guessing and checking answers using GeoGebra (Hannah and Emma, and Leah and Isaac). Both student pairs maintained their interactional pattern as respectively bi-directional and one-directional after the teacher interaction. Lucas's *redirecting* and *progressing actions* could have had an impact on students' continued creative reasoning, where students maintained their bi-directional interaction after the teacher interaction (Philip and Noah). Sophie used *focusing actions*, which possibly impacted the continued creative reasoning of the primary agent (Oscar), which was observed and acted on by the secondary agent (Olivia), where students continued their one-directional interaction after the teacher interaction. The three teachers used *funneling actions* and *focusing actions* (Drageset, 2014; Wood, 1998), resulting in different reasoning by the students in situations where their interaction patterns seemingly remained the same. The study highlights the importance of further research into teacher awareness to facilitate collaborative situations of bi-directional interactions connected to students' shared understanding of mathematical ideas and concepts.

The second study contributes to the analytical design of the present research project by including the teacher interaction aspect. Hence, the second iteration of the analytical design contributes insights into the role of the teacher for design principles related to students' competency for productive interaction patterns. In the study, a teacher's actions and the consequences of those actions, as well as his or her potential actions, were discussed. Based on the events and the discussed potential of various teacher actions in Study 2, a teacher should be aware of 1) the ways in which students are interacting: one-directionally or bi-directionally; 2) students' engagement in plausible and anchored mathematical arguments; and 3) conversations characterized by turn-taking. A teacher should act by 1) asking both students to share their thoughts and 2) encouraging students to share ideas with one another to promote the building of shared agency in a bi-directional interaction. These proposed elements are suggested in Study 2. However, particular teacher actions that contribute to supporting, inhibiting, or facilitating productive bi-directional student interactions are suggested for further research. Particularly important are the types of teacher actions that interplay with students' agency and might facilitate a change in student interaction patterns.

4.3 Study 3: An analytical model for analyzing interactional patterns in creative collaborative mathematical reasoning

The aim of the third study was to develop a tool for assessing students' interaction patterns which was built on how the interactional aspects were connected through students' way of participating (Study 1 and 2). The third study proposed an analytical model called the Creative Collaborative Mathematical Reasoning model (CCMR model) (see Appendix C). The CCMR model was anchored in relevant theory and related frameworks, which were utilized in classroom situations on students' attempted collaborative-learning. The theoretical points were exemplified by glimpses of the same four student dyads as in the previous studies. However, only new excerpts of their conversations about the first and last function problem (Figs. 4 and 6) were used in Study 3.

Results of the conceptual contribution highlighted how the two different interaction patterns, one-directional and bi-directional, may suggest learning opportunities related to the roles in the interaction patterns. Regarding a one-directional interaction, students are likely to experience different learning opportunities due to their participating roles and the opportunities presented through creative reasoning. Since the primary agent is more likely to make arguments based on mathematical properties than the secondary agent, it is likely that that student is presented with more individual learning opportunities. In contrast, students may engage in a bi-directional interaction, where both engage in turn-taking and making plausible arguments, which could strongly suggest a quality interaction with creative reasoning that is important for learning opportunities in mathematics.

Therefore, the quality of student interactions is related to their productivity and the learning opportunities involved. Thus, the CCMR model presents a possible tool for teachers, mathematics educators, and researchers to analyze student interactions to better understand students' methods of participation and foster quality interactions that promote mathematical learning.

In the third study, and also the third iterative step of the analytical design, the CCMR model highlights to two aspects that inform design principles for students' creative collaborative problem solving (see chapter 5): 1) a process focused on student interaction patterns, and 2) the outcome of a CCMR interaction. Together, these aspects emphasize that the processes and outcome of interactions depend on interaction patterns. If acting bi-directionally, certain qualities of the interaction may lead to learning *through* and *of* CCMR. When participating one-directionally, it is more likely that only the primary agent gets to learn through CCMR but not necessarily *of* CCMR because of the missing quality interaction with a peer.

5. The CCMR competency design principles

From the iterative processes of analyzing student–student and teacher–student interactions as three analytical steps outlined above, an overarching focus grew forth: organizing students' quality interactions as competency seen through the interactional facets of mathematical reasoning, collaborative processes, and exercised agency. Thus, this project's research process contributed to developing design principles to achieve learning of and learning through *Creative Collaborative Mathematical Reasoning competency* (CCMR competency, Fig. 7 below).

The CCMR competency model is built on the model developed in Study 3 (Appendix D) and informed by the two other studies. Both models share the characteristics of a bi-directional interaction found in students' collaborative actions and co-reasoning when exercising shared agency. The competency model includes suggestions for teacher actions related to a bi-directional interaction. Furthermore, it emphasizes the connection between quality student interactions and learning outcomes related to not only the inherent learning opportunities of different mathematical products (e.g., fractions, functions, numerical computations) but also the competency itself. These aspects exceed the analytical CCMR model.

The CCMR competency model presents design principles for bi-directional interactions and learning outcomes (see Fig. 7 below). Related to Figure 2 (section 3.1), theory and empirical data formed the design principles for a bi-directional interaction and related teacher role. These further influence the achievement of the practice goals of learning *of* and *through* the CCMR competency, which is seen as the learning outcome. Thus, the design principles comprise the characteristics of quality student interactions, which represent the first step in a process. In the process of establishing and maintaining a bi-directional interaction, students work toward a practice goal, which is the second step in the figure. The practice goal (Euler, 2017) is the outcome of students' engagement with the design principles. Here the practice goal is for students to become competent mathematical producers through reasoning creatively together. A practice goal or learning outcome give the impression of a final product, however, the arrow (step two) indicates a process of continuous development for one's learning. The learning outcome is two-sided:

learning of mathematical products, such as the linear function concept, and learning the competency of creative collaborative problem solving. Thus, the design principles for a bi-directional interaction function as a bridge between theory and empirical data, and the learning outcomes. Moreover, the design principles contribute to theory refinement and local impact (Ryu, 2020). These aspects are further discussed in the following sections on bi-directional interactions (5.1) and learning outcomes (5.2).

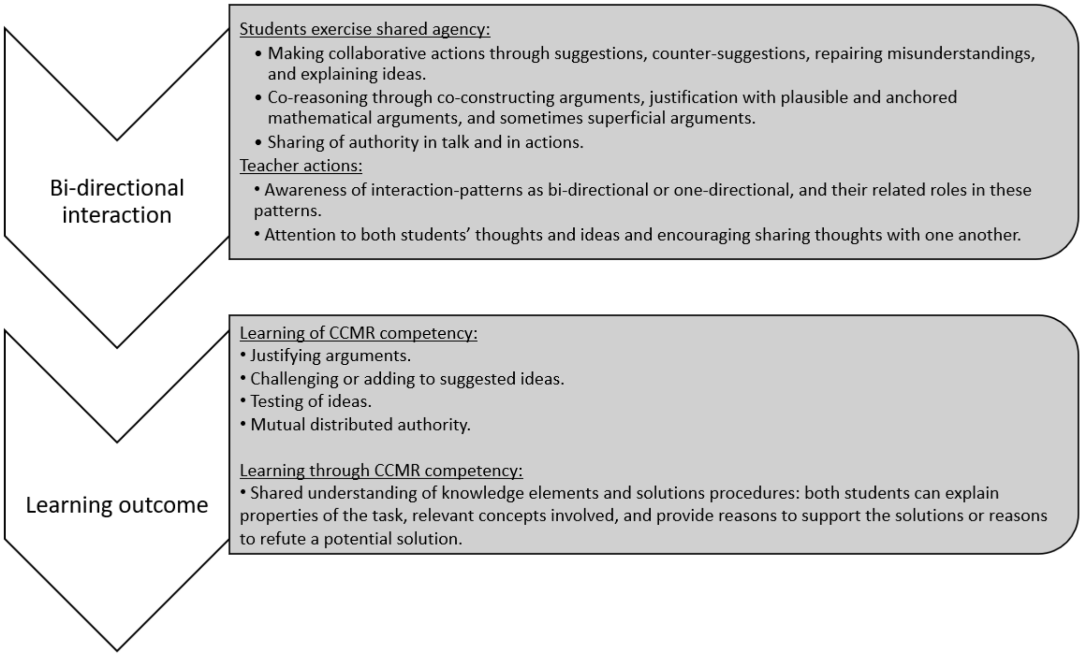


Fig. 7. Design principles of the CCMR competency model

5.1 Bi-directional interaction

After several rounds of analyzing students' interactions, a pattern emerged of bi- and one-directional interaction. Study 1 revealed different roles that students take on related to their exercised agency when attempting to collaborate. Moreover, the two interaction patterns provided insights into students' opportunities and limitations related to their role in the interaction pattern in constructing their own solution procedures, which is important to their mathematical understanding (cf., Lithner, 2017; Mueller et al., 2012; Stockero et al., 2019). After another iteration of analyzing and interpreting data in Study 2, the interaction pattern was further refined in Study 3 into a model for analyzing students' methods of participation to identify quality interactions that promote mathematics learning. Together, the three studies contributed three design principles for collaborative problem solving in which students engage bi-directionally with shared agency:

- Making collaborative actions through suggestions, counter-suggestions, repairing misunderstandings, and explaining ideas.
- Co-reasoning through co-constructing arguments and using justification with plausible and anchored mathematical arguments (and sometimes superficial arguments).
- Sharing of authority in talk and in actions.

The design principles involve intertwined interactional aspects. Therefore, if focusing on one principle, it is likely that another principle will be "activated." For instance, if a student suggests an idea anchored in mathematical properties and further built on by a peer, this includes the first and second principles.

The first design principle emphasizes central collaborative aspects promoting mutuality and coordination in student interactions (Baker, 2015; Sarmiento & Stahl, 2008). Studies 1 and 3 emphasized that synchronicity, mutuality, and coordination are important to students' progress in the problem-solving process and key to strengthening the interaction. This is likely to promote shared understanding. Moreover, a willingness to enter situations of conflicting ideas (Dillenbourg, 1999) connected to frequent engagement in repairing processes was emphasized in Studies 1 and 2 as an important indication of a productive interaction.

The second design principle concerning CMR involves students collaboratively constructing a mathematical meaning to a problem. Study 1 emphasized that when students dealt with conflicting interpretations through processes of repairing, their work was more frequently anchored in reasoning based on the mathematical properties of linear functions compared to the work of other pairs. Study 2 further stresses the importance of both students in a dyad being encouraged to justify arguments and attempt to pose different claims and counterarguments. Thus, a teacher or students themselves could be encouraged to notice when they are using mathematical concepts (either intrinsic or superficial) to justify suggestions or ideas, facilitating a bi-directional interaction.

The third design principle of students' authority is for both students to be responsible for making suggestions, explanations, and disagreeing through talk and actions. Students' methods of collaboration can contribute to students' agency in mathematics (Schoenfeld et al., 2019). Study 3 emphasized that both students in a dyad should act as the mathematical authority; otherwise, the collaborative interaction would rarely be productive. Moreover, Study 3 illuminated the interconnectedness of the dyads' collaborative processes (first principle) and co-reasoning (second principle) for productive interactions contributing to a shared agency.

The design principles in the CCMR competency comprise competency activities (specific ways to understand and do mathematics) (Boesen et al., 2014; Niss, 2003). The first and second principles concern aspects of doing and using mathematics, seen in making suggestions and explanations and in creating reasoning sequences together. The analytical aspects are found in all three principles when judging mathematics and interpreting the social situation of sharing agency. It is essential to students' progress in their CCMR competency that they judge their solution procedure and ideas, which might require them to resolve misunderstandings, follow a chain of reasoning, and evaluate the premises, arguments, and evaluation of a relation: "Is my peer having opportunities to talk, listen and make arguments, and if not, what can I do to make space for my peer's contribution?"

A teacher may promote students' shared agency by sharing authority in the classroom (Langer-Osuna et al., 2020; Mueller et al., 2012). Therefore, the role of the teacher is important, even if students are struggling on their own. Teacher actions that promote shared authority give students opportunities to talk about mathematical problems and let students know that they are expected to suggest ideas, solutions, and strategies for the problem-solving process (Langer-Osuna, 2018). Whether or not a teacher is successful in sharing authority over the problem solving with his students, he requires knowledge about student interactions and how to best interact with student dyads. This knowledge may enable teachers to promote students' awareness of their interactions and emphasize productive interaction aspects for students to aspire to. The second study contributes to an understanding of the role of the teacher related to student dyad interaction patterns, highlighting the following aspects:

- An awareness of interaction patterns as bi-directional or one-directional, and teachers' related roles in these patterns.
- Attention to both students' thoughts and ideas and encouraging sharing thoughts with one another.

The first design principle, a teacher's awareness of dyads' interaction patterns, is important in evaluating whether both students are engaging with the mathematics involved, are in conversation with one another, and are in a position to benefit from learning opportunities through CMR to develop a shared understanding. Study 2 addressed opportunities and limitations of teacher actions related to student interaction patterns. Teacher actions, analyzed using Drageset's (2014) framework, influenced student reasoning, but students' interaction patterns remained the same throughout the entire problem solving session. Teachers used both funneling and focusing actions (Drageset, 2014; Wood, 1998). However, teachers' questions and actions did not appear to influence students' roles and dynamics. Therefore, a perception of the students' interaction patterns as bi- or one-directional could be a fruitful starting point for a teacher's response to promote quality interactions.

This leads to the second design principle: attention to students' thoughts, ideas, and actions. Study 3 emphasized that if a teacher only responds to a primary agent or to only one of the students in a shared agency, only one student is being held accountable. Consequentially, students' mutuality is not strengthened, and teacher authority is likely being distributed to one participant of a dyad. An aspect to consider is teacher involvement in classroom norms. Yackel and Cobb (1996) differentiate between a social norm as an expected explanation of a given task, whereas a sociomathematical norm is an acceptable mathematical explanation. Both aspects are important for individual and collective learning, and students' engagement in both can influence their pattern of interaction. Teacher involvement has the potential to provide students with the necessary resources for social norms and sociomathematical norms. A study found that important teacher guidance for collaborative inquiry involves supporting student contributions with a well-defined structure for mathematical work (Staples, 2007). In consideration of the role of the teacher through social norms, sociomathematical norms, and support for students' contributions, the last design principle emphasizes awareness of both students' contributions, which should illuminate their roles in their interaction pattern, as suggested in the first design principle.

5.2 Learning outcome

Considering for whom and how an interaction is productive in collaborative problem-solving illuminates how aspects of CCMR competency can both promote *learning of* CCMR competency and *learning through* CCMR competency. If the focus is on the *learning of* CCMR competency, as seen in Study 1, Study 2, and Study 3, and related literature, the emphasis is on the CCMR skills in students' bi-directional interactions, where both students are engaged in a mathematical conversation, justifying arguments, challenging or adding to suggested ideas, and testing ideas.

These activities are found in students' interaction processes and are intertwined facets of collaborative processes, mathematical reasoning, and shared agency with a mutual distributed authority. The previous section elaborated on principles facilitating students' bi-directional interactions, expressed through multi-faceted processes of interactional aspects. In this section, the skills are "extracted" from students' bi-directional interactions to emphasize the *learning of* CCMR competency. Thus, the design principles differ in the nature of their components. The suggested design principles for the practice goal of *learning of* CCMR competency are seen in the following student actions:

- Justifying arguments
- Challenging or adding to suggested ideas
- Testing ideas
- Mutual distributed authority

Central for *learning of* the competency is a focus on the interactional accomplishment of collaborative mathematical reasoning (Krummheuer, 2007; Lithner, 2017; Yackel, 2001), including verbal and non-verbal activity in student interactions (Kumpulainen & Mutanen, 1999). The four suggested design principles are based on the principles of bi-directional interactions. The first three principles—justifying arguments, challenging or adding to suggested ideas, and testing ideas—are straightforward skills that students and teachers or teacher educators can understand and recognize. For instance, if students are reflecting on or monitoring their own interactions, they have explicit and prescriptive statements to utilize to achieve the practice goal of learning the CCMR competency. Utilizing prescriptive statements in a monitoring situation could potentially emphasize student awareness of the skills they are learning. Likewise, an observing teacher or teacher educator could assess a student dyad's development of the CCMR competency skills by considering the students' interactions in light of these principles. The design principles could, therefore, function as a tool to be utilized by participants or observers.

The fourth principle, mutual distributed authority, concerns a shared responsibility of the above-mentioned principles. Such as both being responsible for making suggestions, justifying arguments, posing counter suggestions if disagreeing, and questioning explanations or actions. The fourth principle differs from the three other principles in its level of applicability as an indicator to evaluate the interaction, particularly for students or teachers to recognize in a situation. However, if noticing the other three principles, students are probably already having mutual authority in the situation. Nevertheless, the fourth principle is an important social skill to develop for promoting shared agency.

These design principles may also be used as indicators by teachers or teacher educators to assess an interaction as potentially productive for *learning through* CCMR competency. It is likely that students who work together, both mutually reasoning creatively and anchoring their suggestions in mathematical properties (Lithner, 2017) and exercise agency (Gresalfi et al., 2009), are presented with more learning opportunities (Schoenfeld et al., 2019) compared to students who are not invested in the same way in an interaction. The three studies did not examine students' learning outcomes but focused on the possibilities of interaction patterns for establishing, maintaining, and resolving conflicts to develop a shared understanding of a mathematical problem. Consequently, the learning outcomes are based on assumptions anchored in relevant literature connected to the interaction patterns: bi-directional interaction and one-directional interaction. The suggested design principle for *learning through* CCMR competency is seen in students' shared understanding of knowledge elements (Barron, 2000; Roschelle & Teasley, 1995) and solution procedures:

- Both students can explain the properties of the task and the relevant concepts involved and can provide reasons to support the solutions or reasons to refute a potential solution.

The emphasis of the last design principle is again on the mutuality in the dyad: both students must be engaged in the above-mentioned skills of CCMR competency. CPS is defined as a "coordinated attempt between two or more people to share their skills and knowledge for the purpose of constructing and maintaining a unified solution to a problem" (c.f., OECD, 2017, Roschelle and Teasley, 1995, as cited in Sun et al., 2020, p. 2). The suggested design principle is in line with this definition but transcends the general notion with more specific prescriptions for quality interactions. Thus, the prescriptions can be used to evaluate students' learning via the competency when recognizing the CCMR competency skills existing and emerging through the interplay of co-reasoning with relevant concepts.

In summary, the practice goals involve *learning of* and *through* CCMR competency, where the premise is students' bi-directional interaction patterns. When learning mathematics through collaborative problem solving, one must engage in the skills of collaboration, reasoning, and sharing of agency, allowing these skills to be simultaneously developed. Thus, competency development achieved through quality interactions may result in increased knowledge about the mathematics involved and about the competency itself.

6. Concluding thoughts

This project sought to answer the following research question:

How can a design-based research process focusing on aspects of students' interactions and related teacher actions contribute to design principles that support productive student interactions in mathematics?

Connecting and utilizing theories and frameworks as analytical lenses has the potential to lead to theory development (Amiel & Reeves, 2008; Juuti & Lavonen, 2006). However, the aim of this thesis was not to reconcile different frameworks and theories into one coherent theory of mathematics education. The connection of framework is instead pragmatic: refinement of theory and practice forming principles to inform and improve practice (Wang & Hannafin, 2005). Thus, the connection of frameworks may provide insights into productive collaborative student interactions in mathematical problem solving through the educational innovation of design principles. Moreover, the focus of the research results is also pragmatic and concerned with organizing new insights for supporting, assessing, and understanding students' collaborative problem-solving competency. The results of this project present design principles for CCMR competency that expand the CPS competency concept by connecting interaction aspects and promoting a productive interaction pattern. Moreover, the CCMR competency model differs from the other CPS frameworks, as presented by Sun et al. (2020), by identifying specific components of interactions tied to the foundational CPS constructs.

Implementing results from educational research is a well-known challenge (Juuti et al., 2016). One way to meet this challenge is to develop educational innovations, such as the CCMR competency model, in which design principles provide several useful applications. "Useful" here refers to the idea of a practical and straightforward way of utilizing design principles in teaching-learning situations and research purposes. First, DBR offers new educational knowledge to support teachers in their teaching (Juuti et al., 2016). The CCMR model is also useful to teachers, helping them to recognize productive interactions that foster learning in mathematics and inherent competency skills. Teachers should look for design principles for a bi-directional interaction in "Students exercise shared agency" (Fig. 7). Teachers may additionally look for the desired skills in "Learning of CCMR competency." In this way, a teacher can become aware of interaction patterns. They should pay attention to students' methods of participation, which are the emphasized design principles of "Teacher actions" in the CCMR competency model.

The second potential use case is in students' reflections and monitoring of their collaborative work. By recognizing design principles in "Learning of CCMR competency" (i.e., justifying arguments, challenging or adding to suggested ideas), students may develop an awareness of how productive interactions promote learning of competency skills and a deeper understanding of the mathematics involved. A teacher should therefore foster students' awareness of the opportunities afforded by the CCMR model.

The third group for whom this model may be useful is researchers in mathematics education. The model may be used to analyze classroom situations that facilitate student interactions with a focus on collaboration and mathematical reasoning. Moreover, utilizing the design principles for analytical purposes could lead to new theory development of design principles, as well as other interactional aspects to consider in students' quality interactions in collaborative mathematics learning.

The project's strength (simultaneously a limitation, elaborated on below) is a small sample size of students and their teachers, whose interactions were studied closely. The DBR study design also meant that participants could be studied in the naturalistic environment of the classrooms (Anderson & Shattuck, 2012), where the theoretical foundation is anchored in relevant theory (Euler, 2017), and with teachers involved in specific phases (Iversen & Jonsdottir, 2018), which was found to be meaningful and useful to them and myself. In the researcher–teacher interactions, the conversations focused on planning lessons relevant for progress and in line with the curriculum, as well as practicing emphasizing mathematical reasoning and collaborative work in the classroom. The teachers were engaged in the planning and implementation. We planned the three highlighted function problems through discussions of students would respond. We reflected on students' engagement afterwards, as well as the role of the teachers in the situations. In section 3, the second function problem (Fig. 5) is presented. The teachers thought the task was promising, but it facilitated minimal mathematical reasoning, noticed by the teachers and I. Thus, a possible agenda for further research is a focus on task design principles for the CCMR competency model developed through a partnership between researchers and teachers.

The iterative processes led to contextually sensitive design principles (Wang & Hannafin, 2005) in the suggested CCMR competency model (Fig. 7).

Moreover, the argumentation for the generated design principles is properly anchored in a theoretical foundation regarding the interactional aspects of mathematics education. Thus, it is likely that the design principles developed are relevant outside the local context of classroom studies (Barab & Squire, 2004).

Like all studies, this project has limitations. Six pairs out of 33 student pairs and three teachers were chosen for in-depth studies in two problem-solving sessions. Thus, one limitation is that the CCMR competency model was built using a small sample size. Although the iterative processes were anchored in relevant theory and the contextual sensitivity was considered, more student dyads should be studied to bolster support for the CCMR competency model. To further evaluate and strengthen the model, a broader specter of mathematical problems in different areas of mathematics and for different age groups could also be adopted. This could potentially reveal nuances in the interactional patterns identified in students' attempts to collaborate and reason mathematically.

Another issue is the selection of the six pairs. These participants were chosen because of their high level of reasoning competency, which meant that they attempted to explain their thinking and anchored it in mathematics. Moreover, the teachers contributed insights regarding the likelihood of student pairs being verbal and sharing their thoughts with one another. Since the students had recently transitioned from lower secondary school and the teachers did yet not know them, the teachers and I hoped the dyads would provide valuable insights into the interplay of the chosen interactional aspects. However, Lithner (2017) emphasizes that the CMR framework is appropriate for every student, regardless of reasoning skills. Thus, another potential research agenda could be to study student pairs with different levels of reasoning competency. For instance, a study of students who demonstrate low reasoning competency could look at what interaction pattern between the students is observed during mathematical problem solving. Such a study would be a worthwhile exploration of the CCMR competency model's applicability. Additionally, a student pair exhibiting a one-directional interaction that does not change after teacher intervention should possibly be assigned new peers. Thus, another issue to consider is the flexibility of the dyad formation to support potential bi-directional interactions.

The design principles concerning the role of the teacher related to student dyads' interaction patterns could be developed further. Future research could further investigate teacher–student interactions, focusing on teacher actions and emphasizing students' roles and exercised agencies in interactional patterns to improve the CCMR competency model with nuanced teacher action principles.

The DBR process also requires further study. Future research could involve additional collaborative phases with the teachers during some stages of the analysis. For instance, teachers could be involved in interpreting students' CMR sequences connected to teacher interactions. A teacher's beliefs and identity, or orientations, related to the CCMR competency could provide insights that inform their knowledge of how to support productive student interactions. This could thus strengthen the design principles for teacher actions.

7. References

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Appendix A

Coding framework of the three interactional components

Coding framework for the interactional components of collaborative problem solving			
How are students engaged in mathematical reasoning?	**Creative mathematically founded reasoning (CMR)**		
	Creativity	**Plausibility**	**Anchoring**
	• New idea • Recreating a forgotten idea	• Explanation of strategy choice • Explanation of strategy implementation • Explanation of why something is true	• Ideas connected to mathematical properties and concepts • Intrinsic: mathematical concepts and properties
How are students responding to each other?	**Collaborative processes for building and maintaining a shared understanding**		
	Building	**Monitoring**	**Repairing**
	• Accepting ideas • Making suggestions • Stating a problem • Pointing out mathematical properties	• Asking questions • Explaining an idea • Observing and responding to one another's interpretations and ideas	• Negotiations • Correcting conflicting interpretations • Counter-suggestions • Reformulations
	Agency		
How are students participating?	**Shared agency**	**Primary agent**	**Secondary agent**
	• Co-construction of arguments	• Makes the final argument • Assimilates another student's argument • Makes sense of a peer's faulty or flawed argument	• Makes input for the main argument

The headings in the table reflect the interactional categories. The bullet points describe the emphasized interactional aspects within each category and were used as codes in the analysis. The table is organized according to the way in which it was utilized to code student pairs' collaborative problem solving: first for CMR, then for collaborative processes, and finally for exercised agency.

Appendix B

Codes for teachers’ actions in conversations with student pairs

Teachers’ Actions	
Focusing – *Giving attention to students’ thoughts and input*	
Requests for students’ input	
Enlighten detail	• Student explanation • Focus on details • *Gathering information***
Justification*	• Explaining why • Justifying method and/or answer
Apply to similar problems*	• Asking to use knowledge on similar problems
Request assessment from other students*	• Asking other students to evaluate answer/solution
Pointing out	
Recap*	• Repeating an answer • Adding to an answer
Notice	• Highlighting details in a dialogue • Reminding students of new or previous information • *Explanation focus** on a student’s question
Progressing – *Taking action to move the process forward*	
Open progress details	• Open questions with several answers • *Encourage testing of strategy***
Closed progress details	• Closed questions with one answer • Request for details • Request about procedure/steps
Simplification	• Adding information • Hints • Telling students what to do • *Explanation progress***
Demonstration	• Showing a procedure • Doing several steps of a procedure • *Evaluation of solution/strategy* **
Redirecting – *Bringing attention to something else*	
Put aside	• Discarding a student’s suggestion/comment • *Interrupting***
Advising a new strategy	• Suggesting another approach
Correcting questions	• Question changing approach

* Not used in the analysis. **Added actions as a result of the analysis process.

Appendix C

Written informed consent from students and teachers

Samtykke om deltakelse i forskningsprosjekt tilknyttet matematikkfaget 1T

Dette brevet er en forespørsel til deg som elev i matematikkfaget 1T, om ditt samtykke til deltakelse i et forskningsprosjekt tilknyttet funksjonslære i 1T. Prosjektet skal undersøke hvordan elever kan øve sine matematiske resonnementer gjennom å forklare, begrunne og diskutere framgangsmåter og tanker, muntlig og digitalt. Dette er viktige matematiske kompetanser.

Først gjennomføres en matematikktest, som på et senere tidspunkt legger noe av grunnlaget for å sette sammen grupper. Jeg ønsker å observere og gjøre videoopptak av enkelte gruppesamtaler. Jeg er interessert i hvordan gruppedynamikken og lærerens interaksjon bidrar til å øve kompetansene nevnt over. Jeg er *ikke* interesserte i å studere den enkelte students prestasjoner og hvorvidt man «svarer riktig eller galt» på oppgavene. Hensikten er å lage en arena for å øve sentrale matematiske kompetanser som har en tendens til å få liten oppmerksomhet både i grunnskole, videregående skole og i høyere utdanning. Det er nå et økt internasjonalt fokus på hvorfor og hvordan denne type kompetanse kan øves.

All informasjon som blir samlet inn vil bli lagret på hjemmeområdet på PC-en. Kun jeg og min veileder har innsyn i det datamaterialet som er samlet inn, og du kan be om at det skal slettes om du måtte ønske det. Det er frivillig å delta i forskningsstudien og du kan når som helst trekke deg fra studien uten å begrunne dette nærmere. I min rolle som forsker innebærer det at jeg er underlagt strenge etiske regler for hvordan datamaterialet kan brukes. Materialet vil bli behandlet konfidensielt, og vil kun benyttes til forskningsformål. Prosjektet skal etter planen avsluttes august 2027. Ved prosjektslutt slettes videoopptakene.

Jeg håper du vil gi meg den nødvendige tillatelse ved å undertegne og returnere svararket (side 2). Ønsker du å ta del i prosjektet uten å komme med i synsvinkelen fra et filmkamera, legges det til rette for det i undervisningssituasjonen. Ta kontakt for nærmere spørsmål (se kontaktinformasjon under).

Vennlig hilsen
Ellen Kristine S. Hansen
ellen.hansen@nmbu.no

Samtykkeerklæring fra elevene

Leveres innen fredag 01.september

Jeg har lest informasjonen om forskningsprosjektet tilknyttet matematikkundervisningen i 1T. Jeg er kjent med at den frivillige deltakelsen i forskningsprosjektet innebærer dokumentasjon ved en matematikktest og ved hjelp av videoopptak.

Vennligst kryss av:

- ❑ Jeg samtykker til å delta på alt
- ❑ Jeg samtykker til delta, men ikke til å ta matematikktesten
- ❑ Jeg samtykker til å delta i undervisningen, men ikke å bli filmet
- ❑ Nei, jeg samtykker ikke

Underskrift: ______________________________

Sted: ______________________________ Dato: ______________________________

Samtykke om deltakelse i forskningsprosjekt tilknyttet matematikkfaget 1T

Dette brevet er en forespørsel til deg som lærer i matematikkfaget 1T, om ditt samtykke til deltakelse i et forskningsprosjekt tilknyttet funksjonslære i 1T. Prosjektet skal undersøke hvordan elever kan øve sine matematiske resonnementer gjennom å forklare, begrunne og diskutere framgangsmåter og tanker, muntlig og digitalt. Dette er viktige matematiske kompetanser.

Jeg ønsker å observere og gjøre videoopptak av enkelte gruppesamtaler. Jeg er interessert i hvordan gruppedynamikken og lærerens interaksjon bidrar til å øve kompetansene nevnt over. Jeg er *ikke* interesserte i å studere den enkelte students prestasjoner og hvorvidt man «svarer riktig eller galt» på oppgavene. Hensikten er å lage en arena for å øve sentrale matematiske kompetanser som har en tendens til å få liten oppmerksomhet både i grunnskole, videregående skole og i høyere utdanning. Det er nå et økt internasjonalt fokus på hvorfor og hvordan denne type kompetanse kan øves.

All informasjon som blir samlet inn vil bli lagret på hjemmeområdet på PC-en. Kun jeg og min veileder har innsyn i det datamaterialet som er samlet inn, og du kan be om at det skal slettes om du måtte ønske det. Det er frivillig å delta i forskningsstudien og du kan når som helst trekke deg fra studien uten å begrunne dette nærmere. I min rolle som forsker innebærer det at jeg er underlagt strenge etiske regler for hvordan datamaterialet kan brukes. Materialet vil bli behandlet konfidensielt, og vil kun benyttes til forskningsformål. Prosjektet skal etter planen avsluttes august 2027. Ved prosjektslutt slettes video- og lydopptakene.

Jeg håper du vil gi meg den nødvendige tillatelse ved å undertegne og returnere svararket (side 2).
Ta kontakt for nærmere spørsmål (se kontaktinformasjon under).

Vennlig hilsen
Ellen Kristine S. Hansen
ellen.hansen@nmbu.no

Samtykkeerklæring fra læreren

Leveres omgående.

Jeg har lest informasjonen om forskningsprosjektet tilknyttet matematikkundervisningen i 1T. Jeg er kjent med at den frivillige deltakelsen i forskningsprosjektet innebærer dokumentasjon ved hjelp av videoopptak og lydopptak.

Vennligst kryss av:

Jeg kan delta i forskningsprosjektet

❑ Ja, jeg samtykker

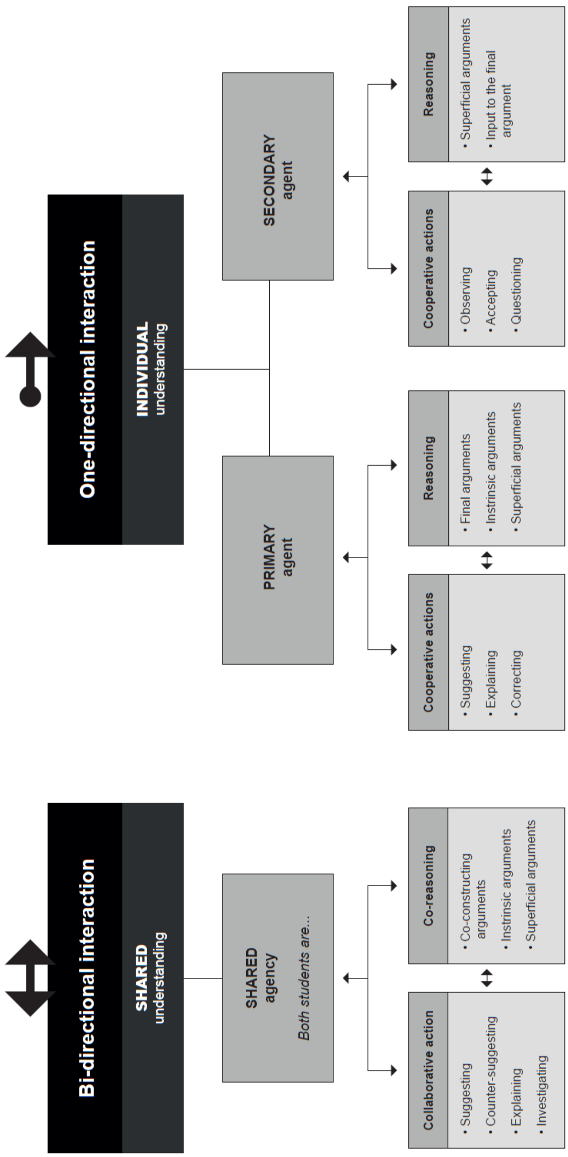
❑ Nei, jeg samtykker ikke

Underskrift: ______________________________________

Sted: ________________________________ Dato: ______________________________

Appendix D

Analytical CCMR model for evaluating collaborative interaction patterns



Errata

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ORIGINAL ARTICLE

Students' agency, creative reasoning, and collaboration in mathematical problem solving

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Abstract

This paper aims to give detailed insights of interactional aspects of students' agency, reasoning, and collaboration, in their attempt to solve a linear function problem together. Four student pairs from a Norwegian upper secondary school suggested and explained ideas, tested it out, and evaluated their solution methods. The student–student interactions were studied by characterizing students' individual mathematical reasoning, collaborative processes, and exercised agency. In the analysis, two interaction patterns emerged from the roles in how a student engaged or refrained from engaging in the collaborative work. Students' engagement reveals aspects of how collaborative processes and mathematical reasoning co-exist with their agencies, through two ways of interacting: bi-directional interaction and one-directional interaction. Four student pairs illuminate how different roles in their collaboration are connected to shared agency or individual agency for merging knowledge together in shared understanding. In one-directional interactions, students engaged with different agencies as a primary agent, leading the conversation, making suggestions and explanations sometimes anchored in mathematical properties, or, as a secondary agent, listening and attempting to understand ideas are expressed by a peer. A secondary agent rarely reasoned mathematically. Both students attempted to collaborate, but rarely or never disagreed. The interactional pattern in bi-directional interactions highlights a mutual attempt to collaborate where both students were the driving forces of the problem-solving process. Students acted with similar roles where both were exercising a shared agency, building the final argument together by suggesting, accepting, listening, and negotiating mathematical properties. A critical variable for such a successful interaction was the collaborative process of repairing their shared understanding and reasoning anchored in mathematical properties of linear functions.

Keywords Agency · Collaboration · Reasoning · Shared understanding

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Introduction

Students' mathematical communication, reasoning, and problem-solving are highlighted in the research literature as important aspects for fostering students' learning of mathematics (Pijls & Dekker, 2011; Seidouvy & Schindler, 2019; Sidenvall, 2019). Therefore, promoting students' mathematical communication such as making reasoning meaningful to oneself and peers, listening to one another, and solving mathematics together (Mueller et al., 2012) are central issues for progress in their mathematical understanding. Students' collaboration and reasoning are key interactional aspects in math-talk and small group work. It is a well-researched area but still highlighted as a topic where more knowledge is needed, because when students are given the opportunity they can construct their own solution procedures, important for their mathematical understanding (cf., Lithner, 2017; Mueller et al., 2012; Stockero et al., 2019).

Collaboration in classrooms has often been studied as an outcome or as a process (Dillenbourg et al., 1996). A research approach on the outcome is often seen as an individual's activity with focus on learning (Child & Shaw, 2018; Dillenbourg et al., 1996), whereas a research approach on collaborative processes emphasize the whole group and in particular the participant interaction (Seidouvy and Schindler, 2019). With the latter view on collaboration, this study defines collaboration as "a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (Roschelle & Teasley, 1995, p. 70). With this focus, the study views productive collaboration enacted when students build and maintain a shared conception of a mathematical problem, which is their shared understanding of the problem at hand (Roschelle & Teasley, 1995). Moreover, for students to pool their knowledge together for a shared understanding, the study sees the processes of *building* by introducing and accepting knowledge, *monitoring* ongoing activity, and *repairing* conflicting interpretations, as central activities for such interaction.

Opportunities for studying situations of collaborative processes are in this study viewed through a problem-solving session, emphasizing active engagement in a learning process (Lithner, 2017). An active learning process is not common in so-called easier learning processes where solution procedures are imitated but rather found in students' problem solving when they attempt to construct their own solution procedures through reasoning (Lithner, 2017). Mathematical reasoning is a central interactional aspect of learning mathematics where arguments are important for the learning process and not only as an outcome of learning mathematics (Yackel, 2001). Therefore, this study views mathematical argumentation and reasoning as an interactional accomplishment and what students "take as acceptable, individually and collectively, and not whether an argument might be considered mathematically valid" (Yackel, 2001, p. 6). In line with this view, including all students at any competence level in mathematics, is Lithner's framework of mathematical reasoning (Lithner, 2008, 2017). From this framework, a student's reasoning is explained as "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2017, p. 939).

In situations of mathematical communication, a student may become a producer of mathematics through their own and joint processes of grappling and exploring mathematics, sometimes hitting the wrong trail, and sometimes making sense of solutions, rather than being a reproducer (Freudenthal, 1991; Schoenfeld, 2013). Schoenfeld (2013) says: "A major issue is, when if ever do students get to develop a mathematical voice? That is, when do they get to propose ideas and answers, defend them, and become recognized as producers of mathematics themselves?" (p. 613). If these situations of mathematical communication include mathematical talk between students, they are in a student–student interaction. How well students interact in pairs and small groups is important for students' progress in becoming a mathematical producer, rather than an imitator who reproduces mathematics without understanding the conceptual parts (Lithner, 2017). Varhol et al. (2020) researched student interactions and found that some specific types of interactions were important for making mathematical progress in algebraic generalization. That study, and other studies on students' group work or pairwise collaboration, contends that quality student interactions and the dynamics of the processes in students' collaborations need to be further explored (Seidouvy & Schindler, 2019; van de Pol et al., 2018; Varhol et al., 2020).

Students engage with one another and take on different roles while interacting. Sometimes a student leads the conversation, other times he or she might listen or withdraw from the conversation. Thus, roles in collaborative work can "change from moment to moment" (Child & Shaw, 2018, p. 1). Hence, another central interactional aspect in group work, when considering students' actions and engagement in reasoning and collaboration, is students' *exercised agency* (Mueller et al., 2012). This study see agency as Gresalfi et al. (2009) define it; "the way in which he or she acts, or refrains from acting, and the way in which her or his action contributes to the joint action of the group in which he or she is participating" (p. 53).

The field needs a better understanding of peer interaction patterns in collaborative mathematical activity. Little is known about the dynamics of students' collaborative interaction in mathematics classroom, and few studies have taken the process view of collaboration (Seidouvy & Schindler, 2019). This study focuses on three particular aspects of students' interactions: collaborative processes, mathematical reasoning, and exercised agency. To investigate conditions for fruitful collaboration, Kuhn (2015) states that "it is essential to understand the underlying mechanisms" (p. 47). Studying interactional aspects separately and seen in interplay in students' interaction patterns may therefore give a better understanding of underlying processes of collaboration, which can furthermore enable collaborative aspects to become teachable. Research on collaboration (Child & Shaw, 2018; Kuhn, 2015) and reasoning (Lithner, 2017) argues that learning is enacted in both instances and that there is a need for insights of the underlying processes causing this learning opportunity. Therefore, unpacking students' social interactions found in the interplay of specific interactional aspects can make collaboration and reasoning more manageable for teachers and students, and consequentially promote productive quality interactions for learning opportunities in mathematics classrooms.

The aim of this study is to give detailed insights of interactional aspects with upper secondary students' roles in their collaboration and reasoning, when attempting to solve a linear function task together. With this aim, the research question is *What are the patterns of interaction for creating a shared understanding through the interplay between students' creative reasoning, collaboration, and exercised agency in a mathematical problem-solving session?*

Theory

Mathematical reasoning

Mathematical reasoning can be defined as "the explicit act of justifying choices and conclusions by mathematical arguments" (Boesen et al., 2014, p. 75). In line with this statement is the framework *Creative mathematically founded reasoning*[1] (CMR, Lithner, 2008), which identifies two major types of reasoning: *creative reasoning* and *imitative reasoning*. The latter reasoning type is seen in students' use of remembered facts and memorized algorithms without considering their meaning. Such path of reasoning has its strength in quickly solving tasks in school mathematics. However, without the conceptual part it may lead to rote learning (Lithner, 2017). Creative reasoning, on the other hand, has the strength of promoting deeper understanding of mathematical procedures and concepts (Lithner, 2008). If engaged with creative reasoning, students are considering mathematical properties with the task they are solving or discussing, which makes it likely to develop an understanding (Lithner, 2017). Creative reasoning is characterized by three aspects: *creativity, plausibility, and anchoring.* These three aspects are interconnected in reasoning as follows: A student may create a new idea or recreate a forgotten one (creativity) using arguments which are meaningful and logical to the student who is employing them (plausibility) and that are based on mathematical properties (anchoring). Creativity is therefore a student's attempt to create or recreate a reasoning sequence that, to some extent, is new to them. A student's reasoning, expressed as arguments, is creative when supported by plausible arguments. Plausible arguments are explanations of strategy choices, implementations of the strategies, and explanation of why a strategy or solution will work or not (Olsson, 2018). The arguments are creative when explanations and suggestions are mathematically anchored justifications (Granberg and Olsson, 2015). Lithner (2008) explains the difference in mathematical property as superficial or intrinsic: "In deciding if 9/15 or 2/3 is larger, the size of the numbers (9, 15, 2 and 3) is a surface property that is insufficient to consider while the quotient captures the intrinsic property" (p. 261). Central CMR-components are presented in Table 1.

[1] In line with Lithner (2008, 2017) and his colleagues studying creative mathematically founded reasoning, this study uses the wording creative reasoning or acronym CMR for linguistically simplicity.

Table 1 Overview of central elements of creative reasoning

Creative mathematically founded reasoning (CMR)		
Creativity	Plausibility	Anchoring
New idea Recreating a forgotten idea	Explanation of strategy choice Explanation of strategy implementation Explanation of why something is true	Ideas connected to mathematical properties and concepts Intrinsic: mathematical concepts and properties

Collaboration

When students work together and attempt to do thinking together, their "emergent interplay of ideas" (Martin & Towers, 2015) promotes a *shared understanding* of the problem. A shared understanding is students' collaborative outcome from engaging in collaborative processes where two or more students work together to solve a problem and attempt to produce a joint outcome (Roschelle and Teasley, 1995). In line with a view on students' merging of ideas into a shared conception for solving a mathematical problem, Roschelle and Teasley (1995) define *collaboration* as a "coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (p. 70).

The way students engage in collaborative processes relates to how well collaboration is maintained and fostered (Child & Shaw, 2016), and collaborative processes are important to study in students' interactions for the dynamics of joint mathematical problem solving. Students' attempts to create and uphold a shared understanding, through coordination of language and actions (e.g., Baker, 2015; Roschelle & Teasley, 1995; Sarmiento & Stahl, 2008), entail what Roschelle & Teasley (1995) call collaborative processes of building, monitoring, and repairing. The collaborative process *building* means to suggest ideas to initiate collaboration or it could be a continuation or ending of collaborative work (Alrø & Skovsmose, 2004; Child & Shaw, 2018; Roschelle & Teasley, 1995). For instance, if a peer accepts the suggested idea, such as a problem-solving strategy or implementation of an algorithm, a student contributes to build a shared understanding. A student could also read out loud and point at the problem to be solved. Sometimes, a peer listens to a suggestion or asks questions about an idea, which is important for *monitoring* a groups' shared understanding (Roschelle & Teasley, 1995). A question about an idea might result in a monitoring action, such as an explanation. If an explanation does not make sense or a suggested idea seems wrong to a peer, then students might experience a discrepancy between viewpoints (Dillenbourg, 1999). But if students try to restore their shared understanding about the problem, they are in the collaborative process of *repairing* (Roschelle & Teasley, 1995). Therefore, important actions for repairing a shared understanding are negotiations and corrections of conflicting interpretations, such as paraphrasing or repeating an utterance in one's own words (Alrø & Skovsmose, 2004).

Through these processes, coordination is seen as mutual exchange of utterances and taken actions, e.g., hand gestures, inputs to a dynamic software program, and explanations and justification of mathematical ideas. Students need to engage in a mutual exchange of ideas and actions to access each other's thinking. In conversations, students can take turns suggesting, questioning, and negotiate ideas. Therefore, a central interactional practice in conversations is *turn-takings* (Roschelle & Teasley, 1995; Sidnell, 2010). Turn-takings may therefore promote students' co-construction of a shared understanding through processes where students build, monitor, and repair the meaning or a strategy for solving a problem (Roschelle & Teasley, 1995).

Trying to fix the differences in opinions (repairing), understanding an explanation (monitoring), or introducing ideas and suggestions as well as accepting them (building) are important social interactions co-existing with students' mathematical reasoning (Granberg & Olsson, 2015). Components of the collaborative processes are presented in Table 2.

Agency

A student can participate in group work by contributing with ideas or by listening to peers. The participation may include actively seeking to solve the task at hand, by making their own attempts, which involves guesses, trials and wrong paths, and investigations that may lead to evaluating their own mathematical production (Freudenthal, 1991). However, another student or the same student, in a different situation or in a different group, could refrain from making suggestions and actions, or active listening. A student may, for various reasons, resist to attend to a collaborating peer. Therefore, the nature of a student's exercised agency will vary within different interactions and situations (Gresalfi et al., 2009).

There are several possible perspectives to study when focusing on students' exercised agency in mathematics. Carlsen, Erfjord, Hundeland, and Monaghan (2016) point to how different cultures, people, and artifacts shape students' actions and decisions, and thus, their exercised agency. Gresalfi et al. (2009) are concerned with students' engagement in classroom activities. This focus concerns students' act of complying or refraining (Sengupta-Irving, 2016), as well as about the given opportunities to act, either from a peer or from a teacher (Langer-Osuna, 2018). The latter aspect concerns distribution of agency (Gresalfi et al., 2009). A teacher may distribute agency through his or her authority to the given group (Engle & Conant, 2002), or there could be a social conflict between group members, consequentially inhibiting students' talk and activities (Langer-Osuna, 2018).

Table 2 Overview of central elements of collaborative processes

Collaborative processes for building and maintaining a shared understanding		
Building	Monitoring	Repairing
Accepting ideas Making suggestions Stating a problem Pointing out mathematical properties	Asking questions Explaining an idea Observing and responding to one another's interpretations and ideas	Negotiations Correcting conflicting interpretations Counter-suggestions Reformulations

The mentioned aspects of teacher authority, social conflicts, or different cultures and artifacts are relevant for studying students' collaboration and mathematical reasoning. However, to focus the agency aspect on student—student interaction, this study views students' exercised agency as their engagement in participation for making mathematical arguments or refraining from making mathematical arguments (Gresalfi et al., 2009).

To investigate students' argumentation when collaboratively solving mathematical problems, a framework on how students' agency is expressed in discursive practices (Mueller et al., 2012) is adopted. This framework is based on the definition on agency from Gresalfi et al. (2009) and suggests that students may exercise different agencies, such as *shared agency, primary agency,* or *secondary agency* (Mueller et al., 2012). See Table 3 for an overview of central elements of agency.

A shared agency is students' co-construction of arguments, where all participants contribute with their ideas. Therefore, ideas, suggestions, and actions are all important elements to building an argument from the ground, and only existing because of all the participants' contributions (Mueller et al., 2012). Contrasting shared agency is students' individual agency, where students are either a primary agent or a secondary agent. A student may act with primary agency when he or she makes the final argument based on correction from a peer or assimilates a peer's argument, or by making sense of a peer's faulty or flawed idea. A secondary agent makes inputs influencing the original argument. These inputs are either corrections or extended or flawed arguments, formed by the primary agent to a final argument (Mueller et al., 2012).

Methods

The study is characterized as an instrumental case study (Stake, 2003)—an in-depth study of the particular case of four student pairs, to advance the understanding of an interplay between students' creative reasoning, collaborative processes, and exercised agency.

Data collection and participants

This article reports on four pairs of students, age 15–16, who collaboratively worked on a function task (Fig. 1). The students were enrolled in their first year of a

Table 3 Overview of central elements of agency

Agency		
Shared agency	Primary agent	Secondary agent
Co-construction of arguments	Makes the final argument Assimilates another student's argument Makes sense of a peer's faulty or flawed argument	Makes input for the main argument

theoretical mathematics program. Data was collected in a Norwegian upper secondary school in 2017, where three mathematics classes (69 students altogether) and their three teachers participated.

In the design process of the larger study, which this case is a part of, the three teachers and the researcher (author) planned and evaluated lessons together emphasizing collaborative work and math-talk. Both prior to and after the planned lessons, the teachers and researcher discussed how to assist and interact with the students in order to promote mathematical reasoning and collaborative work. Students worked together in the same pairs over five consecutive months in their regular classroom setting. Based on the teacher–researcher conversations, the teachers attempted to ask open-ended questions with minimum guiding intervention, aiming to provide students with opportunities to make connections between function representations for understanding the function concept together. The three teachers were considered ordinary and engaged teachers, but not particularly used to organize classrooms for collaborative interactions. Hence, they were previous to the study not particularly aware of their teacher approach to support student pairs or whole classroom discussion for collaborative inquiries. This was a deliberate choice for the study, which this case study is a part of.

The students were organized into 33 pairs based on the following criteria: (1) reasoning competence; (2) understanding of functions; and (3) likeliness to engage in math-talk with one another. The two first criteria were based on the students' scores on a mathematics test. The last criterion was based on conversations with the teachers when considering point 1 and 2.

Due to practicality of observing students through video recordings, two pairs in each of the three classrooms were set as a condition. For an in-depth analysis of interactional aspects in students' collaboration, six pairs were chosen. The six pairs were chosen based on the three criteria above, where two aspects particularly stood out: (1) students should express a high level of reasoning competence, which meant that they attempted to explain their thinking and anchored it in mathematics and (2) the likeliness of student pairs to be verbal and share thoughts with one another. These aspects were discussed with their teachers when making the pairs.

Two out of six student pairs did not exercise turn-takings, a criterion considered important for creating a shared understanding of the problem. These two pairs did not have successful conversations, since they were not engaged in sharing thoughts with one another. The four other student pairs exercised interactions with reasoning, collaborative processes, and different agencies. From the analysis ("Data analysis"), two distinct ways of interacting were seen, two pairs within each interaction pattern.

- Create a straight line $y = mx + c$
- Create another straight line in a way that the corresponding graphs are perpendicular.
- Formulate a rule for when two straight lines are perpendicular.
- Test the rule for other straight lines.

Fig. 1 The function task (reformulated from Olsson (2018))

In this article, four student pairs will illuminate the two typical interaction-patterns. These pairs' interactions are presented in "Results."

The linear function problem

During the time of the study, three main tasks were given. The presented task (Fig. 1) facilitated the richest conversations. In these conversations, students more often anchored their arguments in mathematical properties (Lithner, 2017), such as the slope number being a varying parameter, and students more frequently engaged in all of the collaborative processes of building, monitoring, and repairing of their shared understanding (Roschelle & Teasley, 1995).

The function tasks that were planned emphasized mathematical reasoning and non-routine solving of tasks, where the struggle ought to be more like a challenge to solve, rather than an obstacle (Hiebert & Grouws, 2007; Lithner, 2017; Stein et al., 2008). In order to facilitate a challenge easy to discuss, however not too difficult, tasks without a known procedure for the students to follow were emphasized. Therefore, students were likely not to withdraw from mathematical conversations due to differences in their level of competence in mathematics. The task presented in Fig. 1 had previously and successfully been tested for similar purpose at another school (Olsson, 2018).

An aim with this task was also to facilitate opportunities for the students to connect different function representations to construct their own solutions to the linear function problem. In connecting different function representations, such as graphical and algebraic representations, students can build a more comprehensive function concept (Best & Bikner-Ahsbahs, 2017), rather than view functions as "topics" to be learned in isolation of the others" (Thompson, 1994, p. 24). The function concept is regarded in school mathematics as difficult for students to learn, but very important to understand, since it has a central role in organizing and connecting many mathematical ideas (Michelsen, 2006).

When students solved the function tasks, they were encouraged to use the dynamic software program GeoGebra, which may promote active investigation of different function representations (Olsson, 2019; Preiner, 2008). The strength of the program, to easily adjust representations in the algebraic field or the graphical field, gave students rapid feedback on well-justified suggestions or simple guesses. Students' engagement with GeoGebra for solving a function task may contribute to students' mathematical reasoning (Granberg & Olsson, 2015) and, thus, their way of interacting. However, GeoGebra does not interpret the meaning. Students need to make their own meanings of their findings, which is important for students in producing their own mathematics. In this case study, the unit of analysis is the student–student interaction, unlike Olsson (2018) where GeoGebra additionally was included in the unit of analysis.

Data analysis

Video recordings of the eight students' talk and actions were viewed multiple times, and the first step of analysis comprised denoting longer sequences where students often made justifications and explanations anchored in mathematical properties as sequences of *creative reasoning* (CMR) (Lithner, 2017). These sequences were thus coded as CMR-sequences (creating a solution with plausible arguments anchored in mathematics) and further transcribed.

Then, a second analytical step comprised the coding of each student's utterance in the CMR-sequences with respect to *collaborative processes* (Roschelle & Teasley, 1995). How a CMR sequence was coded for collaborative processes (see Table 2 for details) and for aspects of creative reasoning (see Table 1 for details) is exemplified in "Bi-directional interaction."

The third step in the analytical process was to provide a thick description (Powell et al., 2003) of the four pairs' collaboration and reasoning within their CMR sequences. This step enabled a description of students' participation in the collaborating dyads, which made it possible to characterize students' agency in their conversations as primary, secondary, or shared (Mueller et al., 2012). First, a description was made on the students' interactions when engaged in reasoning, concerning how they attempted to engage with each other. If students constructed a solution sequence where they connected ideas and thoughts making a shared understanding of the current situation, it was recognized as *shared agency*. If students engaged individually, clearly having different roles when suggesting ideas or explaining thoughts, their agency was recognized as *primary or secondary*. After characterizing students' interactions, it was possible to describe students' typical roles in their engagement connected to how they reasoned about linear functions and the ways they collaborated to solve the given task.

Tables 1, 2, and 3 present an overview of characterization of codes for CMR, collaborative processes, and agency, respectively. The headings in the tables are the interactional categories, and the bullet points describe the interactional aspects within each category and were used as codes in the analysis. Excerpts in "Results" outline typical interactions found between the student pairs in their CMR sequences. The excerpts are chosen because of their clear interaction patterns.

Results

Studying students' interactional aspects of reasoning, collaborative processes, and agencies contributed to rich descriptions of students' participation, which further grew into two distinct ways of interacting in the conversation, here named *bi-directional interaction* and *one-directional interaction*. Characteristics for the bi-directional interaction were students who engaged with similar roles; mutually attempting to understand each other's ideas; making suggestions, listening, and negotiating mathematical properties; and mutually driving the problem solving process forward. Such interaction was found in the student pairs: Philip and Noah, and Emma and Hannah. Characteristics for the one-directional interaction were students who engaged with different roles and one student who led the conversation by being the primary reasoner and suggestion-maker for solving the problem, which a peer attempted to understand and occasionally contributed with input to the final outcome of a reasoning sequence. Such interaction was found in the student pairs: Olivia and Oscar, and Leah and Isaac. In the following excerpts, interactional aspects of reasoning, collaborative processes, and exercised agency are italicized followed by a number from the students' utterances in the conversations.

Bi-directional interaction

Philip and Noah had found instances for when two linear functions were perpendicular onto each other. In the challenge that followed, they attempted to make a rule explaining when the functions were perpendicular to each other.

Excerpt 1

1	Noah	I don't know how we should write it [the rule], because…
2	Philip	Okay. It's slope number. What is it called? It's *m*… (writing on the laptop)
3	Noah	Yes
4	Philip	Divided by 1, right. Then we'll have…
5	Noah	No. One divided by *m* is right

Noah *stated* a problem for formulating the rule, thus initiating a new focus in a process characterized as *building* (1). Philip responded by *monitoring* their problem when he *asked* what the slope number was called, which was answered by himself (2). In Philip's turn, he also *built* their shared understanding by *suggesting* a focus *anchored* in the slope number (2). Noah continued to *build* when he *accepted* the initiation of formulating the rule (3). Again, Philip continued to *build* when he *suggested* it was *m* divided by 1 (4). This was not logical to Noah, and he *disagreed* with Philip's suggestion of the rule. Noah made a *counter-suggestion*: 1 divided by *m* (5). Thus, Noah's countersuggestion can be seen as a *repairing* of their shared understanding. Together, Noah and Philip created a new reasoning sequence for making a rule based on the generalized slope number *m*.

To specify the coding procedure for this excerpt: The sequence was characterized as CMR (Lithner, 2017), because in the conversation students' reasoning was *new* to them when they created an expression for the rule. Their explanation involved a connection between the perpendicular functions which was *anchored* in the mathematical property of the slope number *m*. Their solution made sense to the student pair. Thus, it was *plausible* to them. Concerning their collaborative processes (Roschelle & Teasley, 1995); students were, for instance, *building* a shared understanding when *suggesting* a focus of the rule *anchored* in the slope number, and *repairing* when making a *countersuggestion* to the rule.

In the further unfolding problem-solving path, the student pair continued in a turn-taking conversation.

Excerpt 2

6	Philip	Yeah, but is it a rule?
7	Noah	Yes
8	Philip	How we should say something, in a way
9	Noah	A rule. Oh yeah. The slope number on one line is something in connection to the other line. I don't know how to formulate, explain it. (Pause) You have the slope number for one [line] and slope number for the other one. To figure out the second one, I use that formula (pointing at the formula they have written down). So …
10	Philip	No, right. We need to figure out something better to say than the first and the second. Maybe like *a* and *b*, or something
11	Noah	Yes. I can try (writing on the laptop). Slope number for line b is $-m/1$. Wait. -1 divided by the slope number for line *a*. We say that. At line *a*
12	Philip	Yeah, but slope number (pointing at the screen). We can use *m*, as $mx + c$. Instead of "slope number." To add some subject content, concepts, with numbers and letters

Philip and Noah *agreed* upon the rule $-1/m$. However, Philip was initially not sure whether it was a proper mathematical rule (6, 8). Noah *explained* his take on the meaning of a mathematical rule and attempted an *explanation* of the rule they had found (9). Noah was building their shared understanding by *anchoring* his reasoning on how two linear function's slope numbers were connected (9). Together both contributed to generalize their findings (10, 11) into a rule constructed from observations of pairs of perpendicular lines, which they emphasized as "subject content, concepts, with numbers and letters" (12). Their attempt to create a rule was *co-constructed* through their turn-taking conversation.

Thus, Noah and Philip were both actively participating in solving the problem by *making suggestions, observing suggested strategies, taking initiatives, and making counter-suggestions.* The students' line of thought in the excerpts is *collaboratively built*, and their reasoning is *co-constructed* and does not exist without the peer's input. Hence, Noah and Philip exercised *shared agency* creating arguments together through collaborative processes and CMR.

Emma and Hannah began their problem solving by testing different linear functions to be perpendicular to the line $y = 4x + 2$. Their dialogue was focused on varying parameters (slope number and constant), and an anticipation where the lines would appear in the coordinate system. Excerpt 3 shows Emma and Hannah's continuing conversation and struggle about the connection between linear functions for being perpendicular.

Excerpt 3

13	Hannah	*y* equals. I have Caps Lock on. Eh, what should we try?
14	Emma	*x*. 1 divided by 4…. plus 2
15	Hannah	No, it's not the same. I thought, if it were the same [the rule could be] that we only put minus sign in the front. But that doesn't work
16	Emma	I don't know what we have been doing
17	Hannah	Wait. If we take 4 divided by minus… (writing on a paper). No We have. What we actually have been doing… We put number 4 as the denominator in the fraction, in a way. But I don't know

Emma and Hannah struggled to make sense of their own suggestions (15, 16, 17). Hannah *initiated* a new sequence of guess and check, thus collaboratively *building* (13). Emma responded with further *building* when *suggesting* a new linear function: $y = 1/4x + 2$ (14). Her suggestion conflicted with Hanna's expectation of perpendicularity of two linear functions (15). She thought that the pair of slope numbers should have been 4 and −4. Hannah's *correction*, at least for herself, was about the conflicting interpretation of the connection between the linear functions (15). Thus, Hannah *observed* that what she had previously thought did not make sense, which is characterized as *repairing* their progress (15). Emma *observed* that their input did not result in a desired outcome, and she *monitored* their problem situation (16). Hannah continued *monitoring* by attempting a further *explanation* of the algebraic representation, stating indirectly that the slope number of one line was found as a denominator in the slope number of a perpendicular function (17). She vaguely *suggested* that it ought to be a negative slope number but rejected her own suggestion (17).

Emma and Hannah struggled actively together attempting to find a pair of perpendicular lines. Both students addressed the slope numbers of the perpendicular pair of linear functions, and their reasoning sequence was *anchored* in the mathematical property of the slope number, mainly referred to in the algebraic expressions. The students attempted in a joint effort to solve the linear problem with *co-constructed reasoning*. Therefore, their participation in the interaction, seen in the excerpt and before and after, is characterized as a *shared agency*.

The two pairs, Philip and Noah, and Emma and Hannah, typically attempted to make sense of each other's ideas and thoughts. Similar for both pairs were a joint effort and engagement in different aspects of working together: making suggestions, listening to one another, and expressing disagreement with actions or suggestions made by each other or oneself. Although the two collaborating pairs often were engaged in similar ways, Philip and Noah stood out when it came to working together through reasoning and engagement in all the collaborative processes. This pair more often entered processes of repairing of their shared understanding of the function problem, compared to Emma and Hanna.

One-directional interaction

Olivia and Oscar were not engaged in a mutual exchange of ideas and actions for solving the function problem. In the following conversation Olivia and Oscar began their problem solving.

Excerpt 4

1	Oscar	Okay. $y = x + 1$ (Olivia begins to use the laptop). And then you can take $-x + 1$. It should be perpendicular
2	Olivia	Now?
3	Oscar	Yeah. There. It's perpendicular. Um. Rule. Um. You put… They have the same value, but minus in front. I don't know. Constant *x* has the same value, wait…
4	Olivia	It's not the same value, or?
5	Oscar	No. When they have the same constant, but one is positive and one is negative, I don't know. It's kind of a rule. Wait. We should have a rule (looking at the given task). Yes, this works. Right?
6	Olivia	Yes, it's not a formula, so it's okay
7	Oscar	But if the slope values are the same, because the constant can be different. It'll only go higher or lower on the first line-thing (pointing at the laptop screen). The straight line, as it was called

Oscar was the driving force for solving the problem. He made *suggestions*, such as *y* equals $x + 1$ and $-x + 1$, and *stated* problems "They have same value, but minus in front… Constant *x* has the same value" (1, 3). Oscar attempted an *explanation* about the observation of the linear functions, *justified* in the "constant *x*," which was *plausible* to him and *anchored* in mathematical property of the slope number (3, 5). Olivia *accepted* Oscar's explanations and suggestions, by putting her own words to Oscar's ideas: "Yes, it is not a formula, so it's okay" (6) and later she said "Yes, the constant can be different, and it doesn't have to be the same." Thus, the students were *suggesting* and *accepting*, in the process of *building*.

Oscar and Olivia demonstrated the process of *monitoring* when Olivia *asked* if her input looked correct (2), and when she *questioned* the meaning of the numbers having same value (4). Oscar *monitored* when he attempted an *explanation* trying to say that the slope numbers should be the same numbers only with different signs (5). In his last statement, he differentiated between the slope number and the constant (7). Thus, Oscar expressed more details to their rule: the perpendicular lines had to have the same slope numbers with opposite signs and the constant could be arbitrary numbers.

From their interaction and way of participating, Oscar and Olivia had different roles in the *building* and *monitoring* process, which characterized their interactions throughout their problem-solving path. They did not have instances of *repairing*. Oscar exercised agency when *suggesting* and *explaining* his ideas. Thus, he was the initiator and *primary agent* of reasoning in the conversation. Olivia *accepted* Oscar's explanation of the rule, which she distinguished from a formula (6). Oscar further *explained* his thoughts, after Olivia's acknowledgement of his suggestions (7). Olivia was a *secondary agent* who contributed with trying to understand how Oscar was thinking and what she should do to execute his ideas into action in GeoGebra. Their student–student interaction did not demonstrate a mutual and synchronous way of making and maintaining a shared understanding of the problem. Nor did they exercise a shared agency for solving the problem.

Leah and Isaac found a set of linear functions making a perpendicular pair: $y = x + 3$ and $y = -x + 3$. They expressed a relation between the two linear functions: two opposite slope numbers, which they further referred to as either $-x$ and x, or -1 and $+1$. They agreed upon this relation as the rule and Leah initiated testing of their rule for other linear functions.

Excerpt 5

8	Isaac	Okay. Then we create a line. We choose the line y equals, we choose $2x + 3$. Okay. It's probably a bad number. Okay, and then for every x there is 2. Then the negative version of this should be $y = -2x$, and then it should intersect… (writes on the laptop) But it isn't going to… It's always minus… The line should always be $-x$. Like that (writes on the laptop)
9	Leah	Always?
10	Isaac	Yes, because if I use $-2x$, it doesn't work. $y = -x$ and it should intersect…
11	Leah	3
12	Isaac	Are you sure?
13	Leah	Yes. Try (Isaac writes on the laptop)
14	Isaac	Hmm. No. 4
15	Leah	Oh. It's… (Leah uses the laptop). It's because this is f***ed
16	Isaac	Okay, but wait. It doesn't work for every number. Okay, we'll figure it out
17	Leah	I think it's wrong to say it's always $-x$. Try minus… What did you use here? 2?
18	Isaac	This is $2x + 3$
19	Leah	Then you should try $-2x$.

Isaac contended for their new pair of linear functions that always one of the slope numbers had to be $-x$ (8). Leah *questioned* his statement (9), which Isaac replied to by *pointing out* the different variables $-2x$ and $-x$ (10), which is likely his way of *anchoring* the reasoning in the mathematical property of the slope numbers being -2 and -1. They further attempted to *evaluate* their input in GeoGebra (11–15). Towards the end of the turn-taking sequence, Isaac *interpreted* their observations with guessing and checking with GeoGebra and said that it did not work for every case (16). Leah *repeated* his observation and specified that a perpendicular line not always was $-x$ (17), meaning that for a pair of linear functions one line did not have to have a slope number equal to -1. She continued *building* by *suggesting* $-2x$ for a perpendicular line (19), which Isaac previously had observed would not result in a perpendicular line (10).

In Leah and Isaac's conversation, there were mainly collaborative processes characterized as *building* by making *suggestions* for pairwise linear functions (8, 10, 13, 18, 19), which was *monitored* by *observations, evaluations*, and *questions* (9, 12, 14, 16). They did not engage in processes of *repairing*. Both students contributed to the problem-solving process; however, it was primarily Isaac that *suggested* and *made arguments* connected to mathematical properties *anchored* in the algebraic representation of the slope number. Isaac acted as the *primary agent* leading the conversation and making the final arguments. Leah, on the other hand, either *built* by *accepting* Isaac's ideas or *monitored* by *questioning, repeating*, and *observing* how Isaac's ideas and utterances played out in GeoGebra. Therefore, she exercised *secondary agency*.

In the interaction of the two pairs, Olivia and Oscar, and Leah and Isaac, only one student attempted to make sense of the collaborating peer's suggestions. Similar for both pairs were a mutual attempt to solve the function problem where they exercised different roles in the problem-solving session. Moreover, it was Oscar and Isaac who led the conversations, whereas Olivia and Leah attempted to understand their peers' thoughts and actions. The student pair Olivia and Oscar stood out compared to Leah and Isaac in the way that their roles in the interaction were more clearly divided in primary and secondary agency, particularly how Oscar made the reasoning sequences which Olivia tried to understand. The interplay in the interaction patterns found in their attempt of collaboration is in a clear contrast to, particularly, Philip and Noah's well-functioning collaboration.

Discussion

The study's research question is *What are the patterns of interaction for creating a shared understanding through the interplay between students' creative reasoning, collaboration, and exercised agency in a mathematical problem-solving session?*

The case study has analyzed videos of students' attempt to collaborate and engage in mathematical talk to solve a linear function problem. The findings reveal an interplay between exercised agency, reasoning, and collaborative processes, affecting how student–student interactions are expressed. Based on the students' roles in pairwise collaboration, considering interactional aspects, a pattern for their interaction evolved in the analysis process into bi-directional interaction and one-directional interaction.

If students exercise mutual attempts to understand one another (e.g., Mueller et al., 2012; Roschelle & Teasley, 1995), using arguments logically, and anchored in mathematical properties of the reasoning sequence (Lithner, 2017), where both are the driving force of the problem-solving process, they are in a *bi-directional interaction*. Philip and Noah, and Hannah and Emma revealed such an interactional pattern. Students typically negotiated mathematical properties (Lithner, 2017) to make a pair of perpendicular lines, and they often represented functions dynamically. Students' active investigations of mathematics using GeoGebra is seen as support for students' reasoning and activity (e.g., Granberg & Olsson, 2015; Olsson, 2019; Preiner, 2008), and observed in the two student pairs' engagement. Students expressed functions algebraically and graphically, thus, actively engaged with GeoGebra's feedback. Therefore, functions were not operated on as separate topics to be learned (Thompson, 1994), important for a deeper understanding of the function concept (Best & Bikner-Ahsbahs, 2017).

Both student-pairs participated in turn-taking conversations (Sidnell, 2010) building on suggested and explained ideas (Alrø & Skovsmose, 2004). Students exercised coordination and synchronicity in their interaction which characterizes "true" collaboration, where it is likely to achieve a shared understanding (Baker, 2015; Roschelle & Teasley, 1995; Sarmiento & Stahl, 2008).

Philip and Noah's interaction, viewed holistically, showed an interaction pattern where both students engaged in collaborative processes of building, monitoring, and repairing. When Philip and Noah engaged in a repairing process, they more often anchored their reasoning in mathematical properties of linear functions, compared to the other pairs. Moreover, Philip and Noah willingly shared ideas and entered situations with conflicting ideas (Dillenbourg, 1999), such as the formulation of the rule using the connection between an algebraic expression and a graphical representation of the function concept. In the mutual and synchronous interaction, Philip and Noah co-constructed reasoning sequences and participated with shared agency. Therefore, their agency co-existed with collaborative processes and creative reasoning important for shared understanding.

When students exercise different roles in the problem-solving process where the final outcome is expressed repeatedly by one of the students (Mueller et al., 2012), they are in a *one-directional interaction*. In such an interaction, this study suggests that, a primary agent utters creative reasoning, and a secondary agent listens or tries to understand a peer's argumentation. Thus, a student with a secondary agency rarely or never engages in creative reasoning. Moreover, their collaborative processes are characterized by building and monitoring instances, missing the important repairing instances valuable for a possible evolving of students' understanding.

Such a pattern was demonstrated in the student pairs Olivia and Oscar, and Leah and Isaac. In both pairs, one student led the process of solving the task. The co-working student often attempted to understand suggestions or explanations made by the primary agent. Thus, students engaged with different agencies. Here, Oscar and Isaac participated as the primary agents, and Olivia and Leah as the secondary agents. The secondary agents exercised their agency through expressed ideas and questions about the primary agents' ideas. Their input was either assimilated into the final outcome (Mueller et al., 2012) of the reasoning, or considered by the peer who refined the input or neglected it. Concerning collaborative processes for making a shared understanding (Roschelle & Teasley, 1995): Olivia and Oscar or Isaac and Leah did not experience conflicting ideas. Even though they attempted to collaborate, they did not seem to have a shared understanding, and they rarely experienced discrepancies in their ideas, which they would have to repair.

Oscar expressed more details to the solution strategies, and anchored ideas in mathematics (Lithner, 2017) concerning aspects of the rule for making two linear functions perpendicular than the other primary agent Isaac, who often led the conversation and made suggestions for solving the problem. Both Olivia and Leah tried to understand how their peers were thinking. Consequentially, their student–student interaction was not a mutual and synchronous way of creating a shared understanding of the function problem.

Students who engage in a bi-directional or one-directional manner may change roles in a problem solving process (Child & Shaw, 2016). A student might assimilate a peer's input to construct a final argument, which is in this study recognized as a primary agent (Mueller et al., 2012) in a one-directional interaction and as shared agency (Mueller et al., 2012) in a bi-directional interaction. Therefore, characteristics of "assimilating a peer's construct into a final argument" do not indicate what kind of agency a student has exercised. Therefore, to categorize the interactional aspect agency, it is important to study the collaborating pair's turn-takings and other interactional aspects at play, such as collaborative processes and mathematical reasoning.

Concluding thoughts

This study illuminates two interactional patterns from four student pairs: Noah and Philip, Hannah and Emma, Oscar and Olivia, and Leah and Isaac. The limitation of the small sample is acknowledged; however, rich descriptions and the detailed analysis have provided important insights in students' interactions when collaborating and reasoning about functions. More student pairs' interactions should be studied, and their engagement in a broader specter of mathematical problems, to further reveal nuances to interactional patterns found in students' attempt to collaborate and reason mathematically. Researching several instances of students' interactions in different settings may contribute to more nuances in how students, of different age, learning different mathematical topics, in different environments, develop their own mathematical voice (Schoenfeld, 2013; Sengupta-Irving, 2016).

If students interact in a dynamic way, as described above, where both participate in equal roles and show authority over mathematical ideas during the problem-solving process, they might construct a shared understanding from merging an interplay of ideas. Such dynamic structure of pairwise collaboration reveals important components in students' mathematical communication (Sidenvall, 2019), to better understand underlying mechanisms for fruitful collaboration (Child & Shaw, 2018; Kuhn, 2015; Seidouvy & Schindler, 2019). These identified conditions are both being promoters in a problem-solving process, both making reasoning anchored in mathematical properties, and both being engaged in different collaborative processes. Such an interactional pattern in a bi-directional interaction promotes learning of mathematics through quality interactions (Pijls and Dekker, 2011; Varhol et al., 2020).

The contrast to such a dynamic interplay is a monotonic one-directional interplay, where only one student evolves his or her individual problem space. In such instances of one-directional interaction, teacher involvement should be suggested for supporting both students to build (accept and suggest) and monitor (explain and ask question) emphasizing plausible and mathematically founded argumentation.

Reasons for the two patterns of interactions that occurred might have several explanations. One reason might be students' individual experiences and personalities.

Comparing the student pairs in the study, the student pair Philip and Noah stood out and demonstrated a productive interaction, as described above and in results. It is likely that their individual personalities were a good match. Thus, affective patterns or individual personalities come into play and influence students' conversations (Cobb et al., 2009). Other influencing aspects are students' background and disposition for learning, self-confidence and past success in mathematics, and beliefs about their roles and roles of others (Mueller et al., 2012).

A second reason is the given task. Although the given task presents an opportunity for creative reasoning (Granberg and Olsson, 2015), it does not automatically initiate a productive engagement in the other interactional aspects of this study. To promote interactional aspects such as collaboration and agency, a mathematical problem might entail other features than explicitly discussing the slope number and constant in a linear function found in an algebraic and graphical expression. If an individual student has experienced constraints with mathematics, it might prevent them for further engaging in a productive social interaction with a peer. Therefore, it might be worthwhile considering a relatable or meaningful context for students to reason about, and for translating between function representations.

A third reason to consider is classroom norms and teacher involvement in students' interactions. Regarding the first aspect, classroom norms, Yackel and Cobb (1996) differentiate between a social norm as an expected explanation to a given task, whereas a sociomathematical norm is an acceptable mathematical explanation. Both aspects are important for individual and collective learning, and students' engagement in both would influence their pattern of interaction. The second aspect, teacher involvement, has the potential to provide students with necessary resources for social norms and sociomathematical norms. A study found that important teacher guidance for collaborative inquiry happens through supporting student contributions with well-defined structure for mathematical work (Staples, 2007).

An agenda for further research on the three mentioned reasons for the two interactional patterns could be studies of different tasks promoting CMR, collaborative processes, and exercising of shared agency. Another future study could focus on teacher guidance: opportunities and limitations with teacher actions for students' productive interactional pattern. A third study could include interviews with students focusing on individual variables, such as beliefs about one-self as a mathematics learner, self-confidence, and the role of the collaborating peer. A broader understanding of these influential aspects might give a more complete understanding of the underlying mechanisms for productive interactions through the interplay of collaborative processes, mathematical reasoning, and exercising of agency.

If a teacher observes such an interaction pattern, the teacher could facilitate a change of roles: a secondary agent attempts to suggest strategies and explain outcomes or connection anchored in mathematical properties. A teacher's awareness of interactional patterns concerning collaboration processes, reasoning, and agency, has the potential to contribute to students' fruitful interactional dynamic. Moreover, such teacher actions has the potential to model important aspects to facilitate students' synchronicity and coordination pooling knowledge together to construct their own mathematical knowledge for understanding mathematical ideas (Stockero et al., 2019). Therefore, further research on teacher's actions facilitating dynamic interactions between the peers, where students are exercising shared agency, might provide additional insights to foster students' learning of mathematics through quality interactions.

Disclaimer

The manuscript has not been submitted to more than one journal for simultaneous consideration.

The submitted work is original and is not published elsewhere in any form or language (partially or in full). This manuscript has not been split up into several parts. Results are presented clearly, honestly, and without fabrication, falsification, or inappropriate data manipulation.

No data, text, or theories by others are presented as if they were the author's own ("plagiarism"). Proper acknowledgements to other works have been given. The author had permissions for the use of software. Research articles have been cited appropriate and relevant literature in support of the claims made. Author has avoided untrue statements.

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Code availability NVivo was used for coding the videos. All coding procedures are securely stored.

Compliance with ethical standards

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The role of teacher actions for students' productive interaction solving a linear function problem

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ABSTRACT

Many studies in mathematics education have emphasized the importance of attending to students' interactions, particularly, their mathematical reasoning when collaborating on solving problems. However, the question of how teachers can facilitate students' productive interactions for learning mathematics, is still a challenging one. This case study aims to provide detailed insights into opportunities and limitations related to teachers' actions for the productivity of students' interactional patterns solving a linear function problem together. Four student-pairs in the first year of upper secondary school (11th grade) serve as a background on students' interactional patterns, which in this study focused on three interactional aspects: collaborative processes, mathematical reasoning, and exercised agency. The student-pairs' three teachers provide insights on teacher actions observed as different funneling and focusing actions, which elucidated opportunities and limitations in several situations influencing the productivity of students' interactional patterns. The study used purposive sampling in selecting the particular school and three teachers, which were chosen based on acquaintances and willingness to participate in the study. The students' interaction when solving the mathematical problem and the teachers' interaction with the pairs were video recorded and observed by the researchers. The analysis method was a deductive analytical strategy, where specific events of interactions were identified, based on the three interactional aspects combined with teacher actions. Coding schemes on students' interactions were used, as well as on teacher actions. The findings indicate that teachers' actions and questions influenced students' interactions, but mainly their reasoning, and particularly the primary agent's reasoning. Moreover, students who were engaged in interactional patterns called bi-directional and one-directional did not change their ways of interacting after a teacher interaction. Thus, the teachers' actions did not impact students' collaborative processes and agencies in the same way as their reasoning. This study adds to the field of mathematics education by illuminating the importance of teachers being aware of students' roles when they work together, for facilitating a productive interaction for both students in dyads. The study highlights the importance of further research on teacher actions and teacher awareness for facilitating collaborative situations of bi-directional interactions for students' shared understanding of mathematical concepts and ideas.

Keywords: agency, collaboration, interactional pattern, reasoning, teacher actions

INTRODUCTION

A classroom environment focusing on students' interactions and problem-solving is important for building mathematical understanding (e.g., Hufferd-Ackles et al., 2004; Mueller et al., 2012; NCTM, 2014; Stockero et al., 2019). A vast amount of research highlights the importance of students' construction of their own mathematical knowledge for better understanding important mathematical ideas (e.g., Lithner, 2017; Mueller et al., 2012; Schoenfeld, 2013; Stockero et al., 2019). We argue that there are, at least, three central interactional elements important for making students' interactions productive for learning and understanding mathematics. These aspects are *collaborative processes, mathematical reasoning,* and *exercised agency.* This article uses the term *interactional patterns* to describe the intertwinement of these three aspects, and builds on findings from Hansen (2021), which studied students' interactional patterns with the same students as presented here. She found that whether students' interactions are productive or unproductive is connected to how students participate: choosing to engage or refrain from engaging through different types of agencies, including different ways of interacting in those roles. One implication of the study (Hansen, 2021) was the need for a better understanding of teacher actions in light of the findings. This is the aim of the present article, to increase our knowledge on ways teachers act to promote students' productive interactions through guidance and monitoring of mathematical reasoning, collaboration, and distribution of agency.

To promote opportunities for learning, the expected role of the teacher is no longer to be a “dispenser of knowledge”, but to promote a learning environment where students actively engage in problem-solving and construction of their own understandings (Stein et al., 2008). Therefore, classroom activities should provide opportunities for sharing different thoughts, allowing students to respond to, negotiate, and build on each other’s ideas (Norqvist et al., 2019). When students attempt to work together, teachers periodically check in on students’ work or students ask for guidance. In those situations, teachers can engage with students’ interactions, such as their mathematical justifications and suggestions, or with their lack of ideas that halts the progress in their problem-solving.

The literature provides many approaches to studying teachers’ ways of acting to encourage students to engage with mathematics (e.g., Alrø & Skovsmose, 2004; Boaler & Brodie, 2004; Stein et al., 2008). Some approaches emphasize entire practices (Wood, 1998), whole-classroom discussions (Stein et al., 2008), small-group discussions (Webb, 2009), or teachers’ questions more specifically (Boaler & Brodie, 2004). Altogether, there exists a diversity of models with the purpose of gaining insights into teachers’ actions for facilitating mathematics discussions with both the whole classroom, smaller groups, and individual students. Several studies have developed analytical tools to categorize teaching practices. For instance, studies of teacher support in whole-classroom discussions (Staples, 2007; Stein et al., 2008), communicative features in teacher-student dialogues (Alrø & Skovsmose, 2004), or teacher actions in teacher-student interactions (Drageset, 2014).

Our case study involves three Norwegian mathematics teachers’ interactions with dyads of first-year upper secondary students solving a linear function problem in their classrooms. Students discussed and used a dynamic software program, GeoGebra, in their problem-solving. Despite the growing knowledge about teacher actions for encouraging students to engage with mathematics, less is known about the connections between teacher actions and the three interactional aspects emphasized here: reasoning, collaboration, and agency. This article contributes to this area by studying in depth specific teacher actions and how those influence students’ interactional patterns in situations where student pairs attempt to collaborate and reason mathematically, and students’ interactions shape the teacher-student interaction.

Applying a fine-grained model to understand teacher actions in a teacher-student conversation, can give detailed insights to how teachers can facilitate students’ reasoning and argumentation, as well as their collaboration and agency in those situations. In this study we have described teacher actions as suggested by Drageset (2014), outlined in the theoretical framework section. This article aims to give details of teachers’ actions in teacher-student communication connected to students’ interactions, focusing on collaborative processes, mathematical reasoning, and the students’ agency related to linear functions. With this aim, and within the frame of this case, we ask the following research question: *What are the opportunities and limitations of teacher actions for the productivity of students’ interactional patterns?*

Students’ Interactional Patterns

Discussing mathematical ideas with a peer or with a teacher provides opportunities to defend and explain one’s own ideas as well as ask questions regarding another point of view. Thus, sharing ideas and engaging in mathematical thinking may allow students to reason mathematically using arguments to justify ideas. Mathematical argumentation can be regarded as a prerequisite for learning mathematics and simultaneously as an outcome of students’ math-talk (Krummheuer, 2007). Moreover, Krummheuer (2007) highlights students’ arguments in their reasoning as an important interactional aspect that is important as a foundation for both learning mathematics and interacting with a peer in creating a mathematical classroom community. We find support in the perspective on students’ talk as “interactional accomplishments and not as logical arguments” where the focus is on “what the participants take as acceptable, individually and collectively, and not on whether an argument might be considered mathematically valid” (Yackel, 2001, p. 6).

In line with viewing mathematical argumentation as an interactional accomplishment, Lithner (2017) defines mathematical reasoning as not being restricted to only formal or logical proof. In Lithner’s (2017) perspective on mathematical reasoning, students’ building of arguments and exploration of mathematical connections concerns meaningful sequences of thoughts for the individual student. It does not matter whether the reasoning is simple or complex, correct or incorrect, nor the level of competence the student exhibits, as long as the student provides evidence to support the idea (Lithner, 2017).

Therefore, we see mathematical reasoning as “the explicit act of justifying choices and conclusions by mathematical arguments” (Boesen et al., 2014, p. 75). The explicit act concerns reasoning as “the line of thought adopted to produce assertions and reach conclusions in task solving” (Lithner, 2017, p. 939). The latter statement about reasoning comes from a rather recently empirically developed framework of mathematical reasoning (Lithner, 2008, 2017). Students can take different reasoning paths to reach conclusions in problem-solving. Two different paths are identified by Lithner (2008) as two major types of reasoning: creative reasoning and imitative reasoning. The framework outlining these types of mathematical reasoning is called *creative mathematically founded reasoning*[1] (CMR). For students to exercise creative reasoning, their reasoning ought to fulfill three criteria:

1. *Creativity*: creating or re-creating a new solution method,
2. *Plausibility*: arguments supporting strategy choices and implementation, which explain why the conclusion is true, and
3. *Anchoring*: arguments based on mathematical concepts or relationships, which are intrinsic mathematical properties.

Together, these three aspects describe students’ creation of solution methods called creative reasoning. Imitative reasoning, on the other hand, implies students’ copying of procedures or recalling of facts.

[1] In line with Lithner (2008, 2017) and his colleagues studying creative mathematically founded reasoning, this study uses the wording creative reasoning or acronym CMR for linguistically simplicity.

A student group or a pair are frequently in classroom activities encouraged to work together. In literature this is referred to as collaboration or cooperation. We use the word collaboration, and this can simply put be two or more students in joint activity attempting to solve a problem or produce an agreed upon outcome. There are many definitions of collaboration with slightly different wordings and parameters to be fulfilled to identify an interaction as collaborative. Roschelle and Teasley (1995) suggest that "collaboration is a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (p. 70). In line with the same or similar definitions on collaboration used in other studies (e.g., Baker, 2015; Lai et al., 2017; OECD, 2013), this study views collaboration as defined by Roschelle and Teasley (1995), with focus on the collaborative processes enacted to "construct and maintain a shared conception" of a mathematical problem. Roschelle and Teasley (1995) state that students must build a shared conception by introducing and accepting knowledge, monitoring the on-going activity, and repair conflicting interpretations. We see the building, monitoring and repairing as three collaborative processes for collaboratively pooling knowledge together. Kuhn (2015) says that "it is essential to understand the underlying mechanisms" of learning when working together (p. 47). Aspects of students' interactions may therefore give important insights on underlying processes causing productive or unproductive collaboration. A productive collaboration involves engaging and negotiating with others, where all students are involved in constructing the arguments, and where statements and suggestions are challenged, counter-challenged and justified, and decisions are jointly made (Mercer, 2004; Powell, 2006).

Ways of participation when attempting to collaborate are exercised differently depending on several factors. For instance, relations to a collaborating peer and a given topic to solve may impact the likeliness of a fruitful conversation. Therefore, studying students' participation, seen as exercised agency (Gresalfi et al., 2009), in collaborative processes involving mathematical reasoning is another central social aspect of peer interactions. Hansen (2021) discusses students' roles in pairwise collaboration. She found that students were in a *bi-directional interaction* when mutually attempting to understand one another and when both were driving forces of the problem-solving process. Typical for a pair in a bi-directional interaction was a mutual effort where students co-constructed reasoning sequences with such shared agency (Hansen, 2021). Students showed mutuality in all aspects of their interaction: both chose to engage and thus participated with shared agency, they were equally driving the problem-solving process, thus both made plausible and anchored mathematical arguments, and engaged in turn-taking conversations in all stages of collaborative processes. In contrast, if students exercised different roles in a problem-solving process where the final outcome was expressed repeatedly by one of the students, they were in a *one-directional interaction*. In their interaction, both students were engaged, but they expressed agency differently. For instance, the co-working student's role was often to understand suggestions or explanations made by the primary agent. Moreover, questions were expressed by the student who exercised secondary agency and such input was either assimilated into the final outcome of the reasoning by the primary agent, or it was considered and refined or neglected by the primary agent. Consequentially, the primary agent was the main producer of plausible and anchored mathematical reasoning, whereas the secondary agent observed, questioned, or accepted the primary agent's ideas. Cobb (1995) says that a relevant distinction for learning opportunities involve students' explanations noticed as univocal and multivocal, which respectively relates to the concepts of one- and two-directional interaction. When the two interaction types are compared, Cobb (1995) emphasizes that univocal interactions rarely give rise to learning opportunities for either student, whereas multivocal interactions usually are more productive. Thus, learning opportunities are found in productive interactions such as two-directional interactions (Cobb, 1995; Hansen, 2021): depending on concepts that are "taken as shared" for a basis for the mathematical conversation where both students are equally involved. This aligns well with findings from Mercer (2004) and Powell (2006), who speak of exploratory talk and negatory discourse, respectively, as the interaction forms most productive for mathematical learning.

THEORETICAL FRAMEWORK

Considering the interactional aspect of reasoning connected to teacher actions, Maher et al. (2018) present an extensive list of research on teachers' attendance to students' development of mathematical reasoning for productive classrooms. They contend that "one of the most well-established research findings is that teachers' knowledge of students' reasoning is an essential component for student learning" (p. 3). However, the review does not reveal details of what teachers do when attending to students' reasoning. Nor any outline of teacher actions for promoting students' reasoning through teacher-student interaction. Likewise, Ayalon and Even (2016) point to a variety of roles a teacher has in promoting student argumentation, and highlight the importance of prompts, encouragements, and explaining for promoting students' argumentation. The existing advice on teacher guidance promoting students' engagement in CMR is to let students attempt their own construction of their own solutions. To support students' attempt, whether they fail or they to some degree succeed, the suggested teacher actions is general as well: diagnose students' difficulties with the particular task, and provide adapted feedback, but not a solution method (Lithner, 2017). Although there exists extensive research on the importance of students' mathematical reasoning, there is still a challenge to promote teachers' awareness of how classroom situations can facilitate mathematical reasoning (Maher et al., 2018). While it requires knowledge of student reasoning and validation of the quality (Maher et al., 2018), it is important for teachers and teacher educators to know how to act to promote mathematical reasoning.

Further, a teacher's role for students' collaborative work is similar to the role they have for students' reasoning since these aspects are tightly interwoven (Granberg & Olsson, 2015). Teachers' guidance for collaborative work is shaped by students' mathematical ideas and contributions, and their ways of participating (Staples, 2007). However, to successfully engage students in collaborative learning, Staples (2007) suggests that "this kind of teaching requires a deep understanding of mathematics, students' thinking, curricular materials, as well as how students reason and potentially develop proficiency with a mathematical domain and its practices" (p. 212). She outlines teachers' roles and specific teacher strategies to organize a whole-class collaborative inquiry in a model of three main components: supporting students in making contributions; establishing and monitoring a common ground; and guiding the mathematics. The findings emphasize that eliciting, scaffolding, and creating contributions, are important teacher actions in this setting.

In Howe et al.'s (2007) study on the role of the teacher for students' collaborative outcome, they highlight a teacher's optimal intervention through guidance and monitoring, and task design. Both aspects should facilitate proposal and explanation of ideas, and teachers are suggested to be relatively non-directive. Findings of another study, van de Pol et al. (2018), show that students in small groups using teacher support are likely to formulate answers that are more accurate. Moreover, useful teacher support was characterized as being more extensive initially and reducing over time. This is explained as support "provided at the right time when students need support, which can make further processing of new information easier for the students" (van de Pol et al., 2018, p. 4). Untimely support hindered students' abilities to make sense of teachers' feedback, but the use of teacher support proved beneficial to students' learning (van de Pol et al., 2018). Thus, van de Pol et al. (2018) stress the importance of teachers' awareness of how to use support for students in their problem-solving and for processes in group work.

The above-mentioned research contributes to our knowledge on teachers' role in students' collaborative work important for whole-classroom inquiry (Staples, 2007), and for students' collaborative *outcome* (Howe et al., 2007). However, more knowledge is needed on students' collaborative *processes* (our focus) in mathematics classroom (Seidouvy & Schindler, 2019), and there is a need for detailed insights on teacher's role for promoting students' collaborative processes in small-group interactions.

Facilitating reasoning and argumentation in mathematics classrooms is challenging, and there is a need for better understanding how to facilitate these aspects (e.g., Ayalon & Hershkowitz, 2018; Maher et al., 2018; van de Pol et al., 2018). How students share their thinking with one another, such as their reasoning processes (Lithner, 2008), and how they act or refrain from acting in the conversation–their *agency* (Gresalfi et al., 2009; Mueller et al., 2012), are such central aspects of the participants' interaction.

For students to exercise agency (the third interactional aspect in this study) it is not enough to ask students to work collaboratively on mathematical tasks for agency to automatically occur (Mueller et al., 2012). Students are afforded the agency to "author mathematical ideas" in cases where teachers distribute shared authority between students and teacher (Langer-Osuna et al., 2020). Teacher actions for sharing authority, so that students can exercise agency, is to offer students opportunities to address mathematics problems, and holding students accountable to their strategies, solutions, and ideas (Bell & Pape, 2012; Hamm & Perry, 2002 as referred to in Langer-Osuna, 2018). Classroom situations where students are afforded shared agency has the potential for conceptual agency (Cobb et al., 2009), which means students' opportunities for constructing their own meaning and methods (Mueller et al., 2012). Moreover, if students choose problem solving paths and connect mathematical ideas, a teacher is more likely to support students' mathematical learning through shared agency (Cobb et al., 2009). Classroom situations where teachers exercise authority, students are only afforded to exercise disciplinary agency (Mueller et al., 2012). Disciplinary agency is a concept posed by Pickering (1995), a complementary concept to conceptual agency, and explained as "…utilizing established procedures" (Mueller et al., 2012, p. 374). Consequentially, in teacher-student interactions with disciplinary agency, a teacher is responsible for determining the validity of student responses (Cobb et al., 2009).

A detailed framework of teacher actions for the purpose of studying in depth specific teacher actions and how those influence students' interactional patterns in situations of collaboration, reasoning, and exercised agency, however, not linked specifically to the interactional patterns in focus here, was developed by Drageset (2014). The framework, called *redirecting, progressing,* and *focusing actions*, is both empirically and theoretically built (Drageset, 2014). The three main categories, redirecting, progressing, and focusing, elucidate tools and techniques teachers use to make students' thoughts and strategies visible, help students progress in their problem-solving, or redirect students in an alternative direction (Drageset, 2014). The teacher interactions may facilitate different types of student responses (Drageset, 2015, 2019). The main teacher-action categories entail 13 sub-categories built on concepts from theories about mathematical discourse grounded in perspectives on student-centered versus teacher-centered classrooms.

In teacher-student interactions each utterance and every turn of speech and action depend upon the previous turn. Taking turns is a social practice and an important structure of a conversation (Sidnell, 2010). However, Drageset (2019) says that "looking at single comments, or turns, yield a very limited scope" (p. 2). Therefore, it is important to look at the interplay of turns of speech and actions together in conversation sequences. If the teacher actions components are detailed it can provide better understanding on how different turns of teacher actions influence students' interactions. Thus, different teacher actions influence students' interactions when they collaborate, discuss, reason mathematically, and take ownership of a problem. Although teacher actions influence students' collaborative work, teachers' actions are also shaped from interacting with students and from their ways of participating (Staples, 2007). This complex relation needs to be addressed by a fine-grained analytical model to give detailed insights on teacher's role for promoting students' collaborative interactions. This is possible to investigate by the framework of Drageset (2014), since it separates teacher's actions from student's talk and actions. Giving attention to specific students' interactional aspects, as emphasized above, and specific teacher actions provide opportunities to explore how teacher actions are related to students' interactional patterns.

Two overarching categories, funneling and focusing (Wood, 1998), organize the areas of teacher actions in the framework by Drageset (2014). If a teacher is *funneling* students' thinking, it means that "the student's thinking is focused on trying to figure out the response the teacher wants instead of thinking mathematically himself" (Wood, 1998, p. 172). Thus, mainly the teacher is doing the intellectual work. Redirecting and progressing actions are primarily categories of funneling actions where the teacher is the intellectual authority. Drageset (2014) explains *redirecting actions* as corrections exhibited implicitly or explicitly (Alrø & Skovsmose, 2002). Moreover, redirecting actions are categorized as a teacher's attempt to challenge the students (Drageset, 2014), which means "questioning already established knowledge" (Alrø & Skovsmose, 2004, p. 55).

Alternatively, to funneling is what Wood (1998) calls *focusing*, where the intellectual responsibility is with the students, hence to a larger degree than funneling actions are teacher actions for promoting productive interactions as reviewed in the first part of this section, zooming in on reasoning (Ayalon & Even, 2016; Lithner, 2017; Maher et al., 2018), collaboration (Howe et al., 2007; Staples, 2007; van de Pol et al., 2018), and agency (Langer-Osuna, 2018; Mueller et al., 2012).

Thus, we see the three main categories from Drageset (2014), redirecting, progressing, and focusing, as a useful tool to investigate opportunities and limitations of teacher actions for the productivity of students' interactional patterns. Using this framework, with its 13 subcategories, interpreted in light of theories on teacher actions for the emphasized interactional aspects, allows us to dig deeply and in detail into the question of how teacher's guidance for students' collaborative work and reasoning is shaped by students' mathematical ideas and social contributions to the conversations.

METHODS

This study seeks, through an instrumental case study (Stake, 2003), to generate more knowledge concerning the role of teacher actions in students' interactional patterns. The article aims at developing more inclusive theories on the issue at hand (Layder, 1998), based on the details and nuances from the particular case, which is explained in detail through the coding procedures.

Participants and Data Collection

The data was collected in 2017 at an upper secondary school in the eastern part of Norway. In this study, students had recently transitioned from lower secondary school (10th grade) to upper secondary school (11th grade). Children start grade 1 at the age of six and upper secondary starts at grade 11. The larger study, which this case study is a part of, followed three first-year theoretical mathematics classes (69 students in total, from 15-16 years old) and their three teachers over a time span of five months. Whole-class discussions were studied, as well as the collaboration and dialogue between pairs of students (six pairs in particular; two pairs in each class). In both instances, the role of the teacher in the interactions was emphasized. This article focuses on the dialogue between four of the six student-pairs—Emma and Hannah, Philip and Noah, Olivia, and Oscar, and Leah and Isaac—and their teachers. The teachers—Jacob, Lucas, and Sophie—were all engaged in the study, and contributed to the planning and evaluation of their teaching in light of the study's aim (Amiel & Reeves, 2008). The teachers were encouraged to help their students think together and to hold back on their guiding, and both prior to and after the planned lessons, the teachers and researcher discussed how to assist and interact with the students in order to encourage mathematical reasoning and collaborative work.

The study used purposive sampling (Bryman, 2016) in the selection of the particular school and the three teachers. The teachers and the school were chosen based on acquaintances and willingness to participate in the study. The six dyads for the in-depth study were chosen based on conversations with the teachers, and two aspects were particularly emphasized: (1) a high level of reasoning competence based on a test combined with average-to-high score levels for functional understanding and (2) the likeliness of a student-pair to be verbal and share thoughts with one another. Thus, the objects of study were students who willingly talked about mathematics with one another, were likely to reason about functions, and already had some knowledge about functions. The student pairs had previously to this session been presented with relevant concepts for talking about linear functions, such as the slope number and constant. From preceding school years, the students should have become familiar with the coordinate system and straight lines, but the topic of linear function was new to them.

The students' collaboration and the teachers' interactions with the pairs were video recorded and observed by the researchers. The teachers encouraged students to talk with one another, attempting justification of their thoughts and ideas, and use of relevant mathematical concepts. A microphone was placed on a desk in front of the students and connected to a video camera recording their talk and gestures. The laptop screen is not video recorded, but the recordings show when students used it to draw, write or point to the screen. Students' solving of the linear function problem lasted for approximately 45 minutes in each of the three classrooms.

The Linear Function Problem

A productive struggle with important mathematical ideas is central for effective mathematical problem-solving (Lithner, 2017). Lithner (2017) points out that "the focus is on the *particular type of struggle* when students construct task solutions instead of imitating them" (p. 938, italics added for emphasis by authors). The particular type of struggle should emphasize mathematical reasoning and non-routine solving of tasks, where the struggle should be more like a challenge to solve, rather than an obstacle (e.g., Hiebert & Grouws, 2007; Lithner, 2017; Stein et al., 2008). We see an obstacle as a too-difficult problem, which may be a stumbling block for students, whereas a challenge entails a better-adjusted problem for students to solve. Therefore, a fundamental aspect of a teaching design is choosing suitable tasks for facilitating a mathematical discourse to potentially strengthen the teacher-student and student-student interactions.

When designing a problem in this study, we wanted students to be presented with a challenge to connect function representations in order to not view linear function representations as "'topics' to be learned in isolation of the others" (Thompson, 1994, p. 24). Moreover, the linear function problem ought to facilitate an opportunity to discuss and share their own ideas, strategies, and knowledge about functions.

A function can be referred to in different ways: as a dynamic mechanism that performs transformation through an input and output, as the relationship between two variables, and as a rule of correspondence between two sets (Malik, 1980). Functions can appear as a graph, a verbal description, a table, or an algebraic expression. Students need to connect fragments of function representations in order to build a comprehensive understanding of the function concept (Best & Bikner-Ahsbahs, 2017). This is

- Create a straight line $y = mx + c$
- Create another straight line in a way that the corresponding graphs are perpendicular.
- Formulate a rule for when two straight lines are perpendicular.
- Test the rule for other straight lines.

Figure 1. The function task (reformulated from Olsson, 2018).

particularly important in upper secondary school, as the function concept has a central role in organizing and connecting many important mathematical ideas (Michelsen, 2006). An important aspect of understanding the function concept is students' underlying algebraic knowledge of variables (Leinhardt et al., 1990; Lepak et al., 2018). For instance, students are more often familiar with letters as unknowns than with variables. When working with linear functions, such as $y=mx+b$, it is important to interpret the algebraic expression as a "kind of relationship between the letters, as their value changes in a systematic manner" (Küchemann, 1978, p. 26).

Exploring linear functions' parameters m and b through focusing on translations between different representations (e.g., algebraic, graphical, tables) (Akkoç & Tall, 2005; Leinhardt et al., 1990) can be fruitful for in-depth learning. In this article, the function problem in **Figure 1** was discussed by the student pairs and contains a focus on translation between algebraic and graphical representation as well as the importance of variables for understanding the function concept. Attempting to formulate a rule for a pair of perpendicular lines supports a generalization process where students make use of their findings via patterns for making a general relationship between two linear functions—hence, the rule.

Students worked in dyads on one laptop and were encouraged to use the dynamic software program GeoGebra as a tool in their problem-solving. GeoGebra provides tools to create, manipulate, and control mathematical content for students to investigate mathematical relations (Granberg & Olsson, 2015; Hall & Chamblee, 2013). Thus, the given linear function problem presents an opportunity to investigate varying parameters of the slope number and the constant. Changing an algebraic expression may cause GeoGebra to dynamically change the related graphical representation (Preiner, 2008). Thus, students get rapid feedback on performed actions, inputs, and changes in GeoGebra. However, GeoGebra does not interpret the generated information. Therefore, students would have to make sense of dynamic changes between different linear representations. Olsson (2018) found that students successfully solving a task with GeoGebra used given feedback extensively and engaged in reasoning.

Data Analysis

A deductive analytical strategy was used (Yin, 2014), which involved "identifying or creating a suitable video corpus and systematically sampling from it to examine specific research questions" (Derry et al., 2010, p. 10). The videos were watched several times before critical events (Powell et al., 2003) were identified. Events were seen as critical if (1) the dyads had some form of interaction, characterized by the three emphasized aspects and (2) there were teacher interactions tied to these sequences. Four dyads fulfilled these criteria, and each pair's work with the given problem, as well as the interactions they had with the teacher in the process, was transcribed and coded. The teachers' interactions with the dyads were coded using the coding scheme presented in **Table 1**. These codes were based on Drageset's (2014, 2019) framework for redirecting, progressing, and focusing teacher actions, but were slightly revised through an iterative coding procedure between the data set and the theoretically based framework. **Table 1** shows which codes are based on the original framework and which ones are added or revised. The following paragraphs outlines Drageset's (2014, 2019) framework starting with the funneling actions, *redirecting* and *progressing*, which is followed by focusing actions, called *focusing*, and ending with **Table 1**–an overview of the teacher action framework.

A typical redirecting teacher comment is to discard a student's suggestion or comment, called *put aside* in the coding. With such an action, a teacher is not providing help with a presumably pressing question or challenge. The second redirecting action, *advising a new strategy*, means that a teacher's comment is suggesting an alternative approach or way of thinking to solve a problem. The last redirecting action is *correcting questions*, where a teacher's question aims to move a student's focus over to another approach. In summary, redirecting actions are a teacher's strategy for shifting attention to something else.

In line with the funneling manner of actions, a teacher may aim to move a problem-solving process forward. Drageset (2014) explains four actions for attempting to guide students' *progress*. *Open progress details* are teachers' open questions with possibilities for several answers concerning the progress for solving a problem. This action includes questions on how to do, how to think, how to solve, and how to generalize patterns. Thus, an open progress action is aiming at "moving the process forward, but without pointing out the direction" (Drageset, 2014, p. 16). *Closed progress details*, on the other hand, concern how (many, large, much, big, how to do it) and what (it becomes, shall we write, is, to do). Questions typically request details needed to move the process forward, connected to steps in a procedure (Lithner, 2008). These details can be process answers (one step at a time) or details about how the process should go on to reach the answer. Another aspect of teacher action attempting to move the process forward is a *simplification* of the task at hand. To simplify a task, a teacher may change or add information, tell students how to solve it, or give hints to make a task easier (Wood, 1998). A teacher would typically pull a student toward the solution (the Topaze effect, cited in Brousseau, 2006): "It often seems that this involvement is meant to ensure the progress of the class and sometimes these comments appear to come as a consequence of a halted progress. Many of the simplification comments could also be characterized as hints" (Drageset, 2014, p. 15). The last progress action is *demonstration*. A teacher typically demonstrates how to solve the problem or shows students every step in a procedure. It is primarily a monologue given by a teacher that is occasionally broken if students ask questions or if the teacher asks students whether they understand or agree.

Table 1. Codes for teachers' actions in conversations with student-pairs

Teachers' actions	
Focusing–*Giving attention to students' thoughts and input*	
Requests for students' input	
Enlighten detail	• Student explanation • Focus on details • *Gathering information***
Justification*	• Explaining why • Justifying method and/or answer
Apply to similar problems*	• Asking to use knowledge on similar problems
Request assessment from other students*	• Asking other students to evaluate answer/solution
Pointing out	
Recap*	• Repeating an answer • Adding to an answer
Notice	• Highlighting details in a dialogue • Reminding students of new or previous information • *Explanation focus** on a student's question
Progressing–*Taking action to move the process forward*	
Open progress details	• Open questions with several answers • *Encourage testing of strategy***
Closed progress details	• Closed questions with one answer • Request for details • Request about procedure/steps
Simplification	• Adding information • Hints • Telling students what to do • *Explanation progress***
Demonstration	• Showing a procedure • Doing several steps of a procedure • *Evaluation of solution/strategy* **
Redirecting–*Bringing attention to something else*	
Put aside	• Discarding a student's suggestion/comment • *Interrupting***
Advising a new strategy	• Suggesting another approach
Correcting questions	• Question changing approach

Note. *Not used in the analysis; **Added actions as a result of the analysis process

Further, Drageset (2014) divides focusing teacher actions (Wood, 1998) in two categories: *request for student's input* and *pointing out*. A teacher may ask for a student's input through *enlighten details*, *justification*, *applying to similar problems*, or *requesting assessment from other students*. These concepts are related to what Franke et al. (2007) express as access to student thinking. If teachers *enlighten details*, it is a request for students to explain what something means or how something happens. Typically, details are brought into focus. If a teacher asks for a student's *justification* (Cengiz et al., 2011), it is a request for a more thorough explanation, often to validate why the answer found, or the method used, is right.

Another approach focusing on students' thinking is when a teacher is *pointing out* something. Sub-categories of pointing out are *recap* and *notice*. The purpose of the *recap* is to merge information to clarify important elements in a student's explanations. Also, a teacher can repeat a student's answer with the purpose of confirming or ending dialogue, or sometimes adding information to an answer. A *noticing* action (Cengiz et al., 2011) is when a teacher highlights particular aspects, concepts, or details he wants to make a student aware of. Other aspects of noticing are reminding students of new or previous information and adding information.

All teacher utterances in the interactions between the four student-pairs and their teacher were coded, and from every interaction, narratives were written about the students' situation before a teacher interacted with the students, as well as a characterization of the teacher-student interaction. After the teacher left the conversation, written narratives described how the students interacted in the moments that followed. Excerpts presented in the results section represent typical teacher actions for the given teacher-student interactions, combined with the typical way students interacted with each other and their teacher.

Students' Interactions Prior to Teacher Interaction

As outlined in the introduction, four student-pairs gave insights into collaborative interactions concerning their reasoning and processes for creating and maintaining collaboration connected to their agency (Hansen, 2021). Two pairs demonstrated *bi-directional interactions*, whereas the other two pairs interacted in a *one-directional manner* when solving the function problem (**Figure 1**). This section characterizes the interactions found in the four student-pairs without the teacher present in their conversation, based on Hansen (2021). Results section follows the student-pairs' conversations with their teacher presented with a brief summary of the course of events before and after the conversations with their teacher.

In the bi-directional interactions, there were mutual attempts to solve the linear function problem in both student-pairs. The student-pairs, Emma and Hannah, Philip and Noah, engaged in CMR, which was observed particularly through the *plausibility* and the *anchoring* of their arguments (Lithner, 2017). The students made arguments about the linear function that were acceptable and plausible not only to themselves, but to each other. Therefore, the arguments used in creating a shared understanding were an interactional accomplishment (Yackel, 2001). The students who were in a bi-directional interaction mutually justified choices (e.g., choosing different slope numbers or constant) and conclusions (e.g., connecting their general algebraic expression of the rule and the graphical explanation of the rule) by using mathematical arguments (Boesen et al., 2014). Through their turn-taking conversations (Sidnell, 2010), the four students were prompted by their interaction to anchor explanations in mathematical properties when ideas or thoughts were not consistent with a peer or feedback from GeoGebra. These situations were seen in the collaborative processes of monitoring and repairing, which are central to maintain a shared conception of the problem.

Like the two pairs engaged in bi-directional interaction, there were many mutual attempts to solve the linear function problem also among the student-pairs interacting in a one-directional manner. Olivia and Oscar, and Leah and Isaac, were also engaged in CMR (Lithner, 2017), although in a different way than Emma and Hanna, and Philip and Noah. In contrast to collectively co-constructing reasoning sequences, the students interacting one-directionally had a primary agent expressing the final outcome of a reasoning sequence. As seen in Hansen (2021), Isaac and Oscar were the primary agents, whereas Leah and Olivia were the secondary agents. One similarity between the two student-pairs was that both primary agents, Isaac and Oscar, made suggestions for solving the function problem, whereas Olivia and Leah, secondary agents, asked their peer about details concerning ideas and inputs. Also, both pairs focused on the slope numbers of linear functions related to the perpendicularity, focusing on how the parameter *m* was changing in a systematic manner (Küchemann, 1978), which is important underlying algebraic knowledge of variables that is important for building a comprehensive function concept (Lepak et al., 2018). Both pairs used GeoGebra, where they easily manipulated and adjusted different function representations, and they extensively used and attempted to make sense of its feedback.

RESULTS

This section provides the results from each of the four student pairs' interaction with their teacher. The excerpts presented within each subsection are typical teacher actions for the given teacher-student interactions, combined with the typical way students interacted with each other.

Jacob in Interaction with Hannah and Emma–Progressing Actions

Prior to the interaction with the teacher, Hannah and Emma had made a shared attempt to understand the meaning of perpendicular lines by exploring the connection between the algebraic and the graphical expression of the linear functions. Right after their turn-taking conversation, Jacob initiated a teacher-student interaction by getting in contact with the student-pair and asking the student-pair what they had tried out so far in the problem-solving process.

Excerpt 1

(14) Emma	When you say perpendicular, right, should it be 90 degrees?
(15) Jacob	Yes. Because when you measure an angle, any angle, then you should've... 90 degrees, there, yes (pointing to the screen).
(16) Hannah	90 degrees.
(17) Emma	Umm? For any?
(18) Hannah	If you measure here, or here, or here, or here (using the laptop).
(19) Emma	Oh, like that.
(20) Jacob	So, you tried some different things.
(21) Hannah	Yes.
(22) Jacob	What've you done so far?
(23) Emma	Well, I misunderstood at first. I thought it was a type of task like this (using her pencil to illustrate two crossing lines and pointing to a previous task on the piece of paper in front of her).
(24) Jacob	Like, this line here, and this one, aren't those perpendicular? Because here you see... (pointing at the drawn lines on the paper).
(25) Emma	Is it just to take... since we got +4... is it just to take -4x (both students look down at their own papers with graphs in a coordinate system)?
(26) Jacob	Yes, can you see what'll happen if you take -4x?
(27) Emma	Didn't we do that earlier?

(28) Hannah Yes, that might be right.

During the conversation, Jacob seemingly tried to make sense of the students' questions, suggestions, and input. With his actions, our interpretation is that he aimed to move the process forward; thus, it was a *progressing* conversation. For instance, he began an explanation of perpendicularity (15), which was interrupted by another student asking for bathroom permission. Consequentially, Hannah and Emma engaged in a dialogue where Hannah attempted to explain the meaning of perpendicularity (16-19). Furthermore, Emma suggested testing two linear functions with slope numbers 4 and -4 (25). Jacob encouraged testing of the suggested strategy (26). After his question, a short student dialogue followed where they concluded that they had already tested the suggested lines. Jacob stated that their previous attempt did not work and asked an *open progress detail* question, "What's wrong with the line in a way?" Emma replied that it was not 90 degrees. Jacob's response was initiating another *progressing* conversation.

Excerpt 2

(29) Jacob Yes. What would you've to do with the formula?

(30) Hannah It can't be 4*x*, or it must be less of each.

(31) Jacob Yes. What is the number doing? The number in front of x.

(32) Emma Slope number.

(33) Jacob Yes, so here it's the... So, if we try something, what would you change it to, to come closer to the solution?

(34) Emma 2 or something? -2.

(35) Jacob Yes (Hannah writes on the laptop).

(36) Emma Closer at least.

(37) Jacob Yes. You should try to make it even smaller.

(38) Emma -2*x*, then? Because now we're on minus... it's not 90 degrees now either.

(39) Hannah No. Wasn't it what we... Now it goes back. Maybe it's a fraction, then?

Jacob responded with *closed progress details*, hence moving the progress forward by pointing out that *something* should be changed; more specifically, the algebraic representation (29). When Hannah replied that it should be less than 4*x* (30), Jacob guided the conversation's focus to the slope number (31-39). Thus, Jacob used the students' responses to channel a focus on the slope number in the algebraic expression connected to the graphical representation.

When Jacob encouraged them to test a smaller slope number (37), Hannah and Emma continued their dialogue, which Jacob quickly *interrupted* after Hannah's question (39). Such instances happened several times in the conversation above (e.g., 20, 33). In those situations, Jacob guided the student-pair on what seemed to be his chosen solution path. When Jacob interfered, he often made *progressing* actions, such as *open progress details*, *closed progress details*, *evaluation of solution/strategy*, and *encouraging of testing strategy.* In **Excerpt 1** and **Excerpt 2**, Jacob is moving the process forward by channeling the focus to the slope number, which initiate progress for guessing and checking different linear functions in GeoGebra. This consequentially results in less student reasoning and justification for their actions, less collaborative repairing, but a continued shared agency.

After the teacher-student interaction, Hannah and Emma tested for different slope numbers: 1/2, 1/3, and then 1/4. Their inputs resulted in the perpendicular pair of lines: $y=4x+2$ and $y=-1/4x+2$. Hannah and Emma progressed to the third task of formulating a rule for when two linear functions were perpendicular. They tested their rule, but it gave them the wrong result, and they concluded that they should ask for guidance. However, Hannah and Emma did not finish their work, as they did not get any further support in their problem-solving.

Lucas in Interaction with Philip and Noah–Redirecting Actions

Philip and Noah had prior to the conversation with the teacher jointly reflected and agreed upon a shared understanding of an expression of the rule for making linear functions perpendicular to one another. Their teacher, Lucas, was passing by Noah and Philip when Noah signaled a need for teacher support by raising his hand.

Excerpt 3

(13) Noah Umm... This...

(14) Lucas You don't have to test the rule on me (interrupting).

(15) Noah I just...

(16) Lucas You must test the rule (interrupting). Can you use it for different instances?

(17) Noah — But, does the rule look okay? For the second one to be perpendicular with the first...

(18) Lucas — To be picky, this does not look like a rule (interrupting).

(19) — (Noah laughs quietly, but Philip looks serious).

(20) Lucas — Could you read out loud what you have written down here, Philip, inside the frame?

(21) Philip — -1 divided by a is the slope number to the perpendicular line (after reading their rule, both students looked at Lucas).

Noah was barely able to say one word (13) before being *interrupted* by their teacher, Lucas: “You don’t have to test the rule on me” (14). His response is characterized as *redirecting*, and being of an unsupportive character. Noah further attempted to explain their rule, but was cut off a second time (15). Noah tried to ask for teacher feedback on their formula, so he asked, “But, does the rule look okay?” Then he hurried on to continue his explanation of the rule (17), but was *interrupted* a third time (18). Lucas asked whether they could use the rule (16) and for more precisely formulated mathematics (18). Thus, his teacher actions were categorized as *redirecting* and *progressing*.

Both students seemed to think they had found a valid expression for the formula, but they were not given the opportunity to share their findings and questions with their teacher, from whom they seemed to want confirmation and/or guidance. In this situation, Lucas did not show interest in the students’ thoughts, reasoning, or collaboration. Lucas’s tone was harsh, and his interaction cannot be described as welcoming. The student-pair did not seem to be intimidated. However, Philip looked a little disappointed (19).

After the teacher-student interaction, Noah and Philip made an explanation for their rule for perpendicular lines. Noah uttered that the slope number had to be -1 divided by the slope number from line *a*. Philip challenged Noah to write the rule like a formula. They discussed different options and agreed to call the two lines for 1 and 2 in order to make it a “bit more professional mathematics language,” as they put it. They developed their formula to $a_2 = \frac{-1}{a_1}$, where a_2 was the slope number of the second line and a_1 was the slope number of the original line. Philip and Noah had just found this way of writing it when Lucas came by a second time and observed for 10 seconds before they noticed him. When they saw him, Noah began explaining the formula by writing it down on the paper. Lucas responded by saying, “I leave this to you,” then he left the conversation.

In **Excerpt 3**, Lucas noticed students’ findings, which they had written down on a paper. With redirecting teacher action Lucas did not ask for students’ explanations for their findings, but prompted a better mathematical expression of the rule. In the continuing student-student interaction Philip and Noah responded by adjusting their findings into a more precise expressed formula. They continued to anchor their suggestions in mathematical properties through collaborative processes and with shared agency.

Sophie in Interaction with Oscar and Olivia–Focusing Actions

Oscar had prior to the conversation with the teacher suggested a rule for making a pair of perpendicular lines, justified in linear functions with slope numbers with opposite signs. Olivia supported his input and explanation. Olivia and Oscar tested, in line with their formulated rule, another pair of linear functions with slope numbers 2 and -2. Oscar observed the input in GeoGebra and said that “this is absolutely not perpendicular. Then we need a new rule.” Olivia gently asked, while crossing two fingers, indicating perpendicular lines, “Perpendicular... It’s 90 degrees?” Oscar did not notice her question and suggested another line, and that they should try -0.5*x*. Olivia asked again if perpendicular meant 90 degrees. He said yes and evaluated Olivia’s input of the new linear function with the slope number -0.5, which he thought looked “very perpendicular.” They used GeoGebra to check if it was perpendicular, which the tool confirmed.

Their teacher, Sophie, approached the student-pair and asked if they had any success in making perpendicular functions. Both students answered, and Olivia said that they had made two pairs.

Excerpt 4

(8) Sophie — You tried something here (pointing at the laptop screen). When you tried that, what did you think?

(9) Oscar — No. But, first, at least I thought that 2x and therefore -2x would become perpendicular. That didn’t work. It became too steep... or too gentle (makes a hand movement).

(10) Sophie — Why did you think... How did you find that 2x, and therefore -2x, would make a perpendicular line?

(11) Oscar — Umm, because we tried with only x, and then it worked. But it didn’t with 2x. So... Hmm, I don’t know.

(12) Sophie — Yes. You might consider making another pair of lines perpendicular to one another... to look for any relationship. Because now, you have two, right, these two are perpendicular to one another, and these two are perpendicular to one another. So, try to make another pair being perpendicular to one another.

Sophie initiated a conversation with Olivia and Oscar by *gathering information* on what progress they had made and how they had started their problem-solving process. Sophie continued to request the students’ thoughts and input when pointing out performed actions in GeoGebra (8). Oscar replied with what *he* had thought would make a perpendicular pair of linear functions and why. He evaluated the graphical representation and concluded it was either too steep or too gentle (9). Again, Sophie *requested* the students' thoughts and brought details into focus (10). Thus, Sophie *enlightened details* (8, 10) emphasizing the students' reasoning. Oscar explained why they tested 2 and -2 and justified his answer by connecting it to the first linear pair with slope numbers 1 and -1. Sophie ended the conversation by encouraging the student-pair to find several pairs of perpendicular lines (12). Her teacher actions addressed details already highlighted by Oscar. Thus, Sophie brought the students' ideas to the center of attention, which is characterized as *noticing*, a *focusing* action (12). In **Excerpt 4**, Sophie continued the conversation with attention to the students' thoughts and attempted to promote students' reasoning by focusing on their findings when addressing details already highlighted by Oscar. After the teacher-student interaction, Olivia and Oscar continued their same interactional pattern: Oscar continued making suggestions, which Olivia translated into actions on the laptop.

Lucas in Interaction with Leah and Isaac–Progressing Actions

Prior to the conversation with Lucas, Leah and Isaac had found a perpendicular pair of linear functions and agreed upon the relationship entailing a slope number with the same number, only with different signs: -1 and 1. In the following teacher-student conversation, Leah initiated the conversation, asking their teacher, Lucas, if -1 and 1 were opposite numbers.

Excerpt 5

(10) Lucas	I don't know if it [opposite numbers] is a concept.
(11) Isaac	But we want to formulate that the slope number for this line (pointing to the laptop screen) is x, and this is -x.
(12) Leah	Is the negative number of... umm.
(13) Lucas	Yes, rather that.
(14) Isaac	Is the same slope number, only negative.
(15) Leah	Is the negative number of the slope line... err (frustrated). Yeah, but this slope line (pointing), no, this line has the slope number. And this line (pointing) has the negative of the slope number.
(16) Lucas	Yes, that's better than saying opposite numbers. But how can you be sure that they're perpendicular to each other? Can you make sure about that in GeoGebra? Maybe do that first.

Lucas's initial response, "I don't know if it is a concept" (10), partially answered the student-pair's question, but more importantly, Isaac and Leah continued to explain what they meant by opposite numbers and what they attempted to find (11-15). Leah and Isaac engaged in a dialogue discussing the relationship between the linear functions. This presented an opportunity to address mathematical properties, such as slope number, constant, algebraic expression, graphical representation, and coordinate system. However, Lucas was *progress*-oriented and did not use the students' reasoning. Nevertheless, Lucas provided positive feedback on their use of concepts and said that their formulations were better than "opposite numbers" (13, 16). Then he asked them to use GeoGebra to prove the functions' perpendicularity (16). Lucas acted to move the process forward by asking for *open progress details*: "But how can you be sure that they're perpendicular to each other?" Such a question may have several possible answers and be aiming for progress. In the same line, Lucas also said, "Can you make sure about it in GeoGebra? Maybe do that first." Thus, he *simplified* the question by adding a step for the procedure of investigating perpendicularity between the linear functions.

Further into the unfolding situation, Isaac and Leah attempted to use GeoGebra to investigate the linear pair's perpendicularity. However, they did not succeed at first. Lucas responded and said that

> "I think we can say that [they're perpendicular] about the two lines upon each other. However, I think we have a small challenge... (pointing to the laptop screen). If you add a couple of points on every line, then you can make line segments between them. That might be hard work."

Lucas *simplified* by suggesting making another input using GeoGebra to examine the perpendicularity before he left the conversation. Lucas used Isaac and Leah's reasoning to guide them in their problem-solving process of the function problem. He particularly evaluated their input in GeoGebra. Moreover, Lucas focused on *simplifying* Isaac and Leah's reasoning path by giving hints for validating their findings, probably to pull them in the direction of investigating the connection between the slope numbers.

Isaac and Leah were engaged the entire time during the teacher-student conversation. When Lucas left the conversation, they kept their dialogue going. Isaac and Leah managed to use GeoGebra to confirm that the linear functions were perpendicular to each other. In the situation that followed, Isaac said he wanted to make a formulation for their findings, and Leah said they should test for other pairs of lines before attempting to formulate a rule. However, Isaac suggested a rule anyway, and observed that his assumption was not going to work for every number; hence, it would not work for every slope number. For a couple of minutes, Isaac was testing different linear functions in GeoGebra, and Leah observed his actions. Then, Isaac tested two linear functions with the slope numbers 2 and -0.5. However, they did not evaluate their result, and started testing several other linear functions.

In **Excerpt 5**, Lucas acted with progressing actions aiming for an evolving problem-solving process. His actions focused on (1) students' use of the concept slope number and (2) simplifying students' strategies by adding steps to the solution method. Consequentially Leah and Isaac used GeoGebra to confirm that the linear functions were perpendicular to each other. In the further unfolding events Isaac and Leah wanted to pursue different solution paths. Isaac acted with primary agency and neglected Leah's suggestion, which seemed to exclude Leah from making her thoughts visible. Therefore, Leah is placed in a position as a secondary agent where she observed, evaluated, and seemingly attempted to understand Isaac.

DISCUSSION

Initially we asked: What are the opportunities and limitations of teacher actions for the productivity of students' interactional patterns? In the study, three teachers interacted with the four student-pairs. Sophie, who interacted with Olivia and Oscar, primarily focused on the students' thoughts and reasoning for their suggested ideas and actions. Thus, her teacher actions were categorized as *focusing actions* (Wood, 1998). Sophie did *enlighten details* (Drageset, 2014) through her gathering of information and request for the students' explanations. Sophie requested a response from both students in a pair when asking about their performed actions in GeoGebra or when specifically asking about details brought into the conversation by the students. Sophie contributed to making details in the students' mathematical reasoning explicit, thus interacting with powerful teacher moves (Franke et al., 2007) for promoting a learning environment where students actively engage in problem-solving and construction of their own understandings (Stein et al., 2008). In the teacher-student interaction with Olivia and Oscar, Sophie mainly interacted with the primary agent, Oscar. Consequentially, it is likely that it was Sophie's focusing actions that facilitated reasoning from the primary agent, which was translated by the secondary agent into actions in GeoGebra.

Opportunities and limitations in Sophie's actions in teacher-student interactions are related to the category of focusing teacher actions (Drageset, 2014). Sophie's attention to the students' thoughts and input is a foundational aspect of supporting students in learning mathematics through their own attempts to make sense of mathematics and explore mathematical ideas (Norqvist et al., 2019), such as the linear function concept. Sophie provided timely support (van de Pol et al., 2018) when the student-pairs needed feedback in their problem-solving process. However, she primarily interacted with the primary agent and missed the opportunity to support both students in mathematical reasoning and the development of collaborative processes. In the interactions with Olivia and Oscar, Sophie had the opportunity to engage in other aspects of focusing actions, such as *assessment from other students*, where she could have promoted a collaborative interaction. If Olivia were presented with an opportunity to evaluate an answer or an idea, she could have refused to answer or attempted to contribute to the dialogue. For the latter outcome, it would be crucial for Sophie to both provide timely support, probably more extensive in the beginning and reducing over time, as suggested by van de Pol et al. (2018). However, Howe et al. (2007) say that a teacher should be relatively non-directive with the guidance, therefore Sophie could start with requesting Olivia to share details of what they had found, which is *closed progress details*. From initiating a student's talk, Sophie could further have progressed with *focusing* action where Olivia would have had the opportunity to *explaining why* and *justifying* an idea, thus, engaging in collaborative processes of building and monitoring and encouraged to use CMR. Moreover, it could facilitate for a change in the pair's agency dynamics, where potentially Olivia could feel encouraged to participate with shared agency.

Sophie was present and engaged in interacting with the students, but she did not encourage the students to explain to each other, which is emphasized as important for developing students' exploration and autonomy (Hufferd-Ackles et al., 2004) and crucial for constructing and maintaining a shared conception (Roschelle & Teasley, 1995). Olivia and Oscar remained in a one-directional interaction throughout the problem-solving process, which can have an impact on their learning potential. There were probably more learning opportunities for Oscar who frequently used CMR, compared to Olivia, who did not. However, that does not mean that Olivia did not learn the mathematics involved, or other skills, such as using GeoGebra as a useful tool or other social aspects of interacting with a peer. It indicates that students with secondary agency is not engaged with anchoring their arguments in mathematical properties, which potentially can lead to learning of the mathematics involved. However, Sophie's request for the students' input and thoughts could have been a fruitful start for encouraging students to interact with each other, and it would have been interesting to observe Sophie interacting with other student-pairs with different group dynamics than Oscar and Olivia's. Since a teacher's guidance is shaped by students' mathematical ideas and contributions (Staples, 2007), it is probably the case for who and how a teacher reacts and respond to a group's dynamics, such as Sophie's main interaction with the primary agent Oscar.

Lucas was acting with *progressing actions* (Drageset, 2014) when interacting with Leah and Isaac. The conversation began when Leah asked Lucas about "opposite numbers," indicating slope numbers with different signs, to which Lucas answered, but he refrained from elaborating on the answer. Consequentially it possibly made an opening for Leah and Isaac to engage in a math-talk, which became a short discussion about the relationship between the linear functions. It is likely that Lucas entered the conversation at the right time, and he had the opportunity to support the students' learning in a timely manner (van de Pol et al., 2018), but as the conversation progressed, Lucas missed the opportunity to focus his actions on the students' reasoning. He moved the problem-solving process forward with progressing actions (Drageset, 2014): *open progress details* about validating the linear function's pair perpendicularity, and *simplification* by adding steps in a procedure (Lithner, 2008) for solving a sub-question in GeoGebra. A main focus on progressing action limits the students' opportunities to produce mathematics (Schoenfeld, 2013) through building their own theory sequences of arguments (Lithner, 2017). Lucas commented on and evaluated the students' inputs in GeoGebra; therefore, he missed an opportunity to support the students' own interpretations of GeoGebra's feedback, which is important in order for students to construct shared reasoning pathways (Olsson, 2018).

Similar to Olivia and Oscar, Leah and Isaac interacted in a one-directional way. During the conversation with Lucas, both Leah and Isaac were asked questions and explained their interpretation of “opposite numbers.” Thus, both the secondary and primary agent were engaged, unlike in the interaction between Sophie, Olivia, and Oscar, where Oscar mainly responded to Sophie’s teacher actions. Despite the differences seen in the teacher-student interactions concerning the one-directional pattern, we observed that neither of the teachers’ approaches changed the students’ interactional dynamics concerning their agency in the conversations that followed. We propose that there are more opportunities for learning, in general, for the primary agents if a teacher mainly acts toward them. Consequentially, such actions strengthen the one-directional relationship between the students, and the primary agent is given more authority in the development of the collaborative processes and ownership of the reasoning process.

In the teacher-student interaction between Lucas, Philip, and Noah, we observed *redirecting* and *progressing teacher actions* (Drageset, 2014) where Lucas discarded the students’ suggestions and comments about their problem-solving process and findings. His teacher action was to *put aside* the students’ ideas and channel the focus into something else. In this case, Lucas aimed his focus on the student responses concerning more precisely formulated mathematics. However, Lucas’s way of acting could have been an attempt to *challenge* (Alrø & Skovsmose, 2004) the students’ perception of the rule for making linear functions perpendicular. Philip and Noah mutually engaged in problem-solving, making a shared understanding and having shared agency. We believe that their shared agency was a central component of the perseverance of their productive collaborative interaction.

Although a fruitful result concerning Philip and Noah’s continuation of building their shared understanding was seen, we would not expect such an outcome in general, as Lucas’s tone was harsh and his approach could be perceived as intimidating, thus probably influencing the students’ willingness to approach the teacher or collaboratively explore mathematical properties to solve the linear function problem. It seems that Lucas’s way of interacting with Philip and Noah stood out compared to his interactions with Leah and Isaac. Perhaps Lucas saw potential in Noah and Philip’s mathematical thinking, and he wanted to challenge them to make their thinking even more explicit.

Lucas’s way of interacting with Philip and Noah sheds light on how difficult it is, even for teachers as experienced as he, to promote a classroom community facilitating collaboration, shared agency, and mathematical reasoning. There is still a need for better understanding of how teachers can support students’ reasoning for building mathematical understanding (Ayalon & Hershkowitz, 2018; Maher et al., 2018; Stockero et al., 2019), but even more so, there is a need to make experienced and new teachers aware of the impacts their own actions have when interacting with student-pairs or groups.

The teacher-student interaction seen between Jacob, Hannah, and Emma is characterized as *progressing* conversations (Drageset, 2014). Jacob probably aimed at moving the process forward by asking *open progress details* questions and responding with *closed progress details*, as well as *evaluation of solution/strategy* and *encouraging of testing strategy* regarding the students’ inputs in GeoGebra. Several times, Hannah and Emma, who were in a bi-directional interaction, attempted reasoning about mathematical properties of the problem, but this was interrupted by Jacob. With his funneling actions, he seemed to pull the students toward important aspects (the Topaze effect, cited in Brousseau, 2006) of making a pair of perpendicular functions: (1) connecting the algebraic expression with the graphical representation and (2) zooming in on the slope number.

Opportunities for productive interactions with Jacob’s teacher actions were through following a given procedure for what to focus on and how to look for a path to further generalize a rule from a pattern between the linear function pairs in GeoGebra; thus, they were in line with timely support making new information easier for the students to access (van de Pol et al., 2018). Jacob’s support resulted in the students’ testing of different pairs of slope numbers in GeoGebra, moving them closer to discovering a pattern to generalize into a rule. At the same time, primarily leading students toward the solution method without asking them to justify their answers or explain why (*justification action*) limits students’ opportunities to think for themselves and discover central mathematical properties for connecting pieces of knowledge of functions into a robust function (Best & Bikner-Ahsbahs, 2017).

In summary, the teachers’ actions influenced students’ interaction, but mainly their reasoning. The specific teacher actions influenced in following ways: (1) Sophie’s *focusing actions* contributed to new suggestions for solving the problem made by the primary agent, and implemented into action by the secondary agent, (2) Lucas interacted differently with two pairs; first, with *progressing actions* channeling students’ focus to further investigating the connection between the slope numbers, which the primary agent responded to when suggesting a rule, then by guessing and checking in GeoGebra, which was observed by the secondary agent. Secondly, with *redirecting* and *progressing actions* Lucas channeled students’ focus to more precisely formulated mathematics, which resulted in collaboratively built reasoning anchored in mathematical properties of the linear function. However, as previously stated, we believe that the productivity of the interaction was a result of students’ shared agency, and not because of timely or supportive teacher guidance. In the last teacher-student interaction, (3) Jacob’s *progressing action* contributed to channel students’ focus to the slope number, which resulted in guessing and checking in GeoGebra by both students. Moreover, both students were less engaged in anchoring their guesses in intrinsic mathematics. Their conversation included less reasoning and less repairing since they did not make any claims about their different suggestions.

A teacher’s guidance is shaped by students’ contributions (Staples, 2007). The three teachers in this study responded to uttered reasoning and actions made in GeoGebra, and their guidance were consequentially shaped by that. Even though the teachers wanted to encourage collaborative work and reasoning, uttered in conversations, they primarily responded to the mathematical ideas of the conversation, not the social aspect of collaborative processes and exercised agency. Yackel and Cobb (1996) argue that students’ reasoning and sense-making cannot be separated from their participation in making a shared mathematical understanding of the problem. Moreover, Yackel and Cobb (1996) delineate normative norms from sociomathematical norms: where normative norms are “general classroom social norms that apply to any subject matter”, whereas sociomathematical norms are “what counts as an acceptable mathematical explanation and justification” (p. 460-461). We could view social norms as collaborative processes and agency for a collaborative environment, and the sociomathematical norms as students' creative mathematical reasoning. We could further argue that teachers' actions influenced students' sociomathematical norms and not their social norms. Since the students' reasoning was influenced, but their participation was not, consequentially, we highlight the importance of teachers' awareness, not only of the mathematical content in a conversation, but also students' roles in an interaction, which emphasize the importance of both normative norms and sociomathematical norms for students' productive interactions.

When assessing students' collaborative processes, a central focus is the quality of students' interactions (Child & Shaw, 2018; Francisco, 2013), and we propose evaluating students' way of participating through different agencies, as important for understanding opportunities for students' interactions. As teachers or as teacher-educators an indication of productive interaction is to look for (1) turn-takings, where students listen to one another and build arguments depending on actions and ideas of each other. We argue that this is an important aspect for recognizing a shared agency (Mueller et al., 2012). Turn-takings in students' dialogue are critical to establish a shared understanding from an interplay of ideas in a conversation (Barron, 2000; Martin & Towers, 2015; Sidnell, 2010). Another indication to notice is (2) plausible arguments, where both students make their thinking visible through speech and actions, and base their ideas, suggestions, and explanations on mathematical concepts and relations. Thus, it is an indication of coordination of language and actions (Sarmiento & Stahl, 2008), as well as CMR (Lithner, 2017) important for learning through their own, and as mutual, attempts to construct mathematical knowledge (Maher et al., 2018; Norqvist et al., 2019). As observed when a student pair was mutually engaged in plausible arguments, their reasoning was often by both students, anchored in mathematics in all instances of collaborative process. This indicate a particularly important aspect for driving the problem-solving process forward and was often seen in situations of repairing.

CONCLUDING THOUGHTS

Following three teachers' interactions with four dyads, and building on a previous analysis of the dyads' interactional patterns (Hansen, 2021), we have discussed the importance of teacher actions for students' opportunities of productive interactions through mathematical reasoning, collaboration, and agency. Jacob and Lucas acted with *progressing actions* which resulted in less reasoning anchored in mathematical properties and more guessing and checking in GeoGebra (Hannah and Emma, and Leah and Isaac), where both student pairs maintained their interactional pattern as respectively bi-directional and one-directional, after the teacher interaction. However, another outcome from Lucas's progressing actions was seen when he acted with *redirecting* and *progressing actions* which could have had an impact on the continued creative reasoning where students maintained their bi-directional interaction after the teacher interaction (Philip and Noah). Sophie's teacher action was *focusing actions* which also could have impacted creative reasoning by the primary agent observed and acted on by the secondary agent (Olivia and Oscar), where students continued their one-directional interaction after the teacher interaction. Moreover, the students who were initially engaged in a bi-directional or a one-directional interaction did not change their ways of interacting after interacting with their teacher. Neither the nature of the collaborative process nor the students' agency seemed to be particularly influenced by the teachers' actions. One explanation for this pattern might be that, to a large extent, the teachers approached the primary agent in their interactions with the one-directional dyads, thus mainly initiating a conversation between the teacher and the primary agent. On the other hand, we observed that the dyads with a shared agency and productive collaboration maintained this interactional pattern in spite of the teachers' funneling actions.

Although the teachers wanted to encourage collaborative work and reasoning, their actions were shaped by students' mathematical ideas, not the social aspects of collaborative processes and exercised agency. Furthermore, concerning these aspects it is important that teachers notice students' roles, so that teacher actions potentially can promote reasoning in collaborative processes such as monitoring and repairing and co-construction of arguments in shared agencies. These findings can probably apply to different collaborative instances with different mathematical problems, or possibly in other subjects. If applicable to other contexts, a teacher's action for more productive student interactions should evaluate a student's role and aim to promote students' shared agency through turn-takings and making plausible arguments.

It is important for both students in a pair to engage in the following actions: reasoning where they justify their arguments and attempt to pose different claims, where both should be encouraged to make counterarguments or justifications by anchoring the suggestions in mathematical properties. Teacher actions facilitating the mentioned students' actions are important for their engagement in the collaborative processes of monitoring and potentially repairing, promoting students to become authors of mathematical ideas (Langer-Osuna et al., 2020). With such outcomes of teacher actions, we suggest that teacher authority (Langer-Osuna, 2018) could potentially afford students' shared agency through important collaborative processes for progress in problem solving processes and construction of their own mathematical knowledge. However, while we propose the importance of what could have been done to facilitate such outcome, it is yet to be studied what kind of teacher actions, as suggested by Drageset (2014), that would influence a productive interaction seen in students' exercised agency and collaborative processes. This is an issue for further research. We acknowledge a teacher's challenge in observing students' interactions. However, when situations allow for observation, we think the two mentioned aspects, properties for turn-taking and plausible arguments, as often observed in bi-directional interactions (Hansen, 2021), may be an important indication for evaluating opportunities for productive student interactions. Which is a step towards a better understanding of how to facilitate mathematical reasoning and argumentation in classrooms (Maher et al., 2018; van de Pol et al., 2018).

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An analytical model for analyzing interactional patterns in creative collaborative mathematical reasoning

Abstract

Research in mathematics education highlights the importance of students' mathematical reasoning and small-group work for building mathematical understanding. It is important for teachers and teacher educators to observe and interpret students' interactions in these situations to facilitate mathematical reasoning in collaborative work. For such purpose, with this conceptual paper we present an analytical model of students' interactions focusing on their mathematical reasoning, collaborative processes, and exercised agency. Teachers, mathematics educators, and researchers can utilize the model, called the *CCMR model*—Creative Collaborative Mathematical Reasoning model—when observing student-pairs engaged in collaborative problem solving. The analytical model is presented with three interrelated interaction aspects, presented with construct definitions, interplay of interaction aspects, and indicators to evaluate students' interactions. Furthermore, the theoretical points we are making are illustrated with glimpses from students' interactions when solving a linear function problem in their classroom settings. The propositions from utilizing the model suggest that students' way of participating through different agencies (shared, primary, secondary), and particularly their interaction pattern seen as one-directional or bi-directional, may be a stronger indication for understanding the quality of students' interaction when engaged in solving a mathematical problem, than evaluating whether students are mutually engaged (collaborating) or merely dividing the task between themselves (cooperating).

Keywords: agency, collaboration, reasoning, interactions, analytical model

1. Introduction

Mathematical communication, through problem-solving, collaborative work, and reasoning foster students' learning of mathematics (e.g., Erath et al., 2021; Maher et al., 2018; Mueller et al., 2012; Seidouvy & Schindler, 2019). However, talking and listening to each other in a mutual collaboration, as a contrast to only instructing each other and dividing the workload, is not always an easy path to follow. Nevertheless, if given the opportunity, students may experience through reasoning and collaborative work in a problem-solving setting that mathematics make sense to one-self and peers, which are key issues for deeply understanding mathematics (Mueller et al., 2012; Sidenvall, 2019).

Thus, students' interactions are central in the learning of mathematics. A student interaction can be defined as "a complex social phenomenon which is composed of non-verbal and social properties in addition to its verbal characteristics" (Kumpulainen & Mutanen, 1999, p. 455). Sánchez et al. (2013) review six interaction types (based on research of Leikin and Zaslavsky (1997) and Kahveci and Imamoglu (2007)), which involve communication with a peer, a teacher, and learning material, for instance, a device or computer program. Thus, an example of an interaction type is student–learning-material–student: a learning activity involving communication between students (Leikin & Zaslavsky, 1997). How students engage when attempting to work together affect aspects of their interaction: participation patterns, individual learning opportunities, the nature of the mathematical discussion, and the construction of a solution path (Esmonde & Langer-Osuna, 2013, Langer-Osuna, 2016, referred to in Langer-Osuna et al., 2020).

Several studies have focused on different aspects with students' interactions and emphasize the need for more knowledge on how to facilitate for students' productive interactions (e.g., Langer-Osuna et al., 2020; van de Pol et al., 2018). We support the description of 'productive' posed by Engle and Conant (2002): "students' engagement is productive to the extent that they make intellectual progress" (p. 403). Simply put: students' effort is getting them somewhere (Engle & Conant, 2002). Related to this study's focus: a mutual productive interaction concerns building a *shared understanding* of the mathematics involved. Having a shared understanding comes from the notion of mutually working together (Roschelle & Teasley, 1995) to create a shared solution to a problem or having a shared strategy for solving a problem. Explaining shared understanding, they refer to a shared knowledge structure they call a *Joint Problem Space* (JPS), consisting of "… an emergent, socially-negotiated set of knowledge elements" (Roschelle & Teasley, 1995, p. 70). Collaboratively making a shared understanding involves a shared learning process where students share thoughts and ideas through verbal expressions of suggestions, explanations, and disagreement. Martin and Towers (2015) describe a shared understanding as a collective mathematical understanding consisting of a "ever-changing interactive process, where shared understandings exist and emerge in the discourse of a group working together" (p. 6). Moreover, attempting to create a shared understanding of a problem happens if students make their reasoning available for one another. Thus, students' shared understanding coexists with reasoning sequences that are plausible and meaningful to them (Granberg & Olsson, 2015). Creating and maintaining a shared understanding is an important aspect of collaborative problem-solving; thus, it is essential for reasoning together. Consequentially, students' mathematical talk is important for making a productive interaction. A recent review outlines the state of development and research on design principles related to mathematical talk for mathematics learning (Erath et al., 2021). The review recommends further research on students' mathematical talk, with particular emphasis on their mathematical reasoning in collaboration, with a greater focus on the content and mathematical features, and students' understanding, rather than how long their talking turns are or the number of times they speak. Moreover, the complexities of student-student interactions connected to how they work with mathematics in classrooms and use mathematical concepts have often been researched separately (Erath et al., 2021). By analyzing students' interactions when reasoning mathematically, which combine analysis of several dimensions (Erath et al., 2021), could therefore strengthen the knowledge of instructional design for teachers, design of tasks, and curriculums (Erath et al., 2021).

With an aim of developing a tool for assessing students' interaction patterns when they participate in collaborative creative mathematical reasoning, this article presents an analytical model which combines three interactional aspects: mathematical reasoning, collaborative processes, and exercised agency. We see assessing as an action where you evaluate, for instance, the collaborative work and/or the mathematics involved, and its quality. In this study, the quality of students' interactions is related to the productivity and the learning opportunities involved.

The presented model allows a study of students' interactional patterns in their shared learning process, which may have an "important role in the analysis of learning-teaching processes" (Sánchez et al., 2013, p. 247). Thus, making it possible to connect the need for combining different complex aspects seen in students' interactions towards teaching design principles.

The analytical model is exemplified by glimpses of students' interactions solving a linear function problem in their classroom settings, which is building on the empirical work from Author (2021) and Authors (2022). With this focus we build an analytical model for analyzing interactional patterns, called the *CCMR model*, Creative Collaborative Mathematical Reasoning model. The CCMR model illustrates how the interactional aspects are connected through students' way of participating, and how students participate in dyads are an important indicator for assessing their interaction as productive or unproductive.

The analytical model is an important contribution to the field of mathematics education, because of its insights on how interaction aspects are related to students' participation for more or less productive interactions. Moreover, from analyzing students' interactions it is possible for teachers to make assumptions on their learning outcome related to the mathematical topic involved. Thus, the model can be a tool for teachers, mathematics educators and researchers, for analyzing student-interactions to better understand students' ways of participating for quality interactions promoting learning of mathematics.

The suggested CCMR model is based on theory and utilized frameworks in classroom situations of students' attempted collaborative learning, and further elaborated on in the following sections: 2. Theoretical background, where three interactional aspects are introduced and motivated, 3. Building the analytical model, where the empirical basis is outlined and the building of the CCMR model is explained through data analysis and empirical examples, and 4. Concluding thoughts elaborates on assessing students' interactions connected to learning opportunities.

2. Theoretical background

This section expounds the three interactional components we argue are crucial when we want to assess students' interactional patterns as they participate in collaborative creative mathematical reasoning: mathematical reasoning, processes of collaboration, and exercised agency.

2.1 Mathematical reasoning

Mathematical reasoning is an important interactional aspect for learning mathematics (Yackel, 2001), not only to improve argumentation skills, but the processes of justification and argumentation in itself are important aspects for learning mathematics (Krummheuer, 2007). With Yackel's (2001) focus on students' talk as "what the participants take as acceptable, individually and collectively, and not on whether an argument might be considered mathematically valid" (p. 6), this study views reasoning as an interactional accomplishment using arguments to justify, support, and explain mathematical solutions and suggestions. Thus, argumentation is a part of reasoning where one justify thoughts and ideas, aiming for convincing oneself, or someone else, that the reasoning is appropriate and correct (Bergqvist et al., 2008)

In line with this view is a rather recent empirically developed framework of mathematical reasoning which includes students at any competence level (Lithner, 2008, 2017). The framework defines mathematical reasoning as "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2017, p. 939). There are different paths of reasoning in order to reach a conclusion in task-solving, and the framework *Creative Mathematically founded Reasoning*[1] (CMR, Lithner, 2008), identifies two major types of reasoning: creative reasoning and imitative reasoning. Students' *creativity* implies the creation of problem solutions through engaging in reasoning, whereas copying a procedure or recalling a fact is *imitative* reasoning. Therefore, how students' support and explain their thoughts is important to the learning outcome.

Exploring mathematical connections and building an argument, is within the framework, not a matter of having to create a formal or logical proof. As long as the sequence of thoughts makes sense to the student, whether or not it is simple or complex, correct or incorrect, the student should have enough evidence to support the idea and call it creative reasoning (Lithner, 2017). Three criteria must be fulfilled in order to call a reasoning sequence creative (Lithner, 2017): 1) *Creativity* refers to creating a reasoning sequence not experienced previously or re-creating a forgotten one, 2) *Plausibility* are arguments supporting the strategy choice or strategy implementation explaining why the conclusions are true or plausible, 3) *Anchoring* means that arguments are anchored in the intrinsic mathematical properties of the components of the reasoning. Arguments are considered to be *intrinsic* if they are based on mathematical concepts or relations and superficial if based on an appearance and not on underlying mathematics.

Imitative reasoning, on the other hand, can be memorized reasoning or algorithmic reasoning. Memorized reasoning is, for instance, remembering that 1L=1000 cm^3(Lithner, 2008). Algorithmic reasoning is when the students' approach is to solve the task using a given or recalled algorithm. The strength of using an algorithm in school mathematics is the speed and the high reliability when solving a task (Lithner, 2015). However, if the algorithm is used without the conceptual part, such as the consideration of its meaning, it may lead to rote learning (Lithner, 2017).

2.2 Collaborative processes

Collaborative group work provides opportunities for students to share their thinking, negotiate suggestions, and use mathematical reasoning for better understanding of mathematical ideas. When students work together on a mathematical problem, they can discuss how to solve it together or decide to divide the workload. The latter approach is often named cooperation, whereas the former is, in educational research, called collaboration (Baker, 2015). Cooperation can be described as dividing a task into subtasks for individual work; a shared product can be accomplished through sharing of findings and answers produced individually (Staples, 2007). A student pair successfully engaged in collaboration, can occasionally have periods of cooperative work.

[1] In line with Lithner (2008, 2017) and his colleagues studying creative mathematically founded reasoning, this study uses the wording creative reasoning or acronym CMR for linguistically simplicity.

Defining collaboration depends on the research focus: collaboration as means to an end—organizing for learning outcomes of knowledge or other skills—or collaboration as an end in itself (Lai et al., 2017). Research on cooperation and collaboration combined is mainly found in the first view on collaboration (Lai et al., 2017), whereas collaboration as an end in itself often have a process focus (Seidouvy & Schindler, 2019). Studies with a process focus often investigates verbal and social interactions, and students' attempt of making a shared understanding (Seidouvy & Schindler, 2019). Different actions that students take, for instance, suggesting, explaining, listening, evaluating, and negotiating a solution method or a strategy to solve a task, can indicate a collaborative process. With a focus on students' processes attempting to learn something together, collaboration has been defined as a "coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (Roschelle & Teasley, 1995, p. 70). Thus, a productive collaborative interaction, in line with the definition, is found in students' building and maintaining of a shared conception of a mathematical problem (Roschelle & Teasley, 1995). Collaborative processes are explained below anchored in the definition of collaboration.

We have taken the collaborative process focus, i.e., students' different actions for different processes of making and maintaining a shared understanding of a problem, adopting suggested processes from Roschelle and Teasley (1995), which are outlined at the end of this subsection. It practically means to look for how students respond to one another which can indicate whether students have a shared conception from an interplay of ideas. Although collaborative processes and collaborative outcomes have been separated research focus (Lai et al., 2017), a primary focus on processes does not exclude insights on collaborative outcomes, since the aspects are connected (Child & Shaw, 2018). Thus, *how* students mutually engage in discussing and solving the task at hand, which are seen in their actions, influences the learning outcome or a produced artefact.

Students can explore mathematical ideas together, and if willing and engaged, they may share suggestions and thoughts for solution methods. A central aspect of students' activities to construct and maintain a shared conception of a problem is the coordination of language and actions (e.g., Baker, 2015; Roschelle & Teasley, 1995; Sarmiento & Stahl, 2008). This means a mutual exchange of utterances and taken actions for solving a problem, which can be further seen in how students express their thinking through body language, uttered words, and hand gestures, via inputs into a dynamic software program, or via drawings and writings. Such actions are often expressed through turn-taking, which is an important social practice in conversations (Sidnell, 2010). Turn-taking is critical to establishing shared understandings of a problem (Barron, 2000). In collaborative problem-solving, a shared understanding is important and means that students' joint understandings "emerge from the interplay of ideas of individuals as these become woven together in shared action" (Martin & Towers, 2015, pp. 5-6).

The way students engage in collaborative processes relates to how well collaboration is maintained and furthered (Child & Shaw, 2016). Co-constructing a shared understanding happens through processes where students build, monitor, and repair the meaning or a strategy for solving a problem (Roschelle & Teasley, 1995). For instance, *building* a shared conception of a problem means introducing new ideas and establishing an inquiry process (Alrø & Skovsmose, 2004; Child & Shaw, 2018). A suggested idea or an arising thought, such as pointing out a mathematical property or sketching a figure as a solution strategy, can be a starting point or continuation of collaborative work. If the suggested idea is not making sense, a peer could ask questions about the idea, which ultimately could be explained. Having different perspectives, asking questions, and providing explanations are important parts of *monitoring* the collaborative work (Roschelle & Teasley, 1995). Sometimes, students experience divergences in understanding or of strategies to use, which means that they have conflicting ideas. Experiencing divergences in opinions means that students need to *repair* their shared understanding (Roschelle & Teasley, 1995). Important actions for resolving a period of conflicting ideas can include reformulating ideas, such as by paraphrasing or repeating utterances in one's own words (Alrø & Skovsmose, 2004).

2.3 Agency

Acting or resisting to in collaborative problem solving, the issue of students' *agency*, is yet another central interactional aspect in group work (Mueller et al., 2012). Studies of agency focus on how people, different cultures, and artefacts shape actions and decisions (Carlsen et al., 2016). For instance, if agency in mathematics were expressed by a student, she could say, "I can do mathematics, and I'm willing to jump in and give it my best" (Schoenfeld, 2018, p. 503).

Exhibited agency at play in classrooms has been organized into two categories: conceptual and disciplinary agency (Pickering, 1995). Mueller et al. (2012) explain Pickering's (1995) two concepts: "Conceptual agency entails constructing one's own meanings and methods while disciplinary agency involves utilizing established procedures" (p. 374). Thus, disciplinary agency is in line with imitative reasoning (Lithner, 2008), where students' argumentation consists of memorized facts and procedures. As emphasized in previous sections, research shows that students who can create their own solution methods through collaborative work and reasoning, hence displaying conceptual agency, will better understand important mathematical ideas (Maher et al., 2018).

Students' agency can be seen in relation to their afforded responsibility and authority in their problem-solving (Pickering, 1995). Agency has been referred to as "the way in which he or she acts, or refrains from acting, and the way in which her or his action contributes to the joint action of the group in which he or she is participating" (Gresalfi et al., 2009, p. 53). The nature of the agency—an individual's participation in a group or community—will vary between different situations and in different interactions (Gresalfi et al., 2009). Given an opportunity to participate and act, this means that a student can engage in mathematical problem-solving, which is important for developing a mathematical voice and becoming recognized as a producer of mathematics (Schoenfeld, 2013). Therefore, group collaboration can contribute to students' agency in mathematics (Schoenfeld et al., 2019), and it is an important aspect to consider when studying students' collaboration and reasoning.

When studying student pairs' interactions, we describe their way of participating (Gresalfi et al., 2009) connected to conceptual or disciplinary agency as their exercised agency (Pickering, 1995). Using this approach may give further insights on how students participate in collaborative processes and mathematical reasoning. In line with such approach, Mueller et al. (2012) present a framework for students' argumentation connected to expressed agency in different discursive practices when collaboratively solving mathematics problems. Students can exercise *shared agency* when co-constructing arguments (Mueller et al., 2012). In those situations, students simultaneously build arguments up from the ground, and "without one of the participants, the argument would not exist" (Mueller et al., 2012, p. 378). Moreover, students may exercise individual forms of agency, called *primary agency* or *secondary agency.* A student is a primary agent in situations where he or she makes the final argument based on corrections from a peer or by making sense of a peer's faulty or flawed idea. Thus, a secondary agent's input influences the original argument. These forms of input are corrections, extensions, or flawed arguments further formed by the primary agent for a final argument (Mueller et al., 2012).

Therefore, a student can exercise individual or shared agency in a situation when attempting to collaborate. We have adopted the way of describing such agency as being connected to acting or resisting acting in a mathematics conversation (Gresalfi et al., 2009).

3. Building the analytical model

This article contributes to the growing knowledge of students' interactions in collaborative activities by combining the three construct definitions—collaborative processes, mathematical reasoning, and exercised agency—into an analytical model (Fig. 1), presented in section 3.3. We view the three interactional aspects in interplay, assimilating and combining previously developed concepts from literature (Jabareen, 2009; Jaakkola, 2020) with empirical research where three frameworks (Table 1 in section 3.2) were utilized (Author, 2021; Authors, 2022). This has further resulted in the presented analytical model (Fig. 1). Section 3.4 explores the analytical model through empirical examples.

3.1 The context

The empirical examples presented in section 3.4 come from a design-based research study in a Norwegian upper secondary school. The participants were 69 students divided on three mathematics classes with three teachers. The students were 15-16 years old and enrolled in their first year of a theoretical mathematics program. Out of the larger group of students four pairs were chosen for an in-depth study of their interactions when solving different mathematical problems. This article builds on the analysis of these four dyads solving two function problems (Fig. 2 and Fig. 3).

The students and the teachers participated in the study for five consecutive months. During these months, teachers, and researcher (the main author was present) discussed how to facilitate students' collaborative work and mathematical reasoning. The meetings between the teachers and the researcher included lesson planning and evaluation of the lessons. Based on the conversations during these meetings the teachers attempted to ask open-ended questions, aiming for students to think together and engage in math-talk. The three teachers were considered to be engaged and experienced, however not previously used to focus on collaborative work and mathematical reasoning this extensively and for a longer period of time. Moreover, the teachers explained during the meetings that they were not used to being aware of their role and how they supported students in their collaborative interactions.

3.2 Data analysis

A possible starting point to observe and to assess students' interaction pattern, is to ask: 1) How are students engaged in mathematical reasoning? One can look for students' suggested ideas or counter-suggestions anchored in mathematics 2) How are students responding to each other? One can look for suggestions, questions, explanations, and counter-suggestions in the conversation. This can indicate whether students have a shared conception from an interplay of ideas. 3) How are students participating? A possible approach is to look for student roles when they suggest solution procedures to solve a problem. The roles could be where both are leading, or one is repeatedly leading and making decisions. Moreover, it is seen in how they are acting or resisting to in collaborative problem solving. The three guiding questions are three components of an interaction pattern, which we highlight as central components for analyzing students' attempt of a collaborative interaction focusing on how they interact when reasoning mathematically.

Table 1 The coding framework of the three interactional components.

The coding framework for interactional components of collaborative problem solving			
How are students engaged in mathematical reasoning?	**Creative mathematically founded reasoning (CMR)**		
	Creativity	**Plausibility**	**Anchoring**
	• New idea • Recreating a forgotten idea	• Explanation of strategy choice • Explanation of strategy implementation • Explanation of why something is true	• Ideas connected to mathematical properties and concepts • Intrinsic: mathematical concepts and properties
How are students responding to each other?	**Collaborative processes for building and maintaining a shared understanding**		
	Building	**Monitoring**	**Repairing**
	• Accepting ideas • Making suggestions • Stating a problem • Pointing out mathematical properties	• Asking questions • Explaining an idea • Observing and responding to one another's interpretations and ideas	• Negotiations • Correcting conflicting interpretations • Counter-suggestions • Reformulations
	Agency		
How are students participating?	**Shared agency**	**Primary agent**	**Secondary agent**
	• Co-construction of arguments	• Makes the final argument • Assimilates another student's argument • Makes sense of a peer's faulty or flawed argument	• Makes input for the main argument

Indicators of the three construct definitions—mathematical reasoning, collaborative processes, and exercised agency—make up the coding framework presented in Table 1. The coding framework is anchored in literature, as presented in chapter 2. Theoretical background, and tested empirically (Author, 2021; Authors, 2022).

The three frameworks are organized in Table 1 according to the way it was utilized to code four student pairs' collaborative problem solving of the linear function problem presented in Author, 2021 and Authors, 2022. First, the coding procedure began with identifying students' creative reasoning in their conversations. These snapshots of a conversation were called *CMR-sequences*. These sequences entailed a student or a student-pairs' mathematical arguments which were anchored in the mathematical properties of linear functions. This was the starting point of the coding procedure, because of the learning opportunities when reasoning creatively (e.g., Granberg & Olsson, 2015; Lithner, 2017; Olsson, 2018). To categorize students' reasoning as creative, three criteria in the framework must be fulfilled (Lithner, 2017). First, *creativity*: creating a reasoning sequence not experienced previously or re-creating a forgotten one; second, *plausibility*: arguments supporting the strategy choice or strategy implementation explaining why the conclusions are true or plausible; and third, *anchoring*: arguments are anchored in the intrinsic mathematical properties of the components of the reasoning. Arguments are considered to be *intrinsic* if they are based on mathematical concepts or relations and *superficial* if based on an appearance and not on underlying mathematics.

The second step was to identify students' collaborative processes (Roschelle & Teasley, 1995). This step was important for evaluating how students were responding to each other. Initiation of making a shared understanding started with the collaborative process of *building*: suggesting, accepting, and agreeing upon an idea to solve a problem. The collaborative process of *monitoring* was categorized as asking questions or explaining an idea when observing and trying to understand each other's interpretations of a problem. If the suggestions or explanations conflicted with a peer's understanding of the shared understanding, a state of collaborative *repairing* was categorized as negotiating and correcting conflicting interpretations using justifications and counter-suggestions. Thus, the students' dialogue entailed decision-making, negotiations, justifications, suggestions, accepting suggestions, and action-taking to solve the linear function problem. It was particularly the actions of making suggestions and explaining ideas, and posing counter-suggestions, that indicated that students had the opportunities to anchor their arguments in mathematical properties, thus, presenting learning opportunities through CMR.

Coding for exercised agency (Gresalfi et al., 2009; Mueller et al., 2012) was the third step. Detailed descriptions were written about the way they interacted, specifically in terms of the mathematical content of the conversation regarding linear functions and on the ways students acted or resisted from engaging in the mathematical conversations (Gresalfi et al., 2009). The descriptions were used to code students' participating roles for contributing to the solution procedure in the given situation. Therefore, this coding step was not employed on every student utterance or turn but seen as an overall attribute in each excerpt of the conversations. In situations where students co-constructed arguments, their agency was coded as *shared agency* (Mueller et al., 2012). When students engaged in individual acts of agency, it was coded as *primary agency* or *secondary agency* (Mueller et al., 2012). If a student made the final argument, assimilated another student's argument, or made sense of a peer's faulty or flawed argument, then the student's engagement was categorized as being a primary agent in the conversation. A secondary agent was being the co-working peer who gave input for the final argument in the conversation.

3.3 The CCMR model

The CCMR model (Fig. 1) is based on the analysis procedures and findings of students' interaction patterns from empirical work (Author, 2021; Authors, 2022). Authors (2022) argue that whether students' interactions are productive or unproductive is connected to how students participate: choosing to engage or refrain from engaging through different types of agencies (shared, primary, or secondary), including different ways of interacting in those roles. Thus, agency shapes actions (Carlsen et al., 2016), such as collaborative processes and mathematical reasoning (Author, 2021), which altogether can co-construct a shared understanding, or individual understandings, where the outcome is a solution procedure to a given problem. Since agency shapes how students collaborate and reason mathematically, we highlight through the CCMR-model how students participate in dyads are an important indicator for assessing their interaction pattern as productive or unproductive. Moreover, Author (2021) discusses students' roles in pairwise collaboration. She found that students were in a *bi-directional interaction* when mutually attempting to understand one another and when both were driving forces of the problem-solving process. Typical for a pair in a bi-directional interaction was a mutual effort where students co-constructed reasoning sequences with shared agency (Author, 2021). In contrast, if students exercised different roles in a problem-solving process where the final outcome was expressed repeatedly by one of the students, they were in a *one-directional interaction*. In the one-directional interaction, both students were engaged, but they expressed agency differently. For instance, the co-working student's role was often to understand suggestions or explanations made by the primary agent. However, ideas and questions were expressed by the student who exhibited secondary agency. Such input was either assimilated into the final outcome of the reasoning by the primary agent, or it was considered and refined or neglected by the primary agent.

An important indicator of bi-directional interactions is that both students are driving forces of the problem-solving process. This can be observed when both students are choosing to engage in mathematical reasoning, where the dialogue consists of turn-takings and plausible arguments found in all stages of collaborative processes. This is particularly prominent when both students need to or choose to make suggestions, explain ideas, and make counter-suggestions to make progress in their problem solving. Therefore, if both students in a collaborating pair are actively involved in a bi-directional interaction, they are likely to exercise CMR in multiple instances of collaborative processes with co-constructed arguments creating a shared understanding of the problem at hand (Author, 2021). Thus, the interplay of the three interaction components, prominent in co-reasoning when suggesting or making counter suggestions, is a key to a productive interaction promoting students to become producers of mathematics (Schoenfeld, 2013) and for in-depth learning.

An important indicator of a one-directional interaction is students' different roles in communicating. Asking: "Who is the driving force in the conversation and how are arguments built?", may help to evaluate whether students are acting one- or bi-directionally. If the answer is that one participant repeatedly makes the final argument, it is likely that the student pair is acting one-directionally. Moreover, a primary agent uses the secondary agent's monitoring utterances to determine the final outcome (Mueller et al., 2012) of their reasoning sequence. Students acting one-directionally are not mutually engaged in CMR, nor are there turn-taking conversations where both students suggest new ideas.

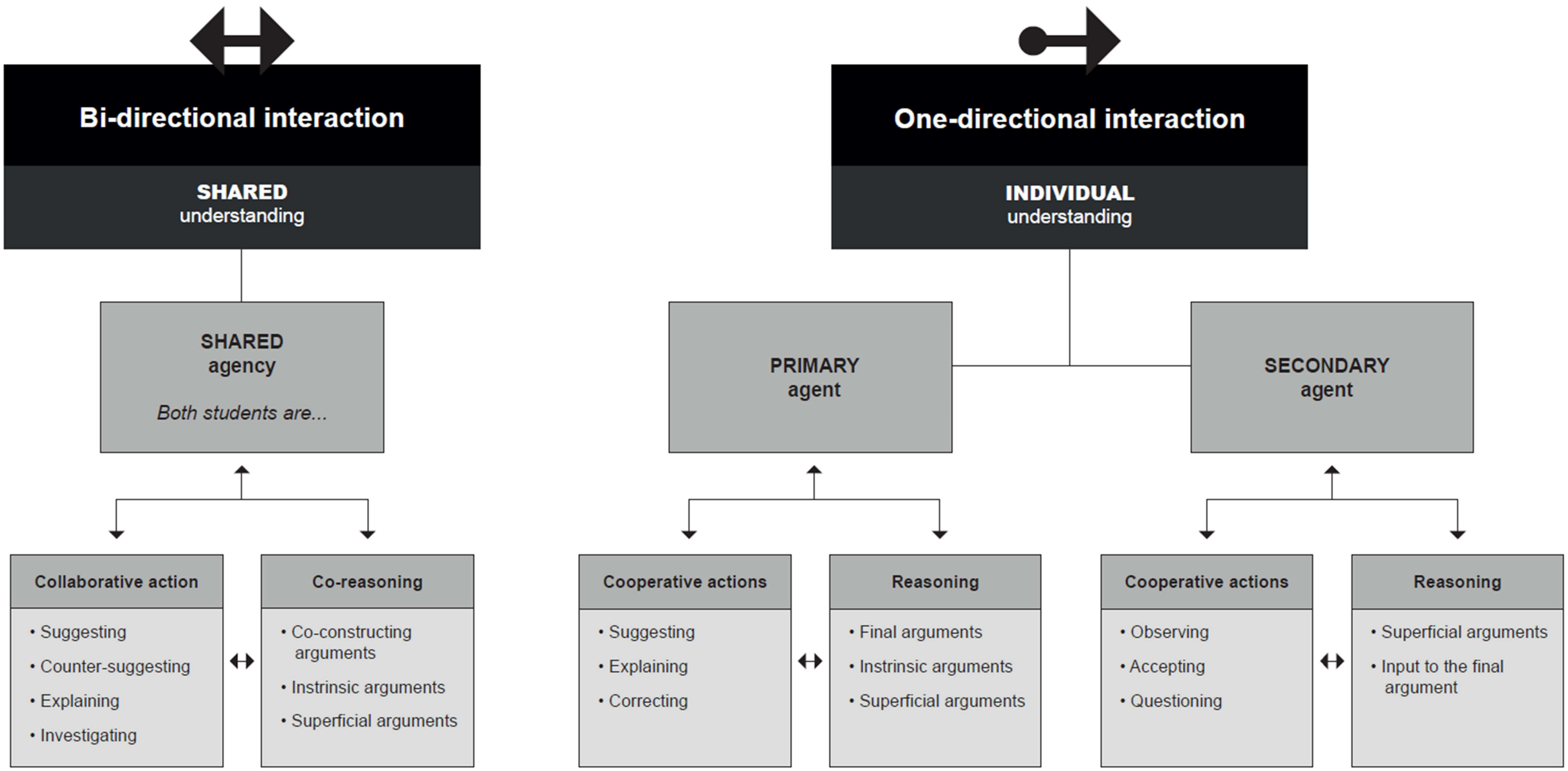

Fig. 1: The CCMR model – an analytical model for evaluating collaborative interaction patterns.

3.4 The CCMR-model in practice – empirical examples

The following four excerpts of students' interactions are from the same student pairs as presented in Authors (2022) and Author (2021). The excerpts are not presented elsewhere, however, two of the four excerpts are from the same problem-solving situation presented in Authors (2022) and Author (2021). Moreover, two excerpts are from another problem-solving situation the students participated in. The students attempted to solve two linear function problems (Fig. 2 and Fig. 3), which presented opportunities for the students to connect different function representations to construct their own solutions to the problem. The problem-solving situations of the two problems were one month apart. Figure 2 was the first problem and Figure 3 the second problem, a continuation of the first function problem. In the last problem students must find (or remember) the connection between perpendicular lines, and how the constant numbers are related to form a square with four linear functions. Thus, both function problems invite connecting algebraic representation and graphical representation. In the following sub-sections central theoretical concepts from the data analysis (Table 1) are written in italics to make the coding procedure more transparent.

- Create a straight line $y = mx + c$
- Create another straight line in a way that the corresponding graphs are perpendicular.
- Formulate a rule for when two straight lines are perpendicular.
- Test the rule for other straight lines.

Fig. 2: The first function task (reformulated from Olsson (2018))

Solve the following problem using GeoGebra.

- Construct four linear functions in such way that they will form a square.
- One of the lines must be parallel with the y-axis.
- Adjust the formulas in GeoGebra's algebra field in such way that you will angel the square less than 90^o clockwise or counterclockwise.

Fig. 3: The second function task based on Granberg and Olsson (2015)

3.4.1 Bi-directional interaction – two student-pairs

Two student-pairs were in bi-directional interactions: Emma and Hannah, and Philip and Noah (see Author (2021) and Authors (2022)).

In function problem 1 Philip and Noah investigated different pairs of linear functions and discussed mathematical properties: coordinate system, slope number, and decreasing and increasing lines. They had differences in opinions about finding a perpendicular linear function to $y = 2x + 5$ and difficulties expressing a rule. During a rapid turn-taking conversation, Philip was *questioning* Noah about why he meant a half of x would make a perpendicular line. Philip *suggested* a perpendicular line with the slope number -2. After talking and investigating the slope number, which they called x, they agreed that $-0.5x$ made a perpendicular line. Regarding the formula, Noah suddenly saw a connection between the perpendicular pair of lines, and *explained* his observation to Philip. Philip *built* on his peer's observation and said, "There is a slope number, what is it called? It's m." They agreed that the solution was 1 divided by m for making perpendicular lines. After their conclusion, Philip said they should test the formula. Yet another discussion occurred when they implemented different slope numbers for their rule, which resulted in *negotiation* of mathematical properties leading to a confirmation of the rule. In the following conversation between Noah and Philip, they went over the formula again and agreed upon their decision. Excerpt 1 shows that Noah and Philip exercised a *shared agency* when negotiating the properties for making a rule, *anchored* in the connection between the generalized expression of the rule and the graphical representation, while engaged in all the *collaborative processes* (building, monitoring, and repairing).

Excerpt 1

1	Noah	So, it's minus... -1 divided by m, which is...
2	Philip	Which is the rule.
3	Noah	Yes.
4	Philip	Is it the same thing we used here as well? Or on the others (pointing at the paper first and then at the screen)?
5	Noah	Yes, minus is always here in front (pointing at the paper). This is... (putting a square around their rule on the paper). This works. So, if we have one line, then the other one must be that, in a way...
6	Philip	$-1/m$ (nodding).
7	Noah	It was a bit tricky.
8	Philip	Is there anything else we could try? Is there something else?
9	Both	No (in choir while looking at the laptop screen).
10	Noah	The slope on one line... Or, 1 divided by the slope number from one line, then we will have, that must be the other one, right?
11	Both	(Nodding).
12	Philip	Good.

Noah summarized their findings and continued to *build* a shared understanding of the rule (1). Philip asked to make sure that their written rule on the paper aligned with the graphical view. Thus, his *monitoring* reply was *anchored* in a connection between the generalized expression of the rule and the graphical representation (4). Noah *explained* and *monitored* their shared understanding (5), which Philip added to, indicating mutuality in their understanding (6). In the conversation that followed, Philip asked if there was anything else to consider when making the rule (8); hence, he engaged in further *monitoring* important to maintain a shared view on the rule.

A month later, Philip and Noah were presented with the second function problem (Fig. 3). They successfully constructed a square with four linear functions. Their focus of the conversation was to figure out how the four linear functions were connected when they attempted to adjust the algebraic expressions tilting the square and keeping all right angels. They did not recall the first function problem, nor did they remember the algebraic expression for a linear function. Their first attempt of a linear expression was $ax + bx + c$. In a conversation with the dyad, teacher Lucas pointed out the algebraic expression by repeating the formula out loud. Philip replied that bx was zero and continued saying that bx would therefore not be necessary to use. Before the conversation in Excerpt 2, Noah and Philip had found pairs of perpendicular lines to be $mx + c$ and $-mx + c$. Their justification was based on one pair of slope numbers: 1 and -1. About this relation Philip meant it was important that m always was a whole number. He did not explain why he thought so, but his opinion followed him to the end of the problem-solving session. Noah did not ask him about it, but later he suggested the slope number $-0{,}5$, which was accepted by Philip. However, it was probably not understood by Philip, since he kept his interpretation about the slope number relation. Noah probably observed that implementing only whole slope numbers in GeoGebra did not provide perpendicular pairs of lines. Thus, their different interpretations kept them engaged in resolving the conflicting understanding and probably wanting to understand the mathematical properties of the relation between the linear functions. However, Lucas ended the problem-solving session, so they did not get the opportunity to make a shared solution of the second function problem.

Excerpt 2

1	Noah	Then adjust the angel less than... So, if we…
2	Philip	If it is whole numbers, then it is a square, right?
3	Noah	We could try minus too; it should be the same (uses the laptop). Or if we do like this (uses the laptop). It must be one.
4	Philip	Must the sides be 90 degrees in a square?
5	Both	(Silent together and looking at the laptop screen)
6	Philip	Or is a rhombus a square? Since four sides have the same length.
7	Noah	I don’t think they have the same length.
8	Philip	They have the same length, don’t they? Here... (uses the laptop)
9	Noah	Wait a moment, if this one… (uses the laptop) They are having the same length. I mean that in a square the degrees (probably meant the angles) are 90 degrees.

10 Philip Yes, it is a square with whole numbers, in a way.
11 Noah Well… (nods)
So, m adjusts only the angle.
12 Philip Mhm. Or… Yes, where it lies somehow

Excerpt 2 shows a *shared agency* between Philip and Noah while they discuss the properties of the linear functions for making a square *anchored* in the slope number, while engaging in all the *collaborative processes* (building, monitoring, and repairing). Noah initiated a *building* of a shared understanding of how to create linear functions that would angle the square less than 90° (1). Philip continued *building* by making sure that they understood the premises for the next step of the task: that a quadrilateral was a square (2, 4, 6). The dyad looked intensely at the laptop screen and judged the length of the lines (5-9). The laptop screen is not visible to us, nor was it recorded; thus, we would think that they were observing the linear functions they had made in GeoGebra. Noah *monitored* the response from Philip and recalled that a quadrilateral in which all sides are equal, and all angles right angles is a square (9). Together Noah and Philip focused on the premises of making a square related to the slope numbers of the linear functions. The premises was not creatively constructed but recalled from memory, however, the creativity of their co-constructed premise was the emphasizing of mathematical properties with a square related to the slope numbers of the linear functions. Moreover, their divergencies in interpretations about the relation between slope numbers for a pair of perpendicular lines contributed to progress in the dyads' problem-solving.

The characteristic of their turn-taking conversations was a *co-construction* of arguments indicating a *shared agency* in the situations and of the given function problem. In a joint effort and synchronicity, Philip and Noah built, monitored, and repaired their shared understanding through reasoning and arguments anchored in intrinsic mathematical properties. Therefore, and most prominent in Excerpt 1, their reasoning sequences were rarely standing alone as individual thoughts. Moreover, individual thoughts and ideas developed into meaningful ideas, suggestions, and investigations that were built and pursued to solve the function problem together. Thus, both students engaged with *shared agency*.

A comparable interaction pattern to Philip and Noah is found in the dyad of Emma and Hannah (Excerpt 3 in Author 2021). Emma and Hannah struggled finding a perpendicular linear function to $y = 4x + 2$. Emma was *building* by *suggesting*, "x. 1 divided by 4…. Plus 2". Hannah responded by saying "No, it's not the same. I thought, if it were the same [the rule could be] that we only put minus sign in the front. But that doesn't work." Hannah expected the slope numbers 4 and -4 would make a pair of perpendicular linear functions, but she reflected on her previous conflicting interpretations with the algebraic and graphical representation. Her actions contributed with the process of *repairing* their problem-solving progress. In the following conversation, Emma and Hannah evaluated and interpreted each other's thoughts, which they responded to using creative reasoning in a turn-taking sequence of collaborative processes. They were both engaged in suggesting, listening, interpreting, disagreeing, and evaluating. Both students engaged in negotiating the mathematical properties for making a perpendicular pair of lines. Hannah and Emma's willingness to participate and investigate the first function problem together illustrates how both can "be in charge" of the problem-solving process. Their reasoning sequence cannot be separated into two individual lines of thoughts, as their reasoning was *co-constructed* with input from one another. Therefore, Hannah and Emma had a *shared agency* when negotiating the properties for making a perpendicular pair of lines.

3.4.2 One-directional interaction – two student-pairs

Two student-pairs demonstrated one-directional interaction: Olivia and Oscar, and Leah and Isaac (see Author (2021) and Authors (2022)).

In the first function problem Leah and Isaac found a set of linear functions making a perpendicular pair: $y = x + 3$ and $y = -x + 3$. Isaac read the task and wondered how to express a rule when he suddenly exclaimed, "Slope number!" Leah immediately *agreed* that the relationship between the perpendicular lines was connected to the slope numbers. In the conversation that followed in Excerpt 3, Leah and Isaac attempted to express the relationship between the two linear functions by *anchoring* the reasoning in the slope numbers of the two linear functions. Leah and Isaac exercised different agencies and different roles in their interaction. Leah acted with *secondary agency* when her reasoning and inputs were mainly assimilated into the final arguments made by Isaac. Isaac had the role of the *primary agent*. Both engaged in all the *collaborative processes* (building, monitoring, and repairing).

Excerpt 3

1	Isaac	...it's $1.\ 1$, right (pointing at the screen)?
2	Leah	No, then it's $-1.\ +1$.
3	Isaac	No, no, no. They have the similar, but one has the negative and one has the positive.
4	Leah	Ahh.
5	Isaac	One goes upwards and one goes downwards.
6	Leah	Yes, I agree. But do both have the same slope number?
7	Isaac	They don't have the same slope number.
8	Leah	No, no, no... Yes, they have...
9	Isaac	However, they have opposite slope numbers, which are alike. I don't know how to formulate it.

Isaac was *building* toward a solution, pointing to the slope numbers "1.1" (1). Leah *disagreed* with his observation. In her opinion, it was about "-1 and $+1$" (2). Both students addressed the slope numbers of the perpendicular pair of linear functions, and the reasoning sequence entailed *anchoring* in the mathematical property of the slope number, mainly referred to in the algebraic expressions. Isaac continued the conversation with *repairing* and expressed that the slope numbers had the same numbers, only with different signs (3). In (6), Leah *accepted* his explanation, and further *monitored* by asking about the meaning of "the same slope number". Isaac elaborated on the relationship, stating that the two linear functions did not have the same slope numbers (7), pointing out that they were the same numbers, only with different signs (9). Leah and Isaac further observed the similarities and differences in the function pair; however, they seemed to struggle to find a way to express their findings.

Leah and Isaac were engaged in the problem-solving process, and both contributed to the conversation. However, Isaac had the role of *primary agent* in his way of "telling" Leah how he was right. Such actions were demonstrated in his *building* suggestions (1, 7), *monitoring* explanations (5, 9), and *repairing* correction of his idea (3). Leah, on the other hand, *built* by *accepting* Isaac's input (6). Moreover, Leah expressed what she observed about the slope numbers, and she asked what Isaac thought. Isaac confidently expressed characteristics about the slope numbers. Leah exhibited *secondary agency*, giving input on the reasoning sequence, whereas Isaac showed *primary agency*, affecting the final outcome of their reasoning. Thus, Leah and Isaac were interacting in a *one-directional* manner. Although Leah monitored Isaac's statement about "same slope numbers" and disagreed with him, indicating a repairing process, she was never in a situation where a mathematical explanation depended on her. Therefore, she mainly engaged in imitated reasoning first expressed by her peer, Isaac.

In the second function problem, Leah and Isaacs engagement was similar to how they participated in the first problem a month earlier. During their problem-solving session Isaac was determined to express the four linear functions by using the function command in GeoGebra. Towards the end of their problem-solving their teacher Lucas explicitly told them that their approach, using the function command, would make it too difficult to solve the problem. Lucas suggested they should "write y equals something". Isaac suggested four linear lines, which made a tilted square. Thus, he solved the last task of the function problem. Right after they ended their problem-solving and started a non-mathematical conversation. In Excerpt 4 the dyad had not yet had the conversation with Lucas. Leah was willing to test Isaac's proposal about the command function in GeoGebra to solve the second task of the function problem.

Excerpt 4

1	Leah	Let’s try the function start value and end value [function command in GeoGebra]. It becomes nothing. (Uses the laptop)
2	Isaac	Just write a normal function.
3	Leah	$mx + c$, right?
4	Isaac	Yes.
5	Leah	But should we use the function-thing?
6	Isaac	Just chose $x + 1$, and chose $x + 0$
7	Leah	$x + 0$. And then start value, end value? (Uses the laptop)
8	Isaac	Yes, you need start something.
9	Leah	5?
10	Isaac	Just chose -5 to -5.
11	Leah	-5 to -5?
12	Isaac	Yes, eh, no, to 5.
13	Leah	Yes, right. Like this?
14	Isaac	OK. If the slope number was 0. No, if the slope number was 1, there. Then it would be parallel with the thing.
15	Leah	Because then it goes one up, right? That’s why the slope number should be 0.
16	Isaac	Yes, OK. Let’s try that. Write $0x$ directly then.
17	Leah	$0x$ plus…? (Looks at Isaac and waits for response)

Leah attempted to drive the problem-solving process forward by *building* on Isaac’s idea of using the function command (1). Thus, she *accepted* his suggestion and willingly tested it out. However, Leah repeatedly requested confirmation for further input and progress about the algebraic expression of a linear function (3), the GeoGebra command (5), the input into the command (7, 9), and of the graphical result (13). Furthermore, Leah *evaluated* Isaacs input by repeating his suggestion (11), thus indicating that it did not make sense to her. Isaac was probably content with the two linear functions: $f(x) = x + 1$ and $f(x) = x$. Then he *pointed out mathematical properties* of the slope numbers, probably for progressing with two new linear functions, to make a tilted square (14). However, it might have been an evaluation of their input, since a slope number of value 1 was already written, but it could have been a suggestion of another linear function with the same slope number with another constant. Nevertheless, Leah *monitored* Isaac’s observations by addressing the slope number’s function and *suggested* another linear function with slope number 0 (15). Isaac used Leah’s suggestion and continued orchestrating the event of actions (16).

In the problem-solving processes Leah and Isaac contributed to the conversations. However, as seen in Excerpt 4, which reflects the essence of their problem-solving conversation, Isaac had the responsibility and Leah requested his feedback and response in proceeding with actions in GeoGebra. Moreover, they rarely explained mathematical properties with the function problem, and mainly focused on testing out different linear functions by using the command function. Thus, there were little to no validation and evaluation of why something was right or not. The dyad participated with the same pattern as previously found in Excerpt 3: Isaac exercised the authority and did not share the responsibility of solving the problem, thus, he exercised *primary agency*. Leah, on the other hand, contributed to driving the process forward by exercising *secondary agency* through *accepting* and *questioning* Isaac's suggestions. Isaac's suggestions were sometimes interpreted by Leah and translated into actions in GeoGebra. With this function problem, even more prominent than in the first function problem, Leah requested input and confirmation from Lucas. Moreover, throughout this conversation was Isaac's condescending tone with Leah. She responded by telling him: "No, you cannot say so" or "You must behave. There are cameras". Although, this study is not reporting on affective aspects, it was distinct that the behavior appeared limiting for a fruitful conversation for addressing mathematical properties.

A similar interaction pattern is found in the dyad of Oscar and Olivia (see Excerpt 4, Author (2021), but here, Oscar was an even more dominant reasoner in the conversation. Olivia and Oscar started their problem-solving path with the first function problem by suggesting two linear functions. Oscar *suggested* a rule and *anchored* it in the mathematical property of the slope number: "They have the same value, but minus in front. I don't know. Constant x has the same value, wait...". Olivia *monitored* the input by *asking*: "It's not the same value, or?". The dyad engaged with the collaborative processes *building* and *monitoring*. Oscar *justified* his reasoning in linear functions with slope numbers with opposite signs. Olivia did not attempt to express suggestions or ideas for further exploration of the linear function concept. She mainly attempted to understand Oscar's ideas and his reasoning. Moreover, Oscar did not show interest in asking Olivia about her thoughts and ideas for a solution method. Therefore, Oscar was the *primary agent* in leading the conversation where his reasoning was *anchored* in mathematical properties for making suggestions and explanations of a rule. Olivia was the *secondary agent* in her attempt to understand Oscar's thinking, while accepting and asking for further details of his ideas. Hence, the dyad was in a *one-directional interaction*.

4. Concluding discussion

With an aim of developing a tool for assessing students' interaction patterns when they participate in collaborative creative mathematical reasoning, we have in this article presented an analytical model, the CCMR model (Fig. 1).

By utilizing the coding framework (Table 1), which means to employ categories of mathematical reasoning, collaborative processes, and exercised agency to analyze students' interactions when they attempt collaborative problem solving, gave insights to characterize how students participate, their roles in the situation, and can possibly indicate whether students are presented with learning opportunities while interacting with a peer.

When characterizing students' interactions, we observe that it is, as many have argued, the mutuality and synchronicity of the collaboration that defines whether students collaborate or merely cooperate (Cobb, 1995; Roschelle & Teasley, 1995; Staples, 2007). However, evaluating students' way of participating through different agencies, seen in their interaction patterns, may be a stronger indication for understanding the quality of students' interaction when engaged in solving a mathematical problem. If students are acting with shared agency bi-directionally they are more likely to attempt to co-construct mathematical knowledge together through creative mathematical reasoning (Author, 2021). Particularly turn-takings (Sidnell, 2010) and plausible arguments (Lithner, 2017) are important indications for recognizing a shared agency. This would strongly suggest a quality interaction where students exercise conceptual agency (Pickering, 1995), and thus, provide an opportunity to better understand mathematical ideas.

In the cases where the student-pairs acted in a one-directional manner, the students did not refrain from acting (Author, 2021). But the student pairs did not mutually work together like the student pairs who were acting bi-directionally. Schoenfeld et al. (2019) express that students' ways of collaborating can contribute to students' agency in mathematics. We believe that students' ways of interacting in a one-directional way is connected to the two categories of agencies, as posed by Pickering (1995): disciplinary agency and conceptual agency. Disciplinary agency is explained as utilizing established procedures (Mueller et al., 2012). In the two student-pairs acting one-directionally, the primary agent engaged in creative mathematical reasoning, and thus, constructed one's own meanings and methods (Mueller et al., 2012). However, the peer, if attempting mathematical reasoning, made mainly superficial arguments, and utilized procedures posed by the primary agent. Thus, it was not a mutual effort regarding creative mathematical reasoning. Therefore, it is the primary agent in a one-directional interaction who is primarily engaged in conceptual agency, which is similar to students with shared agency in a bi-directional interaction.

Students in a one-directional interaction are likely to have different learning opportunities, due to their participating roles and the opportunities presented through reasoning creatively. Since the primary agent more often makes arguments based on mathematical properties compared to a secondary agent, it is likely that the student is presented with more individual learning opportunities. However, a peer might as well have learning opportunities, but not to the same extent since the reasoning mainly is lacking or being superficial. Moreover, if assessing the unit of a one-directional dyad compared to a bi-directional dyad, we assume that the learning opportunities presented in the latter dyad exceeds a one-directional dyad.

If we contrast our assumptions about the learning opportunities with Cobb's (1995) findings, we find some similarities and differences we would like to address. Cobb (1995) highlights that univocal interactions, which resemblances one-directional interactions, rarely provided learning opportunities for either student. For a learning opportunity to take place two central elements for a productive group dynamic should be established, he claims: 1) creating a shared basis for mathematical communication, and 2) routine engagement when interacting where neither student is an authority (Cobb, 1995). In a univocal interaction, that Cobb (1995) describes, one student has the mathematical authority and judge a peer's suggested solutions. Thus, a student with the mathematical authority is likely to experience that she must explain her thinking, and a peer is likely to experience that he has to make an effort to understand a given explanation. This is not productive for either of them, hence, providing an explanation does not in itself give rise to learning opportunities (Cobb, 1995). To create learning opportunities through verbal explanations and helping a peer, a student must clarify and organize his thinking and then explain his solution procedure in a new way (Webb 1989, referred to in Cobb, 1995). Webb's (1989) explanation resembles the concept CMR—new, plausible, and anchored mathematical arguments—and therefore it is a reasonable assumption to say that a one-directional interaction is not unproductive for both students, if a primary and a secondary agent are active in some specific actions. We contend that the CCMR model contributes to highlight such actions that indicate productivity, mostly for the primary agent, but also for the secondary agent. If the primary agent participates in cooperative actions, such as suggesting and explaining combined with reasoning mathematically by making the argument and anchoring it in mathematical properties, we would say that this give rise to individual learning opportunities. And if a peer, the secondary agent, actively observes, ask questions, and contributes to the final mathematical argument (even if superficial), we believe this will provide learning opportunities, although to a lesser and more shallow extent than the secondary agent.

Furthermore, we agree with Cobb (1995) when he says that when a student is the mathematical authority in a group it rarely becomes a productive collaborative interaction. Cobb (1995) further emphasizes that it might be difficult for students to establish a shared understanding where one student is more “conceptually advanced”. Thus, it would be more promising with a starting point where students think they could work with the peer and that neither student has the mathematical authority in the interaction (Cobb, 1995; Langer-Osuna et al., 2020).

We believe that the CCMR model contributes with new insights to the assessment of students’ collaborative interactions through its detailed and nuanced analysis. Hence, we consider the presented model as a useful tool for both teachers and teacher educators in planning, observing, and evaluating teaching involving student-student interactions. Stein et al. (2008) highlight the importance of teachers’ anticipation of-, as well as monitoring of, student responses in orchestrating for productive mathematical discussions. We suggest that the CCMR model can give additional and nuanced details on the interactional aspects of productive discussions.

Moreover, we would not argue that a one-directional interaction is a non-quality interaction, but there might not be as many opportunities for both students to develop an understanding of mathematical ideas. It would therefore be of high importance that a teacher initiates a change in the students’ interaction and the roles they take in the given situation, for both students to construct a shared solution procedure and/or a shared understanding of mathematical ideas.

Further, we will argue for the value of using the presented model as an analytical framework for future research on interactional patterns. One interesting aspect for further research is related to how teachers can support dyads to move or change the interactional pattern they are in. Authors (2022) found that students who established an interactional pattern, such as shared agency in a bi-directional interaction or primary/secondary agency in a one-directional interaction, maintained and progressed in the same interactional pattern before, during, and after a teacher interaction. The teachers involved in the study engaged with both funneling and focusing actions, but students remained in their roles their entire problem-solving process.

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