Hindawi
International Journal of Differential Equations
Volume 2022, Article ID 5549693, 13 pages
https://doi.org/10.1155/2022/5549693

Research Article

Positive Invertibility of Matrices and Exponential Stability of Linear Stochastic Systems with Delay

Ramazan Kadiev [1,2] and Arcady Ponosov [1,2]

[1] *Dagestan Research Center of the Russian Academy of Sciences & Department of Mathematics, Dagestan State University, Makhachkala 367005, Russia*
[2] *Department of Sciences and Technology, Norwegian University of Life Sciences, P. O. Box 5003, NO-1432, Ås, Norway*

Correspondence should be addressed to Arcady Ponosov; arkadi@nmbu.no

Received 27 January 2021; Revised 18 December 2021; Accepted 29 April 2022; Published 31 May 2022

Academic Editor: Giovanni P. Galdi

The work addresses the exponential moment stability of solutions of large systems of linear differential Itô equations with variable delays by means of a modified regularization method, which can be viewed as an alternative to the technique based on Lyapunov or Lyapunov-like functionals. The regularization method utilizes the parallelism between Lyapunov stability and input-to-state stability, which is well established in the deterministic case, but less known for stochastic differential equations. In its practical implementation, the method is based on seeking an auxiliary equation, which is used to regularize the equation to be studied. In the final step, estimation of the norm of an integral operator or verification of the property of positivity of solutions is performed. In the latter case, one applies the theory of positive invertible matrices. This report contains a systematic presentation of how the regularization method can be applied to stability analysis of linear stochastic delay equations with random coefficients and random initial conditions. Several stability results in terms of positive invertibility of certain matrices constructed for general stochastic systems with delay are obtained. A number of verifiable sufficient conditions for the exponential moment stability of solutions in terms of the coefficients for specific classes of Itô equations are offered as well.

1. Introduction

Stability analysis of stochastic delay equations is quite popular due to its numerous applications. It is therefore impossible to give a more or less extensive overview of the topic. Some of the results are summarized in the monograph [1]; others can be found in more recent works of the author of this monograph as well as in many other publications. Indisputably, the main method of stability analysis is based on Lyapunov functions and their generalizations. However, this method may not be applicable in certain situations or may give too restrictive conditions, both in the deterministic and stochastic cases. It is particularly challengeable to use the Lyapunov framework in the case of complicated delays or/and random coefficients and initial conditions (see, e.g., discussion in the recent paper [2]). Yet, many applications require such kinds of models, e.g., in mathematical finance, especially when modeling stock prices, interest rates, or volatilities (see, e.g., [3] and the references therein), stochastic control theory [4–6], semi-Markov systems [7], epidemic models [8], and many others.

Stability analysis used in this paper is based an alternative approach utilizing a regularization technique. This approach has been successfully used by several authors in the theory and applications of deterministic and stochastic delay equations. Its theoretical foundations are presented in the monograph [9], and since then numerous applications of the method have been investigated in a number of papers. Among the recent ones are the publications [10] (the Mackey–Glass equation), [11, 12] (control theory), [13] (neural networks), [14] (hyperjerk systems), and [15] (the stochastic pantograph equation).

The regularization method has its roots in the theory of the input-to-state stability, which usually implies the

Lyapunov stability (see, e.g., [16]). As soon as this relationship between the two types of stability is established, the algorithm starts with choosing a simpler equation (called a reference equation) that is assumed to already have the required stability properties. Resolved and substituted into the original equation, the reference equation produces a new integral equation of the form $x - \Theta x = f(x)$. If the latter equation is solvable (for instance, if $\|\Theta\| < 1$), then stability of the original equation is proven. Thus, the method is similar to Lyapunov's direct method, but instead of seeking a Lyapunov function(al), one aims to first find a suitable reference equation possessing the prescribed asymptotic property. The reference equation is then used to regularize the original equation. The method proven to be particularly efficient when constructing Lyapunov function(al) may be technically difficult (random coefficients and delays, distributed delays, unbounded delays, complicated noises, and so on).

In [13, 14, 17], the regularization method was combined with the estimation technique based on positive invertible matrices. This led to new, verifiable stability conditions in the case of differential equations with variable, in particular, distributed delays. In general, the approach based on $\mathscr{M}$–matrices gives better stability results as the technique utilizing the norm estimation (see, e.g., [15, 17]). In the above publications, verifiable stability conditions were formulated in terms of the coefficients of the systems in question.

The present paper is a continuation of the authors' work started in [16, 17]. Minding future applications of the method (see Section 8), we offer a general framework combining the regularization approach from [9] with the theory of positive invertible matrices. After that, we demonstrate how this scheme can be applied to (rather complicated) systems of Itô equations with variable delays and random coefficients, i.e., in the situations where the Lyapunov-like functionals may be difficult to construct. A motivation to study such systems goes back to several sophisticated stochastic models used, for instance, in population dynamics. Thus, equations for the aggregated state variables derived from the McKendrick–von Foerster equation for structured populations always contain random, distributed delays, random coefficients, and random initial conditions [14], so that the Lyapunov framework may be difficult to apply.

For the sake of simplicity, we chose to describe the framework for the case of linear equations and exponential stability, as it was done in the monograph [9]. Even if the most realistic models are nonlinear, we stress that the methods of the present paper, primarily developed for the linear case, can be directly used to study local stability of stochastic nonlinear equations. Moreover, the regularization technique can be applied to global stability of nonlinear deterministic and stochastic delay equations. This has been clearly shown in many publications based on the regularization method (see, e.g., the monograph [9] and the references therein as well as the recent publications [10, 11, 13, 14, 17]).

The presentation of the regularization method will be incomplete without mentioning the recent publications [18–20], where many profound stability results were obtained. The paper [20] deserves a special attention, as the authors regularize, in our terminology, nonlinear stochastic equations by means of nonlinear reference equations with the known stability properties. This approach is close to the one used in the present paper, but does not exploit $\mathscr{M}$–matrices. It seems to be a fruitful idea to combine both techniques to obtain more general stability results and to cover more applications.

The paper is organized as follows. We start with some preliminary information and the notation (Section 2). Then, we introduce the system of stochastic delay equations we intend to study as well as the main assumptions on it (Section 3). In Section 4, we offer a brief description of the regularization method and its adjustment based on the input-to-state stability and the theory of positive invertible matrices. We remark that the framework described in this section goes far beyond the tasks of the present paper. It can be also used in the case of arbitrary semi-martingales, unbounded delays, nonlinear systems, etc. We chose this level of generality because we intend to apply the framework to many other equations and models in the future. Section 5 contains formulation and the proof of the main result of the paper. It provides sufficient conditions (in terms of the coefficients) of exponential $2p$-stability ($p \geq 1$) of the system introduced in Section 3. In Section 6, the main result is specified for some particular cases of the general system. We believe that even in the case of deterministic equations, we produced some new results. Section 7 contains some examples illustrating the stability conditions offered in the previous section. Finally, in Section 8, we provide a short summary and describe some open problems.

2. Preliminaries

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ be a stochastic basis, where Ω is set of elementary probability events, $\mathscr{F}$ is a σ-algebra of all events on Ω, $(\mathscr{F}_t)_{t\geq 0}$ is a right continuous family of σ-subalgebras of $\mathscr{F}$, and P is a probability measure on $\mathscr{F}$; all the above σ-algebras are assumed to be complete w.r.t. P, i.e., containing all subsets of zero measure.

The following notational agreements are used throughout the paper:

(i) $R = (-\infty, \infty)$ is the set of all real numbers.

(ii) $col(x_1, \ldots, x_n) \in R^n$ is a column vector.

(iii) $|.|$ is an arbitrary yet fixed norm in R^n, $\|.\|$ being the associated matrix norm.

(iv) $\overline{E}$ is the $n \times n$-identity matrix.

(v) μ is the Lebesgue measure on $[0, +\infty)$.

(vi) $\|.\|_X$ is the norm in a normed space X.

(vii) p is an arbitrary real number satisfying $1 \leq p < \infty$.

(viii) $\mathscr{B}(I)$ is the σ-algebra of all Borel subsets of an interval $I \subset R$.

(ix) $(\mathscr{B}_2, \ldots, \mathscr{B}_m)$ is the standard $(m-1)$-dimensional Brownian motion (i.e., the scalar Brownian motions $\mathscr{B}_l$ are independent).

(x) $Z(t) \coloneqq col(t, \mathscr{B}_1(t), \ldots, \mathscr{B}_m(t))$ (only used in Section 4).

(xi) The expectation (the integral w.r.t. the measure P) is denoted by E.

(xii) k^n is the linear space of all n–dimensional, $\mathscr{F}_0$–measurable random values.

(xiii) $k^n_q = \left\{\alpha \colon \alpha \in k^n, \|\alpha\|_{k^n_q} \equiv (E|\alpha|^q)^{1/q} < \infty\right\}$.

(xiv) c_p is the constant from inequality (1) used in the stability conditions.

Recall that a $\mathscr{B}(I) \otimes \mathscr{F}$–measurable stochastic process $\xi(t) = \xi(t, \omega)$, $t \in I$, is called $\mathscr{F}_t$–adapted if $\xi(t, \cdot)$ is $\mathscr{F}_t$-measurable for all $t \in I$.

A $m \times m$-matrix $B = (b_{ij})_{i,j=1}^m$ is said to be nonnegative, resp. positive if $b_{ij} \geq 0$ resp. $b_{ij} > 0$ for all $i, j = 1, \ldots, m$.

The following definition is crucial for what follows.

Definition 1. A matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n$ is called a (nonsingular) $\mathscr{M}$–matrix if $\gamma_{ij} \leq 0$ for $i, j = 1, \ldots, n$, $i \neq j$, and all the principal minors of the matrix Γ are positive.

According to [21], a matrix B is a nonsingular $\mathscr{M}$–matrix if $b_{ij} \leq 0$ for all $i, j = 1, \ldots, m$, $i \neq j$, and there exist positive numbers ξ_i, $i = 1, \ldots, m$, such that one of the following conditions is fulfilled:

(1) $\xi_i b_{ii} > \sum_{j=1, i\neq j}^m \xi_j |b_{ij}|$, $i = 1, \ldots, m$.

(2) $\xi_j b_{jj} > \sum_{i=1, i\neq j}^m \xi_i |b_{ij}|$, $j = 1, \ldots, m$.

In particular, if $\xi_i = 1$, $i = 1, \ldots, m$, in the first of the above conditions, then we obtain the class of strictly diagonally dominant matrices. In [21], one can find plenty of characterizations of nonsingular $\mathscr{M}$–matrices.

One of the most important properties of nonsingular $\mathscr{M}$–matrices states the following.

The inverse of a nonsingular $\mathscr{M}$–matrix is positive. In what follows we will always silently assume that any $\mathscr{M}$–matrix is nonsingular.

The next two lemmas contain inequalities to be used in this paper.

Lemma 1

$$\left(E\left|\int_0^t f(s)d\mathscr{B}_l(s)\right|^{2p}\right)^{1/2p} \leq c_p\left(E\left(\int_0^t |f(s)|^2 ds\right)^p\right)^{1/2p}, \quad (1)$$

for any $\mathscr{F}_t$-adapted stochastic process $f(s)\,(0 \leq s \leq t)$, any $t > 0$, and any component $\mathscr{B}_l(s)\,(1 \leq l \leq m)$ of the Brownian motion $\mathscr{B}$.

Proof. Inequality (1) follows directly from inequality (5) in [22, p.65], where one can find explicit formulas for c_p. □

Lemma 2. *Let $g(s)$ be a scalar function that is square integrable on $[0, \infty)$ and $f(s)$ be an $\mathscr{F}_t$-adapted stochastic process satisfying $sup_{s\geq 0}(E|f(s)|^{2p})^{1/2p} < \infty$. Then,*

$$\begin{aligned}&\sup_{t\geq 0}\left(E\left|\int_0^t g(s)ds\right|^{p}\right)^{1/2p} \\ &\quad\leq \sup_{t\geq 0}\left(\int_0^t |g(s)|ds\right)\sup_{t\geq 0}\left(E|f(t)|^{2p}\right)^{1/2p},\end{aligned} \quad (2)$$

and

$$\begin{aligned}&\sup_{t\geq 0}\left(E\left|\int_0^t (g(s))^2 (f(s))^{2\,ds}\right|^{p}\right)^{1/2p} \\ &\quad\leq \sup_{t\geq 0}\left(\int_0^t (g(s))^2 ds\right)^{1/2}\sup_{t\geq 0}\left(E|f(t)|^{2p}\right)^{1/2p}.\end{aligned} \quad (3)$$

Proof. We only prove inequality (2), as inequality (3) can be justified similarly.

$$\begin{aligned}\sup_{t\geq 0}\left(E\left|\int_0^t g(s)ds\right|^{2p}\right)^{1/2p} &\leq \sup_{t\geq 0}\left(E\left(\int_0^t |g(s)\|f(s)|ds\right)^{2p}\right)^{1/2p} \\ &\leq \sup_{t\geq 0}\left(E\left(\int_0^t \left||g(s)|^{(2p-1)/2p}|g(s)|^{1/2p}|f(s)|\mathrm{d}s\right|\right)^{2p}\right)^{1/2p} \\ &\leq \sup_{t\geq 0}\left(E\left(\left(\int_0^t |g(s)|ds\right)^{2p-1}\int_0^t |g(s)\|f(s)|^{2p}ds\right)^{1/2p}\right) \\ &\leq \sup_{t\geq 0}\left(\left(\int_0^t |g(s)|ds\right)^{2p-1}\int_0^t |g(s)|E|f(s)|^{2p}ds\right)^{1/2p} \\ &\leq \sup_{t\geq 0}\left(\int_0^t |g(s)|ds\right)\sup_{t\geq 0}\left(E|f(t)|^{2p}\right)^{1/2p}.\end{aligned} \quad (4)$$

□

3. The Main System and Formulation of the Problem

We will study exponential stability of the following system of Itô delay differential equations

$$dx(t) = -\sum_{j=1}^{m_1} A_{1j}(t)x\left(h_{1j}(t)\right)dt + \sum_{i=2}^{m}\sum_{j=1}^{m_i} A_{ij}(t)x\left(h_{ij}(t)\right)d\mathscr{B}_i(t)\,(t \geq 0), \tag{5}$$

equipped with the initial conditions

$$x(t) = \varphi(t)\,(t < 0), \tag{6}$$

$$x(0) = b. \tag{7}$$

Remark 1. According to [9], we separate the initial conditions for $t < 0$ and $t = 0$, as we do not require the continuity of the function $\varphi(t)$. This function is assumed to be only bounded and measurable, so that changing its value at countably many points does not change the solution of equation (5). On the other hand, it can be easily checked by examples that changing the value of $x(0) = b$ usually changes the solution of equation (5). That is, the space of initial functions φ and the space of initial values b have different topologies.

The following assumptions are put on (5)–(7) throughout the paper:

(1) $b = col(b_1, .., b_n)$ is an n-dimensional $\mathscr{F}_0$–measurable random variable belonging to the space $b \in k^n$.

(2) $\varphi = col(\varphi_1, \ldots, \varphi_n)$ is an n-dimensional $\mathscr{F}_0$–measurable stochastic process with essentially bounded trajectories, defined on the interval $[-\sigma, 0)$, where $\sigma = \max\left\{\tau_{ij}\colon i = 1, \ldots, m, j = 1, \ldots, m_i\right\}$.

(3) $x = col(x_1, \ldots, x_n)$ is an unknown n-dimensional stochastic process defined for $t \geq -\sigma$; it is $\mathscr{F}_t$-adapted for $t \geq 0$ and $\mathscr{F}_0$-measurable for $-\sigma t < 0$.

(4) $A_{ij} = (a_{sl}^{ij})_{s,l=1}^{n}$ are $n \times n$ –matrices for $i = 1, \ldots, m$, $j = 1, \ldots, m_i$, and the entries of the matrices A_{1j}, $j = 1, \ldots, m_1$, are scalar, $\mathscr{F}_t$-adapted stochastic processes with almost surely Lebesgue integrable trajectories, while the entries of the matrices A_{ij}, $i = 2, \ldots, m$, $j = 1, \ldots, m_i$, are scalar, $\mathscr{F}_t$-adapted stochastic processes with almost surely square integrable trajectories.

(5) h_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$, are Borel measurable scalar functions defined on $[0, \infty)$ satisfying the inequalities $0 \leq t - h_{ij}(t) \leq \tau_{ij}\,(t \in [0, \infty), i = 1, \ldots, m, j = 1, \ldots, m_i)\mu$–everywhere for some constants τ_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$.

Remark 2. It can be proven that under the assumptions 1–5, the initial value problem ((5)–(7)) has a unique (up to the natural equivalence of stochastic processes) continuous, $\mathscr{F}_t$-adapted solution $x(t, b, \varphi)$.

Definition 2. We say that equation (5) is exponentially q–stable $(0 < q < \infty)$ with respect to the initial data, i.e., the initial value x_0 and the "prehistory" function φ, if there are positive numbers K, λ such that all solutions $x(t, b, \varphi)$ of the initial value problem ((5)–(7)) satisfy

$$\left(E|x(t, b, \varphi)|^q\right)^{1/q} \leq K\,\exp\{-\lambda t\}\left(\left(E|b|^q\right)^{1/q} + \operatorname*{ess\,sup}_{t<0}\left(E|\varphi(t)|^q\right)^{1/q}\right)(t \geq 0). \tag{8}$$

In the next section, we describe the method of our analysis. We remark that this description is very general and goes far beyond the particular case of equation (5).

4. Regularization Method in Stability Analysis

In this section, we briefly describe a framework which we use to study stability properties of stochastic functional differential equations. The main idea of this approach is to convert the property of Lyapunov stability to the property of invertibility of certain operators in suitable functional spaces.

In what follows, we put $Z(t) \coloneqq col(t, \mathscr{B}_1(t), \ldots, \mathscr{B}_m(t))$. We remark, however, that the framework described below is valid for any m-dimensional semi-martingale Z (see, e.g., [16]).

We consider a general linear homogeneous stochastic hereditary equation

$$dx(t) = \left(V_h x\right)(t)dZ(t)\,(t \geq 0), \tag{9}$$

equipped with two initial conditions

$$x(s) = \varphi(s)\,(s < 0), \tag{10}$$

$$x(0) = b. \tag{11}$$

Here V_h is a k-linear Volterra operator (see below) defined in certain linear spaces of vector stochastic processes, φ is an $\mathscr{B}(-\infty, 0) \otimes \mathscr{F}_0$-measurable stochastic process, and $b \in k^n$. By k-linearity of the operator V_h we mean the following property:

$$V_h\left(\alpha_1 x_1 + \alpha_2 x_2\right) = \alpha_1 V_h x_1 + \alpha_2 V_h x_2, \tag{12}$$

holding for all $\mathscr{F}_0$-measurable, bounded, and scalar random values α_1, α_2 and all stochastic processes x_1, x_2 belonging to the domain of the operator V_h.

It is easy to see that equation (5) is a particular case of equation (9).

The solution of the initial value problem ((9)–(11)) will be denoted by $x(t, b, \varphi)$, $t \in (-\infty, \infty)$. Below the solution is always assumed to exist and to be unique for an appropriate choice of $\varphi(s), b$.

We also need an adaptation of Definition 1 to the case of equation (9).

Definition 3. For a given real number $q\,(0 < q < \infty)$, we call the zero solution of equation (9)

(i) q-stable (with respect to the initial data b and φ) if for any $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that $E|b|^q + \operatorname{ess\,sup}_{s<0} E|\varphi(s)|^q < \delta$ implies $E|x(t,b,\varphi)|^q \le \epsilon$ for all $t \ge 0$ and all $\mathscr{F}_0$-measurable φ, b.

(ii) Exponentially q-stable if there exist positive constants K, λ such that the inequality

$$E|x(t,b,\varphi)|^q \le K\left(E|b|^q + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^q\right)\exp\{-\lambda t\}, \quad (13)$$

holds true for all $t \ge 0$ and all $\mathscr{F}_0$-measurable φ, b.

Introducing the q-norm in the space Φ_q of the prehistory functions φ by

$$\|\varphi\|_q = \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^q, \quad (14)$$

and minding the norm in the space of the initial values defined in the previous section yield a shorter version of Definition 3, where the expression $E|b|^q + \operatorname{ess\,sup}_{s<0} E|\varphi(s)|^q$ is replaced by $\|b\|_{k_q^n} + \|\varphi\|_q$.

To describe the regularization method, we need to represent (9) and (10) in a slightly different form. Let $x(t)$ be a stochastic process $[0,+\infty)$ and $x_+(t)$ be a stochastic process on $(-\infty,+\infty)$ coinciding with $x(t)$ for $t \ge 0$ and equalling 0 for$t < 0$, while let $\varphi-(t)$ be a stochastic process on $(-\infty,+\infty)$ coinciding with $\varphi(t)$ for $t < 0$ and equalling 0 for$t \ge 0$. Then, the stochastic process$x_+(t) + \varphi-(t)$, defined for $t \in (-\infty,+\infty)$, will be a solution of the (9)–(11) if $x(t)\,(t \in [0,+\infty))$ satisfies the initial value problem

$$\mathrm{d}x(t) = [(Vx)(t) + f(t)]dZ(t)\,(t \ge 0), \quad (15)$$

$$x(0) = b, \quad (16)$$

where $(Vx)(t) := (V_h x_+)(t)$, $f(t) := (V_h \varphi_-)(t)$ for $t \ge 0$. Indeed, by k-linearity we have that $V_h(x_+ + \varphi_-) = V_h(x_+) + V_h(\varphi_-) = Vx + f$ yielding (15). Note that f is uniquely defined by the stochastic process φ, "the prehistory function." Let us also observe that the initial value problem ((15) and (16)) is equivalent to the initial value problem ((9)–(11)) only for f admitting the representation $f = V_h\varphi_-$, where $\varphi\prime$ is an arbitrary extension of the function φ to the real line $(-\infty,\infty)$.

Let B_q be a linear subspace of the space of $(\mathscr{F}_t)$-adapted stochastic processes with trajectories that are almost surely essentially bounded on $[0,\infty)$. The norm in this space is defined by

$$\|f\|_{B_q}^q = \operatorname*{ess\,sup}_{t\ge 0} E|f(t)|^q. \quad (17)$$

As we assume the existence and uniqueness property for equation (21) for all $b \in k_q^n$ and $f \in B_q$, we can denote the corresponding solution by $x_f(t,b)$. Let M stand for the space of all solutions of equation (15), and we define its linear subspace M_p by

$$M_p = \left\{x\colon x \in M, \sup_{t\ge 0} E|x(t)|^p < \infty\right\}. \quad (18)$$

The construction above produces two linear operators:

$$\mathscr{L}_1\colon \varphi \mapsto (V_h\varphi_-)(t), \quad (19)$$

$$\mathscr{L}_2\colon f \mapsto x_f(\cdot,b). \quad (20)$$

The next theorem is crucial for the framework (see, e.g., [12]). It says that the stochastic Lyapunov stability follows from the input-to-state stability ("the stochastic Bohl–Perron theorem").

Theorem 1. *Assume that the linear operators $\mathscr{L}_1\colon \Phi_q \longrightarrow B_q$ and $\mathscr{L}_2\colon B_q \longrightarrow M_q$, defined by (17) and (18), respectively, are bounded. Then, the zero solution of equation (9) is q-stable in the sense of Definition 3.*

We remark that the operator $\mathscr{L}_1$ is, as a rule, bounded, so that the only challenge in application of Theorem 1 is to prove boundedness of the operator $\mathscr{L}_2$. This can be done by the regularization method, also known as "the W-method" [9, 16]. The regularization is usually constructed with the help of an auxiliary or reference equation

$$\mathrm{d}x(t) = [(Qx)(t) + g(t)]dZ(t)\,(t \ge 0), \quad (21)$$

where Q is again a k-linear Volterra operator. The reference equation is, thus, similar to equation (15), but it is supposed to be "simpler" in the sense that the required stability property for this equation is already proven (see condition (2) in Theorem 2 below).

Assuming the existence and uniqueness property for equation (21), we get the following "variation-of-constants" formula for its solutions:

$$x(t) = U(t)x(0) + (Wg)(t)\,(t \ge 0), \quad (22)$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation and W is the corresponding Cauchy operator.

Using representation (22), we can regularize equation (15) in two ways: on the right and on the left. In this paper, we only use the left regularization to be described below. The algorithm based on the right regularization is presented in [16].

Using equation (21), we rewrite equation (15) as follows:

$$\mathrm{d}x(t) = [(Qx)(t) + ((V-Q)x)(t) + f(t)]dZ(t)\,(t \ge 0), \quad (23)$$

or, if we use the representation (22), as

$$x(t) = U(t)x(0) + (W(V-Q)x)(t) + (Wf)(t)\,(t \ge 0). \quad (24)$$

Denoting $W(V-Q) = \Theta$, we obtain the operator equation

$$x(t) = (\Theta x)(t) + U(t)x(0) + (Wf)(t)(t \geq 0). \quad (25)$$

Theorem 2. *Assume that equation (15) and reference equation (21) satisfy the following conditions:*

(1) *The linear operators V, Q act continuously from M_q to B_q.*
(2) *The Cauchy operator W in (22) constructed for reference equation (21) is bounded as an operator from B_q to M_q.*
(3) *The operator $I - \Theta$: $M_q \longrightarrow M_q$ has a bounded inverse.*

Then, the operator $\mathscr{L}_1$: $B_q \longrightarrow M_q$ in (20) is bounded.

Theorems 1 and 2 justify the regularization framework in the analysis of Lyapunov stability for stochastic linear functional differential equations. In fact, all the conditions except condition (3) in Theorem 2 are usually fulfilled, and this is easy to check. The only real challenge is therefore the invertibility of the operator $I - \Theta$. In [16] (see also the references therein), the invertibility of this operator is verified by estimating the norm of the integral operator Θ: if $\|\Theta\|_{M_q} < 1$ in the inequality

$$\|x\|_{M_q} \leq \|\Theta\|_{M_q}\|x\|_{M_q} + K_1\|x(0)\|_{k_q^n} + K_2\|f\|_{B_q}, \quad (26)$$

then equation (6) is q-stable due to Theorem 1. If one also proves that the equation remains q-stable after the substitution $y(t) = \exp(\lambda t)x(t)$ for some positive λ, then equation (9) becomes exponentially q-stable.

However, calculation of the norm $\|\Theta\|_{M_q}$ might be challengeable, especially in the vector case and in the case of random coefficients. In [13], and later in [14, 17] for the stochastic case, it was suggested to use the properties of monotone operators. The main idea was to perform all the estimates componentwise, which is much simpler, and then to check the positive invertibility of a certain matrix. Below we offer a generic description of this method.

Let

$$x(t) = col(x_1(t), \ldots, x_n(t)), \overline{x}_i = \sup_{t\geq 0}\left(E|x_i(t)|^q\right)^{1/q},$$
$$\overline{x} = col(\overline{x}_1, \ldots, \overline{x}_n). \quad (27)$$

Suppose that after estimating each component of vector equation (23), we arrived at the vector inequality

$$D\overline{x} \leq \|x(0)\|_{k_q^n}\overline{e}_1 + \|f\|_{B_q}\overline{e}_2, \quad (28)$$

where D is an $n \times n$–matrix and $\overline{e}_1$, $\overline{e}_2$ are some column n-vectors with nonnegative components. Typically, $D = \overline{E} - T$, where $\overline{E}$ is the $n \times n$ identity matrix, while T and $\overline{e}_i$ replace Θ and $K_i (i = 1, 2)$ in the scalar inequality (26), respectively. Then, we have the following.

Theorem 3. *If D is an $\mathscr{M}$–matrix in the sense of Definition 1, then the operator $\mathscr{L}_2$: $B_q \longrightarrow M_q$ in (20) is bounded.*

Proof. As D is an $\mathscr{M}$–matrix, the matrix D^{-1} is positive, and we can rewrite (28) as

$$\overline{x} \leq D^{-1}\left(\|x(0)\|_{k_q^n}\overline{e}_1 + \|f\|_{B_q}\overline{e}_2\right). \quad (29)$$

Therefore,

$$|\overline{x}| \leq K\left(\|x(0)\|_{k_q^n} + \|f\|_{B_q}\right), \quad (30)$$

where $K = \|D^{-1}\|\max\{|e_1|, |e_2|\}$. As $\|x\|_{M_q} \leq |\overline{x}|$, we conclude from (30) that $x \in M_q$ and $\|x\|_{M_q} \leq K(\|b\|_{k_q^n} + \|f\|_{B_q})$ for some positive K. Thus, the operator $\mathscr{L}_2$: $B_q \longrightarrow M_q$ is bounded.

In the next section, the scheme just outlined is applied to equation (5), i.e., for the system of Itô equations with variable delays and random coefficients. Using the boundedness of all delay functions in equation (5), the substitution $y(t) = \exp(\lambda t)x(t)$ for some positive λ, and Theorems 1 and 3 for q-stability of the equation for $y(t)$, we prove exponential q-stability of equation (5) for $q = 2p$, $p \geq 1$.

We also remark that the second Lyapunov method may be difficult to use in this case. □

5. The Main Result

The following conditions are supposed to be valid throughout this section:

(C1) There exist positive numbers $\overline{a}_{sl}^{ij}, i = 1, .., m, j = 1, \ldots, m_i, s, l = 1, \ldots, n,$, so that the coefficients of equation (5) satisfy the inequalities

$$\left|a_{sl}^{ij}(t)\right| \leq \overline{a}_{sl}^{ij},$$
$$t \in [0, +\infty),$$
$$i = 1, .., m,$$
$$j = 1, \ldots, m_i,$$
$$s, l = 1, \ldots, n, \quad (31)$$

$\mu \times P$–almost everywhere.

(C2) For each $s = 1, \ldots, n$, there is a nonempty subset $I_s \subset \{1, \ldots, m_1\}$ and positive number $\overline{a}_s$, so that

$$\sum_{k\in I_s} a_{ss}^{1k}(t) \geq \overline{a}_s,$$
$$t \in [0, +\infty),$$
$$s = 1, \ldots, n, \quad (32)$$

$\mu \times P$–almost everywhere.

The $n \times n$–matrix C is defined as follows:

$$c_{ss} = 1 - \frac{1}{\overline{a}_s}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1}\tau_{1k}\overline{a}_{ss}^{1k}\overline{a}_{ss}^{1j} + c_p\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sqrt{\tau_{1k}}\,\overline{a}_{ss}^{1k}\overline{a}_{ss}^{ij} + \sum_{j=1, j\notin I_s}^{m_1}\overline{a}_{ss}^{1j}\right] - \frac{c_p}{\sqrt{2\overline{a}_s}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}_{ss}^{ij}\right], \quad s = 1,\ldots,n,$$
$$c_{sl} = -\frac{1}{\overline{a}_s}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1}\tau_{1k}\overline{a}_{ss}^{1k}\overline{a}_{sl}^{1j} + c_p\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sqrt{\tau_{1k}}\,\overline{a}_{ss}^{1k}\overline{a}_{sl}^{ij} + \sum_{j=1}^{m_1}\overline{a}_{sl}^{1j}\right] - \frac{c_p}{\sqrt{2\overline{a}_s}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}_{sl}^{ij}\right], \quad s, l = 1,\ldots,n, s\neq l. \tag{33}$$

Theorem 4. *If conditions (C1) and (C2) are fulfilled and if C is an $\mathscr{M}$-matrix, then equation (5) is exponentially $2p$-stable in the sense of Definition 2.*

Proof. We rewrite equation (5) and the initial condition (6) as a single system:

$$d\overline{x}_s(t) = -\sum_{j=1}^{m_1}\sum_{l=1}^{n} a_{sl}^{1j}(t)\left[\overline{x}_l\left(h_{1j}(t)\right) + \overline{\varphi}_l\left(h_{1j}(t)\right)\right]dt + \sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n} a_{sl}^{ij}(t)\left[\overline{x}_l\left(h_{ij}(t)\right) + \overline{\varphi}_l\left(h_{ij}(t)\right)\right]d\mathscr{B}_i(t)\,(t\geq 0, s = 1,\ldots,n), \tag{34}$$

where $\overline{x}_s(t)$, $s = 1,\ldots,n$, are scalar, $\mathscr{F}_t$-adapted stochastic processes defined on $[-\sigma,\infty)$ satisfying the condition $\overline{x}_s(t) = 0$ for $t<0$, while $\overline{\varphi}_s(t)$, $s = 1,\ldots,n$, are scalar, $\mathscr{F}_0$-measurable stochastic processes defined on $[-\sigma,\infty)$ satisfying the conditions $\overline{\varphi}_s(t) = \varphi_s(t)$ for $t\in[-\sigma,0)$ and $\overline{\varphi}_s(t) = 0$ for $t\in[0,+\infty)$. Due to the assumed property of the existence and uniqueness of the solutions of the initial value problem ((34) and (7)), we can use the notation $\overline{x}(t,b,\overline{\varphi})$ for its solution. It is straightforward to check that $x(t,b,\varphi) = \overline{x}(t,b,\overline{\varphi})$ for $t\geq 0$, where $x(t,b,\varphi)$ is the solution of (5)–(7).

The next step consists in introducing the new variables $y_s(t) = \overline{x}_s(t)\exp\{\lambda t\}$, where $y_s(t)$ is defined on $[-\sigma,\infty)$ and the number λ satisfies the inequalities $0<\lambda<\min\{\overline{a}_s, s = 1,\ldots,n\}$ for all $s = 1,\ldots,n$. Clearly, $y_s(t) = 0$ for $t<0$, and we obtain the following system:

$$dy_s(t) = \left[\lambda y_s(t) - \sum_{j=1}^{m_1}\sum_{l=1}^{n} a_{sl}^{1j}(t)\left[\exp\left\{\lambda\left(t-h_{1j}(t)\right)\right\}y_l\left(h_{1j}(t)\right) + \exp\{\lambda t\}\overline{\varphi}_l\left(h_{1j}(t)\right)\right]\right]dt + \sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n} a_{sl}^{ij}(t)\left[\exp\left\{\lambda\left(t-h_{ij}(t)\right)\right\}y_l\left(h_{ij}(t)\right) + \exp\{\lambda t\}\overline{\varphi}_l\left(h_{ij}(t)\right)\right]d\mathscr{B}_i(t)\,(t\geq 0, s = 1,\ldots,n). \tag{35}$$

Denoting $\eta_s(t) = \sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda(t-h_{1k}(t))\} - \lambda$ for $s = 1,\ldots,n$ and minding $\int_{h_{1k}(t)}^{t} dy_s(\tau) = y_s(t) - y_s(h_{1k}(t))$, $k\in I_s$, transform (35) to the system

$$dy_s(t) = \left[-\eta_s(t)y_s(t) + \sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda(t-h_{1k}(t))\}\int_{h_{1k}(t)}^{t} dy_s(\tau) + \sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda t\}\overline{\varphi}_s(h_{1k}(t)) + \sum_{j=1}^{m_1}\sum_{l=1, l\neq s, j\in I_s}^{n} a_{sl}^{1j}(t)\left[\exp\left\{\lambda\left(t-h_{1j}(t)\right)\right\}y_l\left(h_{1j}(t)\right) + \exp\{\lambda t\}\overline{\varphi}_l\left(h_{1j}(t)\right)\right]\right]dt + \sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n} a_{sl}^{ij}(t)\left[\exp\left\{\lambda\left(t-h_{ij}(t)\right)\right\}y_l\left(h_{ij}(t)\right) + \exp\{\lambda t\}\overline{\varphi}_l\left(h_{ij}(t)\right)\right]d\mathscr{B}_i(t)\,(t\geq 0, s = 1,\ldots,n). \tag{36}$$

Substituting the expression for $dy_s(t)$ from the s-th equation in (35) into the s-th equation in (36), where $s = 1,\ldots,n$, leads to

$$\begin{aligned}dy_s(t) = \Bigg[& -\eta_s(t)y_s(t) + \sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda(t-h_{1k}(t))\}\int_{h_{1k}(t)}^{t}\Bigg\{\Bigg[\lambda y_s(\tau) + \sum_{j=1}^{m_1}\sum_{l=1}^{n} a_{sl}^{1j}(\tau)\left[\exp\left\{\lambda\left(\tau-h_{1j}(\tau)\right)\right\}y_l\left(h_{1j}(\tau)\right)+\exp\{\lambda\tau\}\overline{\varphi}_l\left(h_{1j}(\tau)\right)\right]\Bigg]d\tau \\ & + \sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n} a_{sl}^{ij}(\tau)\left[\exp\left\{\lambda\left(\tau-h_{ij}(\tau)\right)\right\}y_l\left(h_{ij}(\tau)\right)+\exp\{\lambda\tau\}\overline{\varphi}_l\left(h_{ij}(\tau)\right)\right]d\mathscr{B}_i(\tau)\Bigg\} + \sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda t\}\overline{\varphi}_s(h_{1k}(t)) \\ & + \sum_{j=1}^{m_1}\sum_{l=1, l\neq s, j\in I_s}^{n} a_{sl}^{1j}(t)\left[\exp\left\{\lambda\left(t-h_{1j}(t)\right)\right\}y_l\left(h_{1j}(t)\right)+\exp\{\lambda t\}\overline{\varphi}_l\left(h_{1j}(t)\right)\right]\Bigg]dt \\ & + \sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n} a_{sl}^{ij}(t)\left[\exp\left\{\lambda\left(t-h_{ij}(t)\right)\right\}y_l\left(h_{ij}(t)\right)+\exp\{\lambda t\}\overline{\varphi}_l\left(h_{ij}(t)\right)\right]d\mathscr{B}_i(t)\,(t\ge 0, s=1,\dots,n).\end{aligned} \tag{37}$$

Finally, setting $m_s(t,\varsigma) = \exp\left\{-\int_\varsigma^t \mu_s(\zeta)d\zeta\right\}$, $s=1,\dots,n$, and remembering initial condition (7), carry (37) over to the system

$$\begin{aligned}y_s(t) = {} & m_s(t,0)b_s + \sum_{k\in I_s}\int_0^t m_s(t,\varsigma)a_{ss}^{1k}(\varsigma)\exp\{\lambda(\varsigma-h_{1k}(\varsigma))\}\int_{h_{1k}(\varsigma)}^{\varsigma}\lambda y_s(\tau)d\tau\, d\varsigma + \sum_{k\in I_s}\sum_{j=1}^{m_1}\sum_{l=1}^{n}\int_0^t m_s(t,\varsigma)a_{ss}^{1k}(\varsigma)\exp\{\lambda(\varsigma-h_{1k}(\varsigma))\} \\ & \times\int_{h_{1k}(\varsigma)}^{\varsigma} a_{sl}^{1j}(\tau)\left[\exp\left\{\lambda\left(\tau-h_{1j}(\tau)\right)\right\}y_l\left(h_{1j}(\tau)\right)+\exp\{\lambda\tau\}\overline{\varphi}_l\left(h_{1j}(\tau)\right)\right]d\tau\, d\varsigma + \sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n}\int_0^t m_s(t,\varsigma)a_{ss}^{1k}(\varsigma)\exp\{\lambda(\varsigma-h_{1k}(\varsigma))\} \\ & \times\int_{h_{1k}(\varsigma)}^{\varsigma} a_{sl}^{ij}(\tau)\left[\exp\left\{\lambda\left(\tau-h_{ij}(\tau)\right)\right\}y_l\left(h_{ij}(\tau)\right)+\exp\{\lambda\tau\}\overline{\varphi}_l\left(h_{ij}(\tau)\right)\right]d\mathscr{B}_i(\tau)d\varsigma + \sum_{k\in I_s}\int_0^t m_s(t,\varsigma)a_{ss}^{1k}(\varsigma)\exp\{\lambda\varsigma\}\overline{\varphi}_s(h_{1k}(\varsigma))d\varsigma \\ & + \sum_{j=1}^{m_1}\sum_{l=1, l\neq s, j\in I_s}^{n}\int_0^t m_s(t,\varsigma)a_{sl}^{1j}(\varsigma)\left[\exp\left\{\lambda\left(\varsigma-h_{1j}(\varsigma)\right)\right\}y_l\left(h_{1j}(\varsigma)\right)+\exp\{\lambda\varsigma\}\overline{\varphi}_l\left(h_{1j}(\varsigma)\right)\right]d\varsigma \\ & + \sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n}\int_0^t m_s(t,\varsigma)a_{sl}^{ij}(\varsigma)\left[\exp\left\{\lambda\left(\varsigma-h_{ij}(\varsigma)\right)\right\}y_l\left(h_{ij}(\varsigma)\right)+\exp\{\lambda\varsigma\}\overline{\varphi}_l\left(h_{ij}(\varsigma)\right)\right]d\mathscr{B}_i(\varsigma)\,(t\ge 0, i=1,\dots,n).\end{aligned} \tag{38}$$

To obtain estimates for the solutions of (38), we will adopt the following notation:

(i) $\widehat{y}_s = \sup_{t\ge 0}(E|y_s(t)|^{2p})^{1/2p}$, $s=1,\dots,n$.

(ii) $\widehat{\varphi}_s = \operatorname{ess\,sup}_{t<0}(E|\varphi_s(t)|^{2p})^{1/2p}$, $s=1,\dots,n$.

(iii) $\|\varphi\| = \operatorname{ess\,sup}_{t<0}(E|\varphi(t)|^{2p})^{1/2p}$.

In addition, we will use the following inequalities:

(i) $\operatorname{ess\,sup}_{t\ge 0}(E|\exp\{\lambda t\}\overline{\varphi}_l(h_{ij}(t))|^{2p})^{1/2p} \le \exp\left\{\lambda\tau_{ij}\right\}\operatorname{ess\,sup}_{t<0}(E|\varphi_l(t)|^{2p})^{1/2p}$

for $l = 1, \ldots, n$, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$.

(ii) $m_s(t,\varsigma) \le \exp\{-(\overline{a}_s - \lambda)(t - \varsigma)\}, t \in [0,+\infty), \varsigma \in [0,t] P$ – almost surely for all $s = 1, \ldots, n$.

(iii) $\int_0^t \exp\{-(\overline{a}_s - \lambda)(t-\varsigma)\}d\varsigma \le 1/\overline{a}_s - \lambda, s = 1, \ldots, n$.

(iv) $\int_0^t \exp\{-2(\overline{a}_s - \lambda)(t-\varsigma)\}d\varsigma \le 1/2(\overline{a}_s - \lambda), s = 1, \ldots, n$.

Now, from (38) and inequalities (1)–(3), we obtain

$$\begin{aligned}
\widehat{y}_s \le \|b_s\|_{k^1_{2p}} &+ \frac{\lambda}{\overline{a}_s - \lambda}\left[\sum_{k\in I_s} \overline{a}^{1k}_{ss} \exp\{\lambda\tau_{1k}\}\tau_{1k}\right]\widehat{y}_s + \frac{1}{\overline{a}_s - \lambda}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1}\sum_{l=1}^{n} \overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\tau_{1k}\overline{a}^{1j}_{sl}\exp\{\lambda\tau_{1j}\}(\widehat{y}_l + \widehat{\varphi}_l)\right] \\
&+ \frac{c_p}{\overline{a}_s - \lambda}\left[\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n} \overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\sqrt{\tau_{1k}}\,\overline{a}^{ij}_{sl}\exp\{\lambda\tau_{ij}\}(\widehat{y}_l + \widehat{\varphi}_l)\right] + \frac{1}{\overline{a}_s - \lambda}\left[\sum_{k\in I_s}\overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\widehat{\varphi}_s\right] \\
&+ \frac{1}{\overline{a}_s - \lambda}\left[\sum_{j=1}^{m_1}\sum_{l=1, l\ne s, j\in I_s}^{n} \overline{a}^{1j}_{sl}\exp\{\lambda\tau_{1j}\}(\widehat{y}_l + \widehat{\varphi}_l)\right] \\
&+ \frac{c_p}{\sqrt{2(\overline{a}_s - \lambda)}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n}\overline{a}^{ij}_{sl}\exp\{\lambda\tau_{ij}\}(\widehat{y}_l + \widehat{\varphi}_l)\right], s = 1, \ldots, n.
\end{aligned} \tag{39}$$

From the definition, we have that $\widehat{\varphi}_j \le \|\varphi\|$, $j = 1, \ldots, n$. Hence, (39) yields

$$\widehat{y}_s \le \|b_s\|_{k^1_{2p}} + \sum_{l=1}^{n} N_{sl}(\lambda)\widehat{y}_l + M_s(\lambda)\|\varphi\| s = 1, \ldots, n, \tag{40}$$

where

$$\begin{aligned}
N_{ss}(\lambda) :=& \frac{\lambda}{\overline{a}_s - \lambda}\left[\sum_{k\in I_s}\overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\tau_{1k}\right] \\
&+ \frac{1}{\overline{a}_s - \lambda}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1}\overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\tau_{1k}\overline{a}^{1j}_{ss}\exp\{\lambda\tau_{1j}\} + c_p\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\sqrt{\tau_{1k}}\,\overline{a}^{ij}_{ss}\exp\{\lambda\tau_{ij}\}\right. \\
&\left.+ \sum_{j=1, j\notin I_s}^{m_1}\overline{a}^{1j}_{ss}\exp\{\lambda\tau_{1j}\}\right] + \frac{c_p}{\sqrt{2(\overline{a}_s - \lambda)}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}^{ij}_{ss}\exp\{\lambda\tau_{ij}\}\right], s = 1, \ldots, n, \\
N_{sl}(\lambda) :=& \frac{1}{\overline{a}_s - \lambda}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1}\overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\tau_{1k}\overline{a}^{1j}_{sl}\exp\{\lambda\tau_{1j}\}\right. \\
&\left.+ c_p\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\sqrt{\tau_{1k}}\overline{a}^{ij}_{sl}\exp\{\lambda\tau_{ij}\} + \sum_{j=1}^{m_1}\overline{a}^{1j}_{sl}\exp\{\lambda\tau_{1j}\}\right] \\
&+ \frac{c_p}{\sqrt{2(\overline{a}_s - \lambda)}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}^{ij}_{sl}\exp\{\lambda\tau_{ij}\}\right], s, j = 1, \ldots, n, s \ne l, \\
M_s(\lambda) :=& \frac{1}{\overline{a}_s - \lambda}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1}\sum_{l=1}^{n}\overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\tau_{1k}\overline{a}^{1j}_{sl}\exp\{\lambda\tau_{1j}\}\right. \\
&+ c_p\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n}\overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\sqrt{\tau_{1k}}\overline{a}^{ij}_{sl}\exp\{\lambda\tau_{ij}\} \\
&+ \sum_{k\in I_s}\overline{a}^{1k}_{ss}\exp\{\lambda\tau_{1k}\}\left[+\sum_{j=1}^{m_1}\sum_{l=1, l\ne s, j\in I_s}^{n}\overline{a}^{1j}_{sl}\exp\{\lambda\tau_{1j}\}\right] + \frac{c_p}{\sqrt{2(\overline{a}_s - \lambda)}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{l=1}^{n}\overline{a}^{ij}_{sl}\exp\{\lambda\tau_{ij}\}\right], s = 1, \ldots, n.
\end{aligned} \tag{41}$$

Put now $y(t) = col(y_1(t), \ldots, y_n(t))$, $\overline{y} = col(\overline{y}_1, \ldots, \overline{y}_n)$, and $M(\lambda) = col(M_1(\lambda), \ldots, M_n(\lambda))$ and let $C(\lambda) = (c_{ij}(\lambda))_{i,j=1}^n$ be an $n \times n$ –matrix with the entries given by

$$c_{ss}(\lambda) = 1 - N_{ss}(\lambda), s = 1, \ldots, n, c_{sl}(\lambda) = -N_{sl}(\lambda), s, l = 1, \ldots, n, s \neq l. \tag{42}$$

From (40), we then deduce the following estimate:

$$C(\lambda)\overline{y} \leq \|b\|_{k_{2p}^n} \overline{e} + \|\varphi\| M(\lambda), \tag{43}$$

where $\overline{e} = col(1, \ldots, 1)$. Evidently, $C(0) = C$. According to the assumptions of the theorem, C is an $\mathcal{M}$–matrix, so that $C(\lambda_0)$ is also an $\mathcal{M}$–matrix for small $\lambda_0 > 0$. Thus, from (43) and Theorem 3, we obtain

$$|\overline{y}| \leq K\left(\|b\|_{k_{2p}^n} + \|\varphi\|\right), \tag{44}$$

for some constant K. Combining the substitution $x(t, b, \varphi) = \exp\{-\lambda t\} y(t)$ with the inequalities $\sup_{t\geq 0} |(Ey(t))^{2p}|^{1/2p} \leq |\overline{y}|$ and (44), we get the estimate

$$\left(E|x(t,b,\varphi)|^{2p}\right)^{1/2p} \leq K \exp\{-\lambda t\}\left(\|b\|_{k_{2p}^n} + \operatorname{ess\,sup}_{t<0}\left(E|\varphi(t)|^{2p}\right)^{1/2p}\right)(t \in [0,\infty)). \tag{45}$$

Hence, equation (5) is exponentially $2p$–stable in the sense of Definition 2.

The theorem is proven. □

Remark 3. Whether or not the matrix C is an $\mathcal{M}$–matrix can be verified by calculation of its principal minors: if all of them are positive, then C is an $\mathcal{M}$–matrix. Otherwise, one can use the sufficient conditions described right after Definition 1.

6. Some Corollaries

In this section, we produce some sufficient conditions for exponential stability of (5).

Let us start with the following particular case of equation (5):

$$dx(t) = -\sum_{j=1}^{m_1} A_{1j}(t)x\left(h_{1j}(t)\right)dt\,(t \geq 0). \tag{46}$$

We define the $n \times n$–matrix $C^{(1)} = (c_{sl})$ as follows:

$$c_{ss} = 1 - \frac{1}{\overline{a}_s}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1} \tau_{1k}\overline{a}_{ss}^{1k}\overline{a}_{ss}^{1j} + \sum_{j=1, j\notin I_s}^{m_1} \overline{a}_{ss}^{1j}\right], \quad s = 1, \ldots, n,$$

$$c_{sl} = -\frac{1}{\overline{a}_s}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1} \tau_{1k}\overline{a}_{ss}^{1k}\overline{a}_{sl}^{1j} + \sum_{j=1}^{m_1}\overline{a}_{sl}^{1j}\right], \quad s, l = 1, \ldots, n, s \neq l. \tag{47}$$

From Theorem 4, we readily obtain the following.

Corollary 1. *If condition (C1) with $i = 1$ and condition (C2) are fulfilled and $C^{(1)}$ is an $\mathcal{M}$–matrix, then equation (46) is exponentially $2p$–stable in the sense of Definition 2.*

In particular, for the system

$$dx(t) = -A_{11}(t)x\left(h_{11}(t)\right)dt\,(t \geq 0), \tag{48}$$

we obtain the following.

Corollary 2. *If condition (C1) with $i = 1$, $j = 1$ and condition (C2) with $m_1 = 1$, $I_s = \{1\}\,(s = 1, \ldots, n)$ are fulfilled and $C^{(2)} = (c_{sl})$ given by*

$$c_{ss} = 1 - \frac{\left(\overline{a}_{ss}^{11}\right)^2\tau_{11}}{\overline{a}_s}, \quad s = 1, \ldots, n,$$

$$c_{sl} = -\frac{\left(\overline{a}_{ss}^{11}\right)^2\tau_{11} + \overline{a}_{sl}^{11}}{\overline{a}_s}, \quad s, l = 1, \ldots, n, s \neq l, \tag{49}$$

is an $\mathcal{M}$–matrix, then equation (48) is exponentially $2p$–stable in the sense of Definition 2.

Remark 4. Systems (46) and (48) consist of random delay equations or, if the coefficients $A_{1j}, j = 1, .., m_1$ are all nonrandom, they become deterministic delay systems. Even in these cases, the results in the two above corollaries seem to be new.

Let us now consider the following system of stochastic delay equations:

$$dx(t) = -A_{11}(t)x\left(h_{11}(t)\right)dt + \sum_{i=2}^{m}\sum_{j=1}^{m_i} A_{ij}(t)x\left(h_{ij}(t)\right)d\mathcal{B}_i(t)\,(t \geq 0). \tag{50}$$

We define the matrix $C^{(3)} = (c_{sl})$ as

$$c_{ss} = 1 - \frac{1}{\overline{a}_s}\left[\left(\overline{a}_{ss}^{1k}\right)^2\tau_{11} + c_p\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sqrt{\tau_{11}}\,\overline{a}_{ss}^{11}\overline{a}_{ss}^{ij}\right] - \frac{c_p}{\sqrt{2\overline{a}_s}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}_{ss}^{ij}\right], \quad s = 1, \ldots, n, \tag{51}$$

$$c_{sl} = -\frac{1}{\overline{a}_s}\left[\overline{a}_{ss}^{11}\overline{a}_{sl}^{11}\tau_{11} + \overline{a}_{sl}^{11} + \sum_{i=2}^{m}\sum_{j=1}^{m_i}\sqrt{\tau_{11}}\,\overline{a}_{ss}^{11}\overline{a}_{sl}^{ij}\right] - \frac{c_p}{\sqrt{2\overline{a}_s}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}_{sl}^{ij}\right], \quad s, l = 1, \ldots, n, s \neq l. \tag{52}$$

Corollary 3. *If condition (C1) and condition (C2) with $m_1 = 1$, $I_s = \{1\} (s = 1, \dots, n)$ are fulfilled and $C^{(3)}$ is an $\mathscr{M}$–matrix, equation (50) is exponentially $2p$–stable in the sense of Definition 2.*

In particular, we have the following.

Corollary 4. *If $n = 2$, condition (C1) and condition (C2) with $m_1 = 1$, $I_s = \{1\} (s = 1, 2)$ are fulfilled and $c_{11} > 0$, $c_{11}c_{22} > c_{12}c_{21}$, where $c_{sl} (s, l = 1, 2)$ are defined in (51) and (52); then, equation (50) is exponentially $2p$–stable in the sense of Definition 2.*

Indeed, in this case, the matrix $C^{(3)}$ with $n = 2$ is an $\mathscr{M}$–matrix, as its principal minors are positive.

Corollary 5. *If condition (C1) and condition (C2) with $m_1 = 1$, $I_s = \{1\} (s = 1, \dots, n)$ are fulfilled and $C^{(4)} = (c_{sl})$ given as*

$$c_{ss} = 1 - \frac{c_p}{\sqrt{2\overline{a}_s}} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \overline{a}_{ss}^{ij}, \quad s = 1, \dots, n,$$
$$c_{sl} = -\frac{1}{\overline{a}_s} \overline{a}_{sl}^{11} - \frac{c_p}{\sqrt{2\overline{a}_s}} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \overline{a}_{sl}^{ij}, \quad s, l = 1, \dots, n, s \neq l, \tag{53}$$

is an $\mathscr{M}$–matrix, equation (50) is exponentially $2p$–stable in the sense of Definition 2.

In particular, we obtain the following.

Corollary 6. *If $n = 2$, $h_{11}(t) = t\,(t \in [0, \infty))\mu$–almost everywhere, condition (C1) and condition (C2) with $m_1 = 1$, $I_s = \{1\} (s = 1, 2)$ are fulfilled and, finally, if*

$$\frac{c_p}{\sqrt{2\overline{a}_1}} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \overline{a}_{11}^{ij} > 1,$$
$$\left[1 - \frac{c_p}{\sqrt{2\overline{a}_1}} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \overline{a}_{11}^{ij}\right]\left[1 - \frac{c_p}{\sqrt{2\overline{a}_2}} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \overline{a}_{22}^{ij}\right] > \left[\frac{1}{\overline{a}_1}\overline{a}_{1l}^{11} + \frac{c_p}{\sqrt{2\overline{a}_1}} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \overline{a}_{1l}^{ij}\right]\left[\frac{1}{\overline{a}_2}\overline{a}_{2l}^{11} + \frac{c_p}{\sqrt{2\overline{a}_2}} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \overline{a}_{2l}^{ij}\right], \tag{54}$$

then equation (50) is exponentially $2p$–stable in the sense of Definition 2.

This follows from Corollary 5 and the observation that the principal minors of the matrix defined by (51) and (52) are in this case positive.

7. Examples

Consider the deterministic system

$$dx(t) = -\sum_{j=1}^{m} A_j x\left(t - h_j\right) dt\,(t \geq 0), \tag{55}$$

where

$$A_j = \left(a_{sl}^j\right)_{s,l=1}^n, \quad j = 1, \dots, m, \tag{56}$$

are the real-valued $n \times n$–matrices and h_j, $j = 1, \dots, m$, are nonnegative real numbers.

Example 1. Assume that

(i) $\sum_{j=1}^m a_{ss}^j = a_s > 0, s = 1, \dots, n.$

(ii) $c_{ss} = 1 - 1/a_s \sum_{k=1}^m \sum_{j=1}^m h_k |a_{ss}^k||a_{ss}^j|, s = 1, \dots, n.$

(iii) $c_{sl} = -1/a_s [\sum_{k=1}^m \sum_{j=1}^m h_k |a_{ss}^k||a_{sl}^j| + \sum_{j=1}^m |a_{sl}^j|], s, l = 1, \dots, n, s \neq l.$

Then, from Corollary 1, we obtain that equation (55) is exponentially stable with respect to initial data if the $n \times n$–matrix $C^{(5)} = (c_{sl})$ with c_{sl} defined in Example 1 is an $\mathscr{M}$–matrix.

Example 2. Assume that in (55), $h_1 = 0$, $a_{ss}^1 > 0, s = 1, \dots, n$, and the $n \times n$–matrix $C^{(6)} = (c_{sl})$ defined by

$$c_{ss} = 1 - \frac{1}{a_{ss}^1} \sum_{j=2}^{m} \left|a_{ss}^j\right|, \quad s = 1, \dots, n,$$
$$c_{sl} = -\frac{1}{a_{ss}^1} \sum_{j=1}^{m} \left|a_{sl}^j\right|, \quad s, l = 1, \dots, n, s \neq l, \tag{57}$$

is an $\mathscr{M}$ –matrix. Then, from Corollary 1, we obtain that equation (55) is exponentially stable with respect to initial data.

It is straightforward to observe that the entries $c_{sl}, s, l = 1, \dots, n$ of the matrix $C^{(6)}$ defined in (57) satisfy the estimates

$$c_{ss} > \sum_{l=1, l\neq s}^{n} \left|c_{sl}\right|, \quad s = 1, \dots, n. \tag{58}$$

Thus, $C^{(6)}$ is an $\mathscr{M}$–matrix, and equation (55) becomes exponentially stable with respect to initial data.

In particular, the deterministic system

$$dx(t) = -Ax(t)dt\,(t \geq 0), \tag{59}$$

is exponentially stable if $a_{ss}^1 > \sum_{l=1, l \neq s}^n |a_{sl}^1|, s = 1, \dots, n.$

Example 3. Consider the following system of linear Itô equations with constant delays:

$$dx(t) = -\sum_{j=1}^{m_1} A_{1j}x\left(t - h_{1j}\right)dt + \sum_{i=2}^{m}\sum_{j=1}^{m_i} A_{ij}x\left(t - h_{ij}\right)d\mathscr{B}_i(t)\,(t \geq 0), \tag{60}$$

where

$$A_{ij} = \left(a_{sl}^{ij}\right)_{s,l=1}^{n}, i = 1, \ldots, m, j = 1, \ldots, m_i, \tag{61}$$

are the real-valued $n \times n$–matrices and h_{ij}, $i = 1, \ldots, m,\ j = 1, \ldots, m_i$, are nonnegative real numbers.

We define the $n \times n$–matrix $C^{(7)} = (c_{sl})$ by

$$\begin{aligned} c_{ss} &= 1 - \frac{1}{a_s}\left[\sum_{k=1}^{m_1}\sum_{j=1}^{m_1} h_{1k}\left|a_{ss}^{1k}\right|\left|a_{ss}^{1j}\right| + c_p\sum_{k=1}^{m_1}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sqrt{h_{1k}}\left|a_{ss}^{1k}\right\|a_{ss}^{ij}\right|\right] \\ &\quad - \frac{c_p}{\sqrt{2a_s}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\left|a_{ss}^{ij}\right|\right], \quad s = 1, \ldots, n, \\ c_{sl} &= -\frac{1}{a_s}\left[\sum_{k=1}^{m_1}\sum_{j=1}^{m_1} h_{1k}\left|a_{ss}^{1k}\right|\left|a_{sl}^{1j}\right| + c_p\sum_{k=1}^{m_1}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sqrt{h_{1k}}\left|a_{ss}^{1k}\right|\left|a_{sl}^{ij}\right| + \sum_{j=1}^{m_1}\left|a_{sl}^{1j}\right|\right] \\ &\quad - \frac{c_p}{\sqrt{2a_s}}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\left|a_{sl}^{ij}\right|\right], \quad s, l = 1, \ldots, n, s \neq l. \end{aligned} \tag{62}$$

If $C^{(7)} = (c_{sl})$ is an $\mathscr{M}$–matrix and

$$\sum_{j=1}^{m_1} a_{ss}^{1j} = a_s > 0, \quad s = 1, \ldots, n, \tag{63}$$

then system (60) is exponentially $2p$–stable in the sense of Definition 2.

Example 4. Consider the system

$$dx(t) = -A_{11}x(t)dt + \sum_{i=2}^{m}\sum_{j=1}^{m_i} A_{ij}x\left(t - h_{ij}\right)d\mathscr{B}_i(t)\,(t \geq 0), \tag{64}$$

and define the $n \times n$–matrix $C^{(8)} = (c_{sl})$ by

$$\begin{aligned} c_{ss} &= 1 - \frac{c_p}{\sqrt{2a_{ss}^{11}}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\left|a_{ss}^{ij}\right|, \quad s = 1, \ldots, n, \\ c_{sl} &= -\frac{\left|a_{sl}^{11}\right|}{a_{ss}^{11}} - \frac{c_p}{\sqrt{2a_{ss}^{11}}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\left|a_{sl}^{ij}\right|, \quad s, l = 1, \ldots, n, s \neq l. \end{aligned} \tag{65}$$

From Theorem 4, it follows that if $C^{(8)}$ is an $\mathscr{M}$–matrix and $a_{ss}^{11} > 0,\ s = 1, \ldots, n$, then equation (64) is exponentially $2p$–stable in the sense of Definition 2. In particular, $C^{(8)}$ becomes an $\mathscr{M}$–matrix if

$$1 - \frac{c_p}{\sqrt{2a_{ss}^{11}}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\left|a_{ss}^{ij}\right| > \sum_{l=1}^{n}\left[\frac{\left|a_{sl}^{11}\right|}{a_{ss}^{11}} + \frac{c_p}{\sqrt{2a_{ss}^{11}}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\left|a_{sl}^{ij}\right|\right], s = 1, \ldots, n. \tag{66}$$

8. Conclusions and Outlook

We described a general framework to study exponential stability of linear functional differential stochastic equations combining the regularization technique with the theory of $\mathscr{M}$–matrices. We demonstrated how this techniques can be directly applied to stochastic Itô equations with variable delays and random coefficients, i.e., in the cases where the Lyapunov-like functionals may be difficult to find. In addition to the general stability result, we provided several specific stability conditions, including those for deterministic systems with delays. The results clearly show that introducing the time delays into a system deteriorates its stability properties in most cases. However, in some special situations (see Corollaries 4–6 and Example 4), our methods provide delay-independent stability conditions.

The suggested framework is by no means thought to replace the Lyapunov method and its stochastic counterparts. Rather, our method may serve as an alternative in the situations where a direct application of the Lyapunov method seems to be more difficult (some examples can be found in the introductory section). On the other hand, our approach is not yet well developed to study global stability of nonlinear differential equations, and the main reason for that is the absence of the complete theory of the stochastic input-to-state stability, which the regularization method is based upon. Therefore, the efficiency of the method should be investigated further. For this purpose, it may be appropriate to study the following problems:

(i) Stability in the first approximation. This is an important problem in many models, for instance, in connection with extinction of populations in the random environment. The techniques developed in the present paper can be directly applied to this problem.

(ii) Asymptotic stability that is not exponential stability. This property is typical for equations with unbounded delays, for instance, for the stochastic pantograph equation [15]. The regularization method in its conventional setting has been used in this case, so that it is quite realistic to assume that combining it with the theory of $\mathscr{M}$–matrices will not be difficult.

(iii) Extension of the analysis based on $\mathscr{M}$–matrices using nonlinear reference equations, as it is suggested in the paper [20]. Such constructions are believed to be challengeable, as it is known from the deterministic theory of functional differential equations [9]. However, the breakthrough ideas from [20] can help to overcome some of the obstacles, and not only in the stochastic case, but also in the deterministic case.

(iv) Applications to specific time-continuous stochastic models from various fields, e.g., stochastic networks, population dynamics, and analysis of neural fields. In this case, a graphical comparison with the existing results should be an essential part of the analysis.

(v) It is highly desirable to apply the theory of $\mathscr{M}$–matrices to analyse time-discrete stochastic systems, especially those arising in the control theory. A popular and challengeable example is given by the Roesser model described by linear, time-delayed systems. The Lyapunov function approach, frequently used to study these systems (see, e.g., [6, 23]), may be complemented by the techniques developed in the present paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to thank the Department of Sciences and Technology, Norwegian University of Life Sciences, for partial financial support.

References

[1] X. Mao, *Stochastic Differential Equations & Applications*, Horwood Publishing ltd, Chichester, England, 1997.

[2] A. N. Bishop and P. Del Moral, "Stability properties of systems of linear stochastic differential equations with random coefficients," pp. 1–16, 2018, https://arxiv.org/abs/1804.09349.

[3] E. Chin, S. Ólafsson, and D. Nel, *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Wiley, New Jersey, NY, USA, 2014.

[4] Q. Meng and Y. Shen, "Optimal control for stochastic delay evolution equations," *Applied Mathematics and Optimization*, vol. 74, no. 1, pp. 53–89, 2016.

[5] M. Song and X. Mao, "Almost sure exponential stability of hybrid stochastic functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 458, no. 2, pp. 1390–1408, 2018.

[6] X. Zhang, H. Wang, V. Stojanovic et al., "Asynchronous fault detection for interval type-2 fuzzy nonhomogeneous higher-level markov jump systems with uncertain transition probabilities," *IEEE Transactions on Fuzzy Systems*, p. 1, 2021.

[7] W. Qi, G. Zong, and W. X. Zheng, "Adaptive event-triggered SMC for stochastic switching systems with semi-markov process and application to boost converter circuit model," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 68, no. 2, pp. 786–796, 2021.

[8] S. Cai, Y. Cai, and X. Mao, "A stochastic differential equation SIS epidemic model with regime switching," *Discrete & Continuous Dynamical Systems - B*, vol. 26, no. 9, pp. 4887–4905, 2021.

[9] N. V. Azbelev and P. M. Simonov, *Stability of Differential Equations with Aftereffect*, Taylor & Francis, London, UK, 2002.

[10] L. Berezansky and E. Braverman, "A note on stability of Mackey-Glass equations with two delays," *Journal of Mathematical Analysis and Applications*, vol. 450, no. 2, pp. 1208–1228, 2017.

[11] L. Berezansky and E. Braverman, "On stability of delay equations with positive and negative coefficients with applications," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 38, no. 2, pp. 157–189, 2019.

[12] A. Domoshnitsky and R. Shklyar, "Positivity for non-Metzler systems and its applications to stability of time-varying delay systems," *Systems & Control Letters*, vol. 118, pp. 44–51, 2018.

[13] L. Berezansky, E. Braverman, and L. Idels, "New global exponential stability criteria for nonlinear delay differential systems with applications to BAM neural networks," *Applied Mathematics and Computation*, vol. 243, pp. 899–910, 2014.

[14] L. Idels, R. I. Kadiev, and A. Ponosov, "Stability of high order linear Itô equations with delays," *Applied Mathematics*, vol. 9, no. 3, Article ID 83417, 2018.

[15] R. I. Kadiev and A. Ponosov, "Lyapunov stability of the generalized stochastic pantograph equation," *Journal of Mathematics*, vol. 2018, Article ID 7490936, 9 pages, 2018.

[16] R. I. Kadiev and A. Ponosov, "Input-to-state stability of linear stochastic functional differential equations," *J Funct Spaces*, vol. 2016, Article ID 8901563, 12 pages, 2016.

[17] R. I. Kadiev and A. V. Ponosov, "Positive invertibility of matrices and stability of Itô delay differential equations," *Differential Equations*, vol. 53, no. 5, pp. 571–582, 2017.

[18] Q. Zhu, "Stability analysis of stochastic delay differential equations with Lévy noise," *Systems & Control Letters*, vol. 118, pp. 62–68, 2018.

[19] Q. Zhu, "Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control," *IEEE Transactions on Automatic Control*, vol. 64, no. 9, pp. 3764–3771, 2019.

[20] Q. Zhu and T. Huang, "Stability analysis for a class of stochastic delay nonlinear systems driven by G-Brownian motion," *Systems & Control Letters*, vol. 140, Article ID 104699, 2020.

[21] R. J. Plemmons, "M-Matrix characterizations. I – nonsingular M-matrices," *Linear Algebra Appl*, vol. 18, no. 2, pp. 175–188, 1977.

[22] R. Liptser and A. Shiryaev, *Theory of Martingales*, Springer, Berlin, Germany, 1989.

[23] P. Cheng, S. He, X. Luan, and F. Liu, "Finite-region asynchronous H∞ control for 2D Markov jump systems," *Automatica*, vol. 129, Article ID 109590, 2021.