DEMAND SYSTEMS AND FREQUENCY OF PURCHASE MODELS[*]

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ABSTRACT

Understanding the frequencies of purchase and the average purchased quantities is important for marketing strategies such as the loss-leader pricing strategy. We develop a microeconomic model where the total purchases of goods are determined by demand systems for the frequencies of purchases and the average quantities purchased, conditional on positive purchase frequencies. An econometric model is developed to estimate the two systems by a Bayesian estimation method, which allows for an unrestricted covariance structure within each system. An empirical example demonstrates how the estimated model can be used to formulate a profitable loss-leader pricing strategy for fish in France.

Keywords: Demand systems, fish, purchase frequencies, multivariate gamma and truncated Poisson log-normal distributions.

JEL: C34, D12, M31.

1. INTRODUCTION

A widely-used marketing strategy is the loss-leader pricing strategy (e.g., Kemp, 1955; Hess and Gerstner, 1987; Lal and Matutes, 1994; Ellison, 2005; Chen and Rey, 2012; In and Wright, 2014). This strategy implies that a retailer reduces the price of a product below its marginal cost to attract customers to the store. In many situations, consumers buy multiple categories of goods and find it convenient to buy them from a single store. Therefore, a decrease in the price of one category may lead a customer to transfer many of their category purchases to this store (Thomassen et al., 2017). According to Chen and Rey (2012), the loss-leader strategy is mainly used by large retailers who are competing with smaller retailers with a limited variety of products. In this situation, large retailers may find it profitable to sell some products for prices below the marginal costs, and other products for prices above the marginal costs.[1]

Knowledge about purchase frequencies and purchased quantities can be used to formulate profitable loss-leader pricing strategies. For example, by reducing the price of a frequently-purchased product that is price elastic in the frequency of purchase, a retailer can attract more customers to the store. By simultaneously increasing the price of another product with high and positive cross-price elasticity in the frequency of purchase with respect to the first product, the retailer can increase profits.

Meghir and Robin (1992) presented a model for a consumer who chooses the purchasing frequency as well as the purchased quantities on each purchasing occasion. They assumed that any zero purchases are generated from infrequency of purchases and would be recorded as positive purchases given a longer observation period. Consequently, observed purchases differ from desired purchases, so they developed a two-step estimator to estimate desired purchases. In the first step, frequencies of purchase were estimated by a generalized linear model to obtain weight parameters,[2] which were used to calculate the desired purchases. In the second step, the desired purchases were used to estimate the purchased quantities within a demand system.

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[1] Consumer purchase frequencies have frequently been modeled by count data models such as the Poison and negative binomial models to estimate brand success, brand loyalty and store choice (e.g., Keng and Ehrenberg, 1984; Uncles et al., 1995; Bhattacharya, 1997; Uncles and Lee, 2006). Count data models such as the Poisson and negative binomial models have also been used to study demand issues in environmental economics health economics, and finance. Examples include Smith (1988), Creel and Loomis (1990) and Hellerstein (1991), who all used count data models in the estimation of the demand for recreation; Deb and Trivedi (2002), who estimated the number of doctor visits; Munkin and Trivedi (1999) and Wang (2003), who estimated the demand for health care; and Davutyan (1989), who estimated the elasticities of important factors such as bank failures. However, a demand system was not used in any of these applications.

[2] Generally, a Poisson or negative binomial model would be preferred for such an estimation.

We extend the work of Meghir and Robin (1992) in two ways. First, they did not derive their model from a constrained optimization problem, so we develop a theoretical model based on a constrained optimization problem. Our model results in a frequency of purchase model of the form that was used by Meghir and Robin (1992). We follow them and divide the consumer's purchase decision into a decision of frequency of purchase and a decision of quantity to purchase conditional on the purchase frequency. Total purchased quantity of a good is then given as the product of the frequency of purchase and the average quantity purchased. The consumer's choice variables are therefore how often to buy different products and how much to buy, on average, of the products on each occasion. In our model, homogeneity of degree zero holds for purchase frequencies, average quantities and total quantities. Furthermore, the associated matrices of substitution effects are symmetric.

Second, Meghir and Robin (1992) focused on how to adjust for the frequency of purchase to obtain consistent parameter estimates in a demand system under an infrequency of purchase assumption.[3] Their frequency of purchase model was a basic Poisson system. In this system, homogeneity and symmetry were not imposed, the problem of zero inflation was not addressed[4] and the covariance structure was assumed to be zero. We extend their count data estimation framework by: (i) accounting for homogeneity and symmetry in the frequency of the purchase demand system; (ii)

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[3] In applied demand analysis for disaggregative products, zero purchases represent a major problem. Usually, it is assumed that zero purchases either represent traditional corner solutions, infrequency of purchase or non-preference for a product (e.g., Wales and Woodland, 1983; Tiffin and Arnoult, 2010). To account for zero purchases in demand system estimation, a two-step model is usually estimated. In the first step, it is estimated whether a product is purchased. The results of this step are used to correct the estimated demand system in the second step to obtain consistent parameter estimates. However, the actual frequencies of purchase are not estimated.

[4] When the share of zeros in a data set significantly exceeds the amount predicted by conventional distributions, such as the Poisson, it is called zero inflation.

accounting for zero inflation by assuming a truncated data-generating process for the counts; and (iii) allowing for an unrestricted covariance structure within the two demand systems. We assume a truncated multivariate Poisson log-normal (TMPLN) distribution for the counts and a multivariate gamma log-normal (MGLN) distribution for the average quantities. To estimate these distributions, we use Bayesian estimation methods, specifically a random walk Metropolitan simulation algorithm.

We provide an empirical illustration on the potential usefulness of this type of model. Our example uses French scanner data for purchases of fresh fish, and the demand system includes fresh salmon, fresh white fish and other fresh fish. Our results indicate that fresh salmon is a good loss-leader candidate. Large retailers who sell a wide variety of products, including a large selection of fresh white fish, could price salmon below its marginal cost and advertise it to attract customers to the store, and then discreetly price other products above the market price to compensate for the loss on salmon. Furthermore, a smaller retailer who sells only fresh salmon and white fish may be unable to compete with these prices and subsequently driven out of the market.

This paper is organized as follows. In Sections 2 and 3, our theoretical and statistical models are developed, respectively. In Section 4, our data set and empirical specifications are described. In Section 5, our empirical results are presented and an illustrative example on how the estimated values can be used to formulate a profitable loss-leader pricing strategy is provided. In Section 6, we conclude.

2. THEORETICAL MODEL

We follow the specification and notation in Meghir and Robin (1992) as far as possible. However, we divide the decision of how much to purchase of each good $x =$

$(x_1, x_2, \dots, x_M)$ into two parts: the frequencies of purchase $n = (n_1, n_2, \dots, n_M)$ and the average quantities purchased on each occasion, $q = (q_1, q_2, \dots, q_M)$. By definition, the identity $x(nq) \equiv n_i q_i$ holds for $i = 1,2, \dots, M$.[5] The consumer is assumed to obtain utility from the purchased goods, leisure l and frequency of purchase, and the utility function is specified as $v(x, n, l) = v(nq, n, l)$ where $nq = (n_1 q_1, n_2 q_2, \dots, n_M q_M)$. The utility function is assumed to be strictly quasiconcave in n, q and l.[6]

The consumer has wage income $y = wk$, where w is the hourly wage rate and k is the number of hours spent at work. The consumer may also have other types of income R, which we assume is exogenously given. The consumer's budget constraint is $y + R = p'nq$, where $p = (p_1, p_2, \dots, p_M)$ is the price vector. The consumer has a constant time endowment T, which can be allocated between leisure, work and purchasing goods. The time spent on purchasing goods is given by the function $g(n)$, which is assumed to be increasing in n.[7] The consumer's time constraint is $T = l + k + g(n)$. The consumer's utility maximization problem is specified as:

$$\max_{n,q,l}\{v(x(nq), n, l): wT + R = p'nq + wl + wg(n), n > 0, q > 0, l > 0\}. \quad (1)$$

It follows from the first-order conditions of Equation (1) that the solution satisfies:[8]

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[5] We assume that n is a latent variable of actual observed purchase frequencies, which can take any positive value, and, consequently, we can take the derivatives with respect to n.

[6] Meghir and Robin (1992) provide two arguments for positive marginal utilities in frequencies of purchase. First, there is a benefit in the form of saved space by having to hold smaller stocks of various goods. Second, freshness is important for many types of food, including fish. The costs of frequent purchases will be reflected as lost leisure time.

[7] The consumer may purchase many goods at each occasion, and the function $g(n)$ is increasing at a decreasing rate with the number of products purchased.

[8] For simplicity, corner solutions are not considered in the theoretical model. The zeros in the data are assumed to be generated by infrequency of purchase and non-preference. The derivation and interpretation of the first-order conditions of Equation (1) are given in Appendix 1. Furthermore, the wage rate is assumed to be equal to the opportunity cost of time, which is spent on either leisure or shopping.

$$\frac{\partial v/\partial x_i}{\partial v/\partial x_j} = \frac{p_i}{p_j}\ \forall i,j \tag{2}$$

$$\frac{\partial v/\partial l}{\partial v/\partial q_i} = \frac{w}{p_i}\ \forall i,j \tag{3}$$

$$\frac{\partial v/\partial n_i}{\partial v/\partial l} = \frac{\partial g}{\partial n_i}\ \forall i \tag{4}$$

$$\frac{\partial v/\partial n_i}{\partial v/\partial n_j} = \frac{\partial g/\partial n_i}{\partial g/\partial n_j}\ \forall i,j. \tag{5}$$

Equations (2) and (3) are the standard first-order conditions of the consumer utility maximization problem with leisure. Equation (4) implies that the ratio between the marginal utility of shopping frequency for good *i* and the marginal utility of leisure equals the marginal cost of time spent on purchasing good *i*. Equation (5) implies that the ratio of marginal utilities of shopping frequencies for goods *i* and *j* are equal to the ratio of marginal costs of time spent on purchasing goods *i* and *j*.

The solution of the first-order conditions of Equation (1) results in three sets of uncompensated demand functions: $n(p,w,R)$, $q(p,w,R)$ and $l(p,w,R)$. The total purchased quantities, $x(p,w,R)$, are found by substituting $n(p,w,R)$ and $q(p,w,R)$ into the identity $x(nq) \equiv n_i q_i$.

The dual problem to the consumer's utility maximization problem (1) is: $\min_{n,q,l}\left\{R = p'nq - w\big(T - l - g(n)\big) \colon v(x(nq),n,l) = v^*, n > 0, q > 0, l > 0\right\}$. The solution to the first-order conditions of this problem results in three sets of compensated demand functions: $n^c(p,w,v^*)$, $q^c(p,w,v^*)$ and $l^c(p,w,v^*)$.[9] The compensated

[9] The derivation of the first-order conditions of the minimization problem is given in Appendix 1.

demand functions for total purchased quantities, $x^c(p, w, v^*)$, are found by substituting $n^c(p, w, v^*)$ and $q^c(p, w, v^*)$ into the identity $x_i^c \equiv n_i^c q_i^c$.

The conditions for homogeneity of degree zero in prices and income, symmetry and negativity for the compensated and uncompensated demand functions are summarized in:[10]

Proposition 1.1

The demand equations $n(p, w, R)$, $q(p, w, R)$, $l(p, w, R)$ and $x(p, w, R)$ are homogeneous of degree zero in (p, w, R).

Proposition 1.2

The matrix of compensated substitution effects for $x^c(p, w, v^*)$ is symmetric and negative semidefinite.

Proposition 1.3

The matrix of compensated substitution effects for the product $n^c(p, w, v^*)q^c(p, w, v^*)$ is symmetric and negative semidefinite.

3. STATISTICAL MODEL

The frequency of shopping $n_i = (n_{i11}, n_{i12}, \dots, n_{iKT})$ is assumed to follow a discrete distribution, $f_{Ni}(n_i|\beta_i, C)$, for $n = 0,1,2, \dots$ where β_i is a vector of parameters and C is a matrix of explanatory variables. Other basic notations are as follows: *N* represents a random variable and n_{ikt} is an observed value of *N*, where the subscript *i* denotes the product; $k = 1,2, \dots, K$ denotes the household; and $t = 1,2, \dots, T$ denotes the time period. The average purchases $q_i = (q_{i11}, q_{i12}, \dots, q_{iKT})$ are only observed after a trip to

[10] These propositions are proven in Appendix 1.

the shop. Thus, the variable $q_i|n_i > 0$ is assumed to follow a continuous distribution, $f_{Qi|n_i>0}(q_i|\alpha_i, C, n_i > 0)$, defined only over positive values, where α_i is a vector of parameters. The interpretations of Q and q_{ikt} are as for N and n_{ikt}. The data-generating process for the average quantity purchased is represented by the following two-part model:

$$f_Q(q_i|\alpha_i, C) = \begin{pmatrix} \Pr(N = 0|\beta_i, C) & \text{if } q_i = 0 \\ \Pr(N > 0|\beta_i, C) f_{Qi|n>0}(q_i|\alpha_i, C, n_i > 0) & \text{if } q_i > 0 \end{pmatrix} \tag{6}$$

The decisions to purchase a good and how much to purchase in each trip are likely related, and it is desirable to model them as stochastically correlated. Furthermore, the demand for one good may be related to the demand for other goods, and it is important to allow for correlation between equations within each of the two systems. To allow for these correlations, random effects are introduced to both densities, $f_{Ni}(n_i|\beta_i, C, b_{Ni})$ and $f_{Qi|n>0}\left(q_i\middle|\alpha_i, C, n > 0, b_{Qi}\right)$, where b_{Ni} and b_{Qi} are random effects assumed to follow a multivariate normal distribution:

$$\begin{bmatrix} b_N \\ b_Q \end{bmatrix} \Bigg| D \sim MVN\left(\begin{bmatrix} 0^M \\ 0^M \end{bmatrix}, \begin{bmatrix} D_N & D_{NQ} \\ D_{NQ} & D_Q \end{bmatrix} \right) \tag{7}$$

where $b_N = (b_{N1}, \dots, b_{NM})$, $b_Q = \left(b_{Q1}, \dots, b_{QM}\right)$ and D is an unrestricted block-covariance matrix.[11] The joint probability density function for n_i and q_i is given as:

$$p(n_i, q_i|\beta, \alpha, D, C) =$$

[11] The conditional means of n_i and q_i are given as follows: $\mathrm{E}(n_i|\beta_i, C, b_{Ni})$ and $\mathrm{E}\left(q_i\middle|\alpha_i, C, b_{Qi}\right) = \Pr(N > 0|\beta_i, C)\mathrm{E}\left(q_i\middle|\alpha_i, C, n_i > 0, b_{Qi}\right)$. The marginal effects of $\mathrm{E}\left(q_i\middle|\alpha_i, C, b_{Qi}\right)$ are given by: $\frac{\partial \mathrm{E}\left(q_i\middle|\alpha_i, C, b_{Qi}\right)}{\partial c_i} = \frac{\partial \mathrm{E}\left(q_i\middle|\alpha_i, C, n_i > 0, b_{Qi}\right)}{\partial c_i} \Pr(N > 0|\beta_i, C)$. The corresponding elasticities are given by: $\frac{\partial \mathrm{E}\left(q_i\middle|\alpha_i, C, b_{Qi}\right)}{\partial c_i}\frac{c_i}{q_i}$, and $\frac{\partial \mathrm{E}(n_i|\beta_i, C, b_{Ni})}{\partial c_i}\frac{c_i}{n_i}$.

$$\int \prod_{t=1}^{T} f_{Ni}(n_{ikt}|\beta_i, C, b_{Nikt})\, f_{Qi|n>0}(q_i|\alpha_i, C, n_i > 0)\phi(b_{ik}|0, D)db_{ik}. \tag{8}$$

The product operator is inside the integral because b_N and b_Q each have one draw for the $2T$ random variables $n_{ik1}, n_{ik2}, \dots, n_{ikT}$ and $q_{ik1}, q_{ik2}, \dots, q_{ikT}$, respectively. Thus, there is a new draw for each cluster, but not for each time period within a cluster. The likelihood is then given by:

$$L = \prod_{k=1}^{K} \prod_{i=1}^{M} p(n_i, q_i|\beta_i, \alpha_i, D, C) \tag{9}$$

Because the joint density $p\left(n_i, q_i \middle| \beta_i, \alpha_i, D, C\right)$ does not have a closed form solution, the likelihood L cannot be optimized with conventional Newton methods, and we therefore use simulation methods.

3.1. Distribution Assumptions

To account for the large share of zeros in the data, we assume that the frequency of purchase, n_i, is generated from $n_i|n_i > 0, \beta_i, C, \sim$ truncated $\text{Poisson}(\mu_i)$, where $\mu_i = \exp(C\beta_i + b_{Ni})$. This results in a multivariate truncated Poisson log-normal distribution:

$$f_{Ni}(n_i|n_i > 0, \beta, D_N, C) = \int \frac{\exp\left(-\mu_i(b_{Ni})\right)\left(\mu_i(b_{Ni})\right)^{n_i}}{\left[1 - \exp\left(-\mu_i(b_{Ni})\right)\right]n_i!} \phi(b_{Ni}|0, D_N)db_{Ni} \tag{10}$$

where $\phi(b_{Ni}|0, D_N)$ is the multivariate normal distribution for b_{Ni}, with covariance matrix D_N. We assume that the average quantity of purchase, q_i, is generated from $q_i|\alpha_i, C, n_i > 0 \sim \text{Gamma}(\kappa_i, \eta_i)$, where the mean of the gamma distribution is specified as $\kappa_i\eta_i = \exp\left(C_i\alpha_i + b_{Qi}\right)$. This specification results in a multivariate gamma log-normal distribution:

$$f_{Qi}(q_i|\alpha_i, C, n_i > 0) = \int \frac{q_i^{\kappa_i - 1} \exp(-q_i/\eta_i)}{\eta_i^{\kappa} \Gamma(\kappa_i)} \phi(b_{Qi}|0, D_Q) db_{Qi} \tag{11}$$

where $\phi(b_{Qi}|0, D_Q)$ is the multivariate normal distribution for b_{Qi}, with covariance matrix D_Q.

3.2 Priors and Markov Chain Monte Carlo Sampling

We assume uninformative priors, which is a common practice (see, e.g., Chib and Winkelmann, 2001). Let $\beta \sim \mathrm{N}(\beta_0, B_0^{-1})$, $\alpha \sim \mathrm{N}(\alpha_0, A_0^{-1})$, $\kappa \sim \mathrm{Gamma}(k_0, s_0)$, $D_N^{-1} \sim$ Wishart (v_{N0}, R_{N0}), and $D_Q^{-1} \sim$ Wishart (v_{Q0}, R_{Q0}), where $\beta_0, B_0, \alpha_0, A_0, k_0, s_0, v_{N0}, R_{N0}, v_{Q0},$ and R_{Q0} are known hyperparameters, and $\mathrm{Wishart}(v_{o0}, R_{o0})$ is the Wishart distribution with v_{o0} degrees of freedom and a scale matrix R_{o0}, where $o = N, Q$. By Bayes' theorem, the posterior density of the two parts of the model are proportional to the following expressions:

$$\phi(\beta|\beta_0, B_0^{-1}) f_W(D_N^{-1}|v_{N0}, R_{N0}) \prod_{k=1}^{K} \prod_{i=1}^{M} f_N(n_{ik}|\beta, b_{Nik})\, \phi(b_{Nik}|0, D_N) \tag{12}$$

$$\phi(\alpha|\alpha_0, A_0^{-1}) f_W(D_Q^{-1}|v_{Q0}, R_{Q0}) \prod_{k=1}^{K} \prod_{i=1}^{M} f_Q(q_{ik}|\alpha, b_{Qik})\, \phi(b_{Qik}|0, D_Q) \tag{13}$$

where f_W is the Wishart density. We construct Markov chains using the blocks of parameters b_N, b_Q, β, α, D_N and D_Q and the full conditional distributions:

$$[b_N|n, \beta, D]; \quad [\beta|n, b_N]; \quad [D|b_N] \tag{14}$$

$$[b_Q|q, \alpha, D]; \quad [\alpha|q, b_Q]; \quad [D|b_Q] \tag{15}$$

The blocks of parameters $[\beta|n, b_N]$ and $[\alpha|q, b_Q]$ are separated into smaller blocks to facilitate convergence. The simulation output is generated by recursively simulating these distributions using the most recent values of the conditioning variables at each step. The mathematical derivation of the Markov chain Monte Carlo sampling of b_N, b_Q, β, α, D_N and D_Q can be found in Appendix 2.

4. DATA AND SELECTED FUNCTIONAL FORM

French scanner data for the purchases of fresh fish as recorded by Kantar Worldpanel for the period 2005–2008 are used. The data were from a rotating panel, i.e., when households drop out, new households are selected to take their place. The numbers of participants recording fresh fish purchases were 3,291 in 2005, 3,234 in 2006, 3,165 in 2007 and 4,479 in 2008, which show significant rotation of households in 2008. Purchased quantities and total expenditures were recorded. To calculate the associated unit prices, we followed Allais et al. (2010) and Bertail and Caillavet (2008) and divided the total expenditures by quantities. The data are weekly and include many zero purchases. To reduce the number of zero purchases, weekly purchases were aggregated to yearly purchases, and the panel structure was accounted for by using random effects, as discussed in the previous section. Demand for salmon has been increasing during recent years and along with white fish, if canned tuna is excluded, has been the most frequently purchased type of fish in France, while fresh fish has been the most common product form (Xie and Myrland, 2011). Therefore, we included fresh salmon, fresh white fish and other fresh fish in our demand system.

In Table I, the purchase frequencies for fresh salmon, fresh white fish and other fresh fish over the 4-year survey period are presented. Between 14% and 45% did not purchase each of the three types of fish. Many households with a positive purchase of one fish type only purchased this type once or twice over the 4 years. However, more than 10% of the households purchased white fish more than 10 times and almost 24% purchased other fish more than 10 times.

(Table I about here)

Table II shows summary statistics of the variables used in our empirical model. The average frequencies of purchase and the average quantities were quite low because of the large share of zeros in the data set. However, the maximum frequency was 172 times for the other fish. The average purchase of each type of fish was around 500 g and the maximum average purchase of each type of fish was around 9 kg. The unit prices and total expenditures on fresh fish were divided by the average French consumer price index (CPI) to impose homogeneity of degree zero for frequencies of purchases, average quantities and total quantities.[12]

(Table II about here)

Total quantities purchased are specified as:

$$x_i = \exp\big(b(p, w, R, \theta)\big) \tag{16}$$

where the function $b(\cdot)$ is linear in its parameters $\theta = (\theta_1, \theta_2, \dots, \theta_n)$. We decomposed $b(\cdot)$ into a frequency part $f(p, w, R, \beta)$ and an average quantity part $z(p, w, R, \alpha)$, where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ are parameter vectors. Using this decomposition, Equation (16) becomes:

$$x_i = n_i q_i = \exp\big(f(p, w, R, \beta)\big)\exp\big(z(p, w, R, \alpha)\big). \tag{17}$$

[12] The average CPI for 2005–2008 was 90.28, and the relative price of 0.13 for salmon and 0.14 for white fish reported in Table 2 correspond to €11.74 and €12.64 per kg of salmon and white fish, respectively.

As is conventional when estimating count data models, the conditional expectation is defined as a semi-logarithmic function. The purchase frequency demand function for each fish type is thus given by:

$$\mathrm{E}(n_{ik}|C_i) = \exp\left(\beta_{ik} + \sum_{s=1}^{M} \beta_{is}(p_{sk}/CPI) + \xi_i\big((R_k + y_k)/CPI\big) + b_{Nik}\right) \tag{18}$$

for $i = 1,2, \ldots, M$ fish categories and $k = 1,2, \ldots, K$ households. The price of fish category *s* for household *k* is denoted by p_{sk}. The total fish expenditure of household *k* is given by $R_k + y_k$, where *CPI* is the French consumer price index, and b_{Nik} is the random effect described in Section 3. For the demand system to be consistent with economic theory, homogeneity and symmetry were imposed on the parameters.[13] LaFrance and Hanemann (1989) and LaFrance (1990) derived these restrictions for the semi-logarithmic functional form. In our case, we must have $\beta_{is} = 0 \; \forall \; i \neq s$ and $\xi_i = \xi \; \forall \; i$. To allow for additional explanatory variables c_{zk}, we define β_{ik} as follows:

$$\beta_{ik} = \eta_{0i} + \sum_{z=1}^{Z} c_{zk}\eta_{iz} \tag{19}$$

The average quantity demand function for each good is given by:

$$\mathrm{E}(q_{ik}|C_i) = \exp\left(\alpha_{ik} + \sum_{s=1}^{M} \alpha_{is}(p_{sk}/CPI) + \gamma_i\big((R_k + y_k)/CPI\big) + b_{Qik}\right). \tag{20}$$

Corresponding to the restrictions on Equation (18), the restrictions $\alpha_{is} = 0 \; \forall \; i \neq s$ and $\gamma_i = \gamma \; \forall \; i$ were imposed on Equation (20), and α_{ik} is defined as β_{ik} in Equation (19).

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[13] Negativity was not imposed due to computational difficulties in that it required non-negativity constraints.

The own-price e_{ni} and total expenditure elasticities ε_{ni} for the frequency of purchase were calculated as:

$$e_{ni} = \frac{\partial \mathrm{E}(n_i|C_i)}{\partial p_i}\frac{p_i}{n_i} = \beta_{ii}p_i \tag{21}$$

$$\varepsilon_{ni} = \frac{\partial \mathrm{E}(n_i|C_i)}{\partial (R+y)}\frac{(R+y)}{n_i} = \theta(R+y). \tag{22}$$

The own-price e_{qi} and total expenditure ε_{qi} elasticities for the average quantity purchased were calculated as:

$$e_{qi} = \frac{\partial \mathrm{E}(q_i|C_i)}{\partial p_i}\frac{p_i}{n_i} = \alpha_{ii}p_i \tag{23}$$

$$\varepsilon_{qi} = \frac{\partial \mathrm{E}(q_i|C_i)}{\partial (R+y)}\frac{(R+y)}{n_i} = \gamma(R+y). \tag{24}$$

Under the restrictions above, all the total substitution effects are zero, and the uncompensated cross-price elasticities are also zero.[14] However, the compensated price elasticities are not zero. These elasticities can be useful for calculating welfare effects, and the elasticity formulas and the estimated values are provided in Appendix 1. In our case, all of the compensated cross-price elasticities were very small numerical values.

5. EMPIRICAL RESULTS

Table III shows the posterior summary for the fresh salmon equations based on the TMPLN and MGLN distributions. The purchase frequencies were estimated by the TMPLN model, and the average purchases were estimated by the MGLN model. Only

[14] No total substitution effects follow from the imposition of symmetry on a semi-logarithmic functional form. It might be of interest for future research to estimate a system without symmetry restrictions to compare the differences between the compensated and uncompensated cross-price elasticities.

the signs and statistical significances of the estimated parameters in the table have an interpretation.

All parameter estimates in the frequency of purchase model were significant except for the time dummies for 2007 and 2008. The signs of the estimated parameters were as expected. The time dummy for 2006 shows that the frequency of purchase was smaller than that in 2005. The Geweke *Z*-test of Markov chain stationarity was not rejected at the 5% level except for the time dummy for 2006.[15]

All of the parameter estimates in the equation for average purchases were significant and were generated from a stationary Markov process. The time dummies show that average quantities have increased for all years, relative to 2005. Kappa is the shape parameter of the gamma distribution and was highly significant.[16]

Table IV shows the posterior summary for the fresh white fish equations. All parameters in the frequency of purchase model were significant except for the 2008 dummy. Furthermore, we did not reject that they were generated from a stationary Markov process. The time dummies showed that the frequency of purchase decreased relative to 2005. All parameter estimates in the equation for average purchases were significant and their Markov chains were stationary, except for the constant. As for salmon, the time dummies showed an increase relative to 2005.

Table V shows the posterior summary for the other fresh fish equations. All of the parameter estimates of both models were significant and were generated from a

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[15] The Geweke *Z*-values are normally distributed. For an application of the Geweke convergence test, see, e.g., Nylander et al. (2008).

[16] Many probability distributions are not a single distribution, but a family of distributions with one or several shape parameters, which allow the distributions to take on different shapes. These distributions are very useful in modeling since they are flexible and can fit many different datasets.

stationary Markov process. The frequency of purchase decreased over time, whereas the average quantity demanded increased relative to 2005.

Table VI shows the cross-equations covariance matrix for the models, i.e., the covariance matrix of the random effects. We define the demand equation for salmon as number 1, the demand for white fish as number 2 and the demand for other fish as number 3. Then, Sigma11 denotes the variance of the random effects for salmon, Sigma22 for white fish and Sigma33 for other fish. Correspondingly, Sigma12 denotes the covariance between the random effects of equations one and two and so on.

All parameter estimates in both systems were significant and were generated from a stationary Markov process. The significant covariance estimates demonstrated the importance of allowing for an unrestricted covariance matrix.

Table VII shows the own-price and total expenditure elasticities with respect to frequencies of purchase, average purchases and total purchases. The values of the own-price elasticities for total quantities purchased are almost equal for the three types of fish, whereas the own-price elasticities for purchase frequencies and average quantities are more unequal. Such differences could be used to find a loss-leader product.

(Tables III–VII about here)

Example of a Loss-Leader Pricing Strategy

We include a simple example showing how fresh salmon could be used as a loss-leader product. We assume a retailer has a large variety of goods and that they want to create a profitable pricing strategy that also benefits their competitive position relative to nearby fish retailers. Let the retailer lower the price of salmon by 20% and raise the price of fresh white fish by 20%. As described above, the uncompensated cross-price effects would be zero in our estimated system, and we only have to take account of the own-price effects. However, as discussed in the meta-study of Cornelsen et al. (2016), the cross-price effects are likely to be much smaller than the own-price effects for many food products, and they will only reinforce or work in the opposite direction of the own-price effects.

The key numbers and calculations used in the example are summarized in Table VIII. As reported in Table VII, the own-price elasticity with respect to frequency of purchase for salmon was –0.05, and a 20% price reduction would increase the frequency of purchase by 1% from 1.96 to 1.98. The own-price elasticity with respect to average quantity was –0.18, and a 20% price reduction would increase the average quantity purchased from 421.78 to 436.96 g. For white fish, the 20% price increase would reduce the frequency of purchase from 3.77 to 3.69 and reduce the average quantity from 435.03 to 425.46 g.

As shown in Table II, the sample averages of the relative prices of white fish and salmon were 0.14 and 0.13, respectively. Multiplying these relative prices by the average CPI of 90.28 resulted in nominal average prices of €12.64 for white fish and €11.74 for salmon. After the 20% price reduction for salmon, the new price became €9.39, and after the 20% price increase for white fish, the new price became €15.17.

The per customer revenue of sales of salmon given the initial price was €9.71, while it was €8.12 after the 20% reduction in price. The per customer revenue of sales of white fish given the initial price was €20.73, while it was €23.82 after the 20% increase in price. The per customer total revenue from sales of salmon and white fish increased from €30.44 to €31.94, i.e., an increase of 4.9%. Given that no changes in total costs were caused by these price changes, the profit increased by €1.50 per customer.

The above effect on revenue only included the direct effects on revenue from sales of the two categories of fish. However, this loss-leader pricing strategy may also convince customers to transfer more of their other category purchases to this store (Thomassen et al., 2017). Most customers not only purchase fresh fish, but also may increase purchases of other products such as meat, bakery, fruits, or toothpaste.

(Table VIII about here)

6. CONCLUSIONS

The microeconomic model presented by Meghir and Robin (1992) assumed that total purchased quantities could be expressed as the product of the average purchased quantities and the frequencies of purchase. We follow their specification and further develop the model so that homogeneity of degree zero and symmetry hold for purchase frequencies, average purchased quantities and total purchased quantities. Furthermore, we extend their count data estimation framework in three ways: (i) we account for homogeneity and symmetry in the count data demand system; (ii) we account for the problem of zero purchases by assuming a truncated data-generating process for the counts; and (iii) we allow the covariance structure to be unrestricted within the two demand systems. We assume a multivariate Poisson log-normal distribution for the counts and a multivariate gamma log-normal distribution for the average quantities. To estimate these distributions, we use a random-walk Metropolitan simulation algorithm.

The proposed estimation method is illustrated by using scanner data for French fish purchases to estimate the demand for fresh salmon, fresh white fish and other fresh fish. The results indicate that salmon can be a loss-leader category for large retailers who want to increase store traffic and, thereby, sales of products that can be sold with a markup above the market price. Future research will focus on providing a more elaborate empirical example of the loss-leader pricing strategy with different product groups.

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Table I. Purchase frequencies

No. of Purchases	Salmon Frequency	Salmon Percent	White Fish Frequency	White Fish Percent	Other Fish Frequency	Other Fish Percent
0	6326	44.64	4126	29.11	1982	13.99
1	2784	19.64	2738	19.32	2357	16.63
2	1514	10.68	1654	11.67	1603	11.31
3	1001	7.06	1135	8.01	1223	8.63
4	685	4.83	871	6.15	953	6.72
5	473	3.34	635	4.48	741	5.23
6	337	2.38	489	3.45	580	4.09
7	254	1.79	402	2.84	549	3.87
8	151	1.07	342	2.41	446	3.15
9	131	0.92	248	1.75	345	2.43
≥10	516	3.65	1532	10.84	3393	23.96

Table II. Summary statistics for fish purchases

Variable	Mean	Std. Dev.	Min	Max
Salmon (frequency)	1.96	3.39	0.00	52.00
White fish (frequency)	3.77	6.03	0.00	87.00
Other fish (frequency)	7.43	11.25	0.00	172.00
Salmon (average quantity in g)	421.78	608.24	0.00	9600.00
White fish (average quantity in g)	435.03	480.61	0.00	8537.70
Other fresh fish (average quantity in g)	552.08	467.96	0.00	8100.00
Real price of salmon	0.13	0.04	0.01	0.75
Real price of white fish	0.14	0.05	0.02	0.71
Real price of other fish	0.12	0.05	0.01	0.68
Real total expenditures on fresh fish	0.98	1.33	0.00	15.57

Table III. Posterior summary for salmon based on the TMPLN and MGLN distributions

	Frequency			Average Quantity		
Variable	Mean	t-value	Geweke Z	Mean	t-value	Geweke Z
Constant	–0.53	–16.00	1.07	6.66	369.87	–0.71
Price	–0.37	–1.97	–1.17	–3.02	–28.73	0.61
Expenditure	0.09	62.41	1.75	0.07	19.59	–0.25
Time06	–0.05	–3.80	2.10	0.11	9.30	–0.50
Time07·10	–0.07	–0.62	0.91	1.26	10.99	–0.03
Time08·100	–0.06	–0.05	0.15	14.27	12.63	–0.32
Kappa	-	-	-	2.36	136.60	–0.84

Notes: TMPLN = truncated multivariate Poisson log-normal and MGLN = multivariate gamma log-normal. Time06, Time07 and Time08 are annual dummy variables, which take the value of 1 in the indicated years. For the ease of reading, some of these dummy variables are scaled in this table. The Geweke *Z* provides the *Z*-value for a test of stationarity of the Markov chains. Kappa is the shape parameter of the gamma distribution.

Table IV. Posterior summary for white fish based on the TMPLN and MGLN distributions

	Frequency			Average Quantity		
Variable	Mean	t-value	Geweke Z	Mean	t-value	Geweke Z
Constant	–0.12	–5.94	0.35	6.35	402.04	–2.57
Price	–0.69	–6.84	–0.60	–1.38	–14.40	1.70
Expenditure	0.09	62.41	1.75	0.07	19.59	–0.25
Time06	–0.04	–5.66	–0.38	0.06	5.15	0.35
Time07	–0.04	–6.18	0.57	0.05	4.56	0.75
Time08	–0.01	–1.80	–0.16	0.06	5.55	1.35
Kappa	-	-	-	2.06	105.80	1.14

Notes: TMPLN = truncated multivariate Poisson log-normal and MGLN = multivariate gamma log-normal. Time06, Time07 and Time08 are annual dummy variables, which take the value of 1 in the indicated years. The Geweke *Z* provides the *Z*-value for a test of stationarity of the Markov chains.

Table V. Posterior summary for other fish based on the TMPLN and MGLN distributions

	Frequency			Average Quantity		
Variable	Mean	t-value	Geweke Z	Mean	t-value	Geweke Z
Constant	0.32	24.00	–0.04	6.43	351.16	1.08
Price	–0.81	–10.55	0.24	–1.22	–10.50	–1.32
Expenditure	0.09	62.41	1.75	0.07	19.59	–0.25
Time06	–0.05	–15.47	–0.41	0.06	5.06	–1.72
Time07	–0.05	–15.42	–0.80	0.11	8.99	–1.80
Time08	–0.07	–18.67	1.18	0.13	10.48	–1.69
Kappa	-	-	-	2.22	123.39	–0.15

Notes: TMPLN = truncated multivariate Poisson log-normal and MGLN = multivariate gamma log-normal. Time06, Time07 and Time08 are annual dummy variables, which take the value of 1 in the indicated years. The Geweke *Z* provides the *Z*-value for a test of stationarity of the Markov chains.

Table VI. Posterior summary cross-equations covariance matrix based on TMPLN and MGLN distributions

	Frequency			Average Quantity		
Variable	Mean	t-value	Geweke Z	Mean	t-value	Geweke Z
Sigma11	0.49	18.65	–0.55	0.29	37.95	0.21
Sigma12	0.17	11.53	–0.05	0.13	24.04	0.90
Sigma13	0.17	14.53	–0.92	0.31	40.83	–0.33
Sigma21	0.17	11.53	–0.05	0.13	24.04	0.90
Sigma22	0.40	23.74	–0.23	0.21	36.01	2.37
Sigma23	0.21	21.79	–0.64	0.14	23.59	0.07
Sigma31	0.17	14.53	–0.92	0.31	40.83	–0.33
Sigma32	0.21	21.79	–0.64	0.14	23.59	0.07
Sigma33	0.26	29.50	–0.90	0.34	38.48	–0.56

Note: TMPLN = truncated multivariate Poisson log-normal and MGLN = multivariate gamma log-normal. Sigma11, Sigma22 and Sigma33 represent the variances of the random effects of the demand for salmon, white fish and other fish, respectively. The other Sigmas represent the corresponding covariances. Geweke *Z* provides the *Z*-value for a test of stationarity of the Markov chains.

Table VII. Own-price and total expenditure elasticities

	Frequency					
	Salmon		White Fish		Other Fish	
	Estimate	t-value	Estimate	t-value	Estimate	t-value
Own price	–0.05	–1.97	–0.10	–6.84	–0.10	–10.55
Total expenditure	0.09	62.41	0.09	62.41	0.09	62.41
	Average Quantities					
	Salmon		White Fish		Other Fish	
	Estimate	t-value	Estimate	t-value	Estimate	t-value
Own price	–0.18	–28.73	–0.11	–14.40	–0.30	–10.50
Total expenditure	0.03	19.59	0.04	19.59	0.07	19.59
	Total Quantities					
	Salmon		White Fish		Other Fish	
	Estimate	t-value	Estimate	t-value	Estimate	t-value
Own price	–0.24	–8.77	–0.21	–7.50	–0.21	–20.51
Total expenditure	0.12	42.65	0.13	53.85	0.14	81.57

Table VIII. Numerical example loss-leader pricing strategy

Product	Elasticity[a]	ΔChange[b]	Frequency/Quantity Initial[c]	Frequency/Quantity New[d]	Revenue Initial[e]	Revenue New[f]
Salmon						
frequency	–0.05	+ 1.00	1.96	1.98		
Average quantity	–0.18	+ 3.60	421.78	436.96		
Total					9.71	8.12
White fish						
frequency	–0.10	–2.00	3.77	3.69		
Average quantity	–0.11	–2.20	435.03	425.46		
Total					20.73	23.82
Total fish					30.44	31.94

Notes:

[a] The elasticity estimates are the estimates from Table VII.

[b] ΔChange is the percentage change in frequency or average quantity given a 20% price decrease for salmon and a 20% price increase for white fish.

[c] The initial frequencies and average quantities refer to the frequencies and average quantities as reported in Table II.

[d] The new frequency and average quantities are the new values after the 20% price decrease for salmon and the 20% price increase for white fish.

[e] Total revenue in € from sales of salmon and white fish given initial frequencies and quantities.

[f] Total revenue in €from sales of salmon and white fish given new frequencies and quantities.

Appendix 1. The Microeconomics Model

The Utility Maximization Problem

The consumer's utility maximization problem is specified as:

$$\max_{n,q,l}\{v(x(nq),n,l): wT + R = p'nq + wl + wg(n), n > 0, q > 0, l > 0\} \tag{1}$$

with the associated Lagrangian problem:

$$\mathcal{L} = v(x(nq),n,l) + \lambda\big[R - p'nq + w\big(T - l - g(n)\big)\big]$$

The first-order conditions to this Lagrangian problem are:

$$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{\partial v}{\partial x_i}n_i - \lambda p_i n_i = 0 \tag{2}$$

$$\frac{\partial\mathcal{L}}{\partial n_i} = \frac{\partial v}{\partial x_i}q_i + \frac{\partial v}{\partial n_i} - \lambda p_i q_i - \lambda w\frac{\partial g}{\partial n_i} = 0 \tag{3}$$

$$\frac{\partial\mathcal{L}}{\partial l} = \frac{\partial v}{\partial l} - \lambda w = 0 \tag{4}$$

From Equations (2) and (4), we have:

$$\frac{\partial v}{\partial x_i}\frac{1}{p_i} = \lambda \tag{5}$$

$$\frac{\partial v}{\partial l}\frac{1}{w} = \lambda \tag{6}$$

By substituting for the first λ in Equation (3) by Equation (5), we get:

$$\frac{\partial v}{\partial x_i}q_i + \frac{\partial v}{\partial n_i} - \frac{\partial v}{\partial x_i}\frac{p_i q_i}{p_i} - \lambda w\frac{\partial g}{\partial n_i} = 0$$

$$\frac{\partial v}{\partial x_i}q_i + \frac{\partial v}{\partial n_i} - \frac{\partial v}{\partial x_i}q_i - \lambda w\frac{\partial g}{\partial n_i} = 0$$

$$\frac{\partial v}{\partial n_i} - \lambda w\frac{\partial g}{\partial n_i} = 0$$

$$\frac{\partial v/\partial n_i}{\partial g/\partial n_i} = \lambda w \tag{7}$$

It follows from Equations (5), (6) and (7) that the necessary conditions for a maximum are:

$$\frac{\partial v/\partial x_i}{\partial v/\partial x_j} = \frac{p_i}{p_j} \ \forall i,j \tag{8}$$

$$\frac{\partial v/\partial l}{\partial v/\partial q_i} = \frac{w}{p_i} \ \forall i,j \tag{9}$$

$$\frac{\partial v/\partial n_i}{\partial v/\partial l} = \frac{\partial g}{\partial n_i} \ \forall i \tag{10}$$

$$\frac{\partial v/\partial n_i}{\partial v/\partial n_j} = \frac{\partial g/\partial n_i}{\partial g/\partial n_j} \ \forall i,j \tag{11}$$

The solution of the first-order conditions is given by three sets of uncompensated demand functions: $n(p,w,R)$, $q(p,w,R)$ and $l(p,w,R)$. The total purchased quantities are given by $n(p,w,R)\ q(p,w,R) = x(p,w,R)$.

The Dual Cost Minimization Problem

The consumer's cost minimization problem is given by:

$$\min_{n,q,l}\{R = p'nq - w\big(T - l - g(n)\big) \colon v(x(nq),n,l) = v^*, n > 0, q > 0, l > 0\} \tag{12}$$

with the associated Lagrangian problem:

$$\mathcal{L} = p'nq - w\big(T - l - g(n)\big) + \lambda[v^* - v(x(nq),n,l)]$$

The first-order conditions to this Lagrangian problem are:

$$\frac{\partial \mathcal{L}}{\partial q_i} = p_i n_i - \lambda\frac{\partial v}{\partial x_i} n_i = 0 \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial n_i} = p_i q_i + w\frac{\partial g}{\partial n_i} - \lambda\frac{\partial v}{\partial x_i} q_i - \lambda\frac{\partial v}{\partial n_i} = 0 \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial l} = w - \lambda \frac{\partial v}{\partial l} = 0 \tag{15}$$

From Equations (13) and (15), we have Equations (5) and (6) above. By substituting for the first λ in Equation (14) by Equation (5), we get:

$$p_i q_i + w \frac{\partial g}{\partial n_i} - \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_i}} p_i q_i - \lambda \frac{\partial v}{\partial n_i} = 0$$

$$p_i q_i + w \frac{\partial g}{\partial n_i} - p_i q_i - \lambda \frac{\partial v}{\partial n_i} = 0$$

$$w \frac{\partial g}{\partial n_i} - \lambda \frac{\partial v}{\partial n_i} = 0$$

$$\frac{\partial v / \partial n_i}{\partial g / \partial n_i} = \frac{w}{\lambda} \tag{16}$$

It follows from Equations (5), (6) and (16) that the necessary conditions for a minimum are given by Equations (8), (9), (10) and (11) above. The solution of the first-order conditions is three sets of compensated demand functions: $n^c(p, w, v^*)$, $q^c(p, w, v^*)$ and $l^c(p, w, v^*)$. The compensated demand function for total purchased quantities is $n^c(p, w, v^*) q^c(p, w, v^*) = x^c(p, w, v^*)$.

Formulas for Compensated Price Elasticities

Under symmetry restrictions on a semi-log system, there are zero uncompensated cross-price elasticities; however, there are non-zero compensated cross-price elasticities that reflect the changes in total expenditures in the Slutsky equation. These compensated elasticities can be useful for calculating welfare effects in the model. The compensated price elasticity between products *i* and *j* for household *k*, e_{ijk}, are calculated as:

$$e_{ijk} = \frac{p_{ik}n_{ik}}{(R_k + y_k)} \frac{\partial n_{jk}}{\partial (R_k + y_k)} \frac{(R_k + y_k)}{n_{jk}} = s_{nik}\omega_{njk} \tag{17}$$

where n_{ik} is the purchase frequency of product *i* by household *k*; s_{nik} is the total expenditure share of product *i* by household *k*, in terms of purchase frequencies; and ω_{njk} is the total expenditure elasticity of product *j* by household *k*, in terms of purchase frequencies.

The compensated price elasticities for average purchased quantities are calculated as:

$$\varepsilon_{ijk} = \frac{p_{ik}q_{ik}}{(R_k + y_k)} \frac{\partial q_{jk}}{\partial (R_k + y_k)} \frac{(R_k + y_k)}{q_{jk}} = s_{qik}\omega_{qjk} \tag{18}$$

where s_{qik} is the total expenditure share of product *i* by household *k*, in terms of average quantities; and ω_{qjk} is the total expenditure elasticity of product *j* by household *k*, in terms of average quantities.

The estimated compensated elasticities for our model are provided in Table A1 at bottom of this Appendix. As we can see, compensated from the table, the cross-price effects are small.

Proposition 1.1

The demand equations $n(p, w, R)$, $q(p, w, R)$, $l(p, w, R)$ and $x(p, w, R)$ are homogeneous of degree zero in (p, w, R).

Proof of Proposition 1.1

i) For the constraint of Equation (1) and any scalar $\rho > 0$, we have:

$$\{(n, q, l)' \in \mathbb{R}_+^{2M+1} : \rho R \geq \rho p'x - \rho w(T - l - g(n)), x \equiv nq\}$$

$$= \{(n,q,l)' \in \mathbb{R}_{+}^{2M+1}: R \geq p'x - w(T - l - g(n)), x \equiv nq\}$$

Thus, when *p*, *w* and *R* increase by the same percentage ρ, the choice set and the objective function are unchanged and, therefore, the optimal choices of *n*, *q* and *l* are unchanged. Q.E.D.

ii) For any scalar $\rho > 0$,

$$\{(n,x,l)' \in \mathbb{R}_{+}^{2M+1}: \rho R \geq \rho p'x - \rho w(T - l - g(n)), x \equiv nq\}$$

$$= \{(n,x,l)' \in \mathbb{R}_{+}^{2M+1}: R \geq p'x - w(T - l - g(n)), x \equiv nq\}$$

Q.E.D.

Proposition 1.2

The matrix of compensated substitution effects for $x^c(p,w,v^*)$ is symmetric and negative semidefinite.

Proof of Proposition 1.2

First, we need to show that the expenditure function $c(p,w,v^*)$ associated with Equation (1) is concave in p and w. For concavity, fix the utility level at $\bar{v}$, and let $p'' = \rho p + (1-\rho)p'$ and $w'' = \rho w + (1-\rho)w'$ for $\rho \in [0,1]$. Suppose that x'', n'' and l'' are optimal solutions to the expenditure minimization problem when prices are p'' and wages are w''. If so:

$$c(p'',w'',\bar{v}) = p''x'' - w''(T - l'' - g(n''))$$

$$= \rho p x'' + (1-\rho)p'x'' - \rho w(T - l'' - g(n'')) - (1-\rho)w'(T - l'' - g(n''))$$

$$=\rho\left[px'' - w\left(T - l'' - g(n'')\right)\right] + (1-\rho)\left[p'x'' - w'\left(T - l'' - g(n'')\right)\right]$$

$$\geq \rho c(p, w, \bar{v}) + (1-\rho)c(p', w', \bar{v})$$

where the inequality follows from $v(x'', n'', l'') \geq \bar{v}$. The definition of the expenditure function implies that $px'' - w\left(T - l'' - g(n'')\right) \geq c(p, w, \bar{v})$ and $p'x'' - w'\left(T - l'' - g(n'')\right) \geq c(p', w', \bar{v})$.

Next, we need to show that for all p and v^*, the compensated demand $x^c(p, w, v^*)$ is the derivative vector of the expenditure function with respect to p. That is, $x^c(p, w, v^*) = \partial c(p, w, v^*)/\partial p_i$ for all $i = 1, \ldots, M$. This follows directly from the Envelope Theorem. Thus:

$$\frac{\partial c(p, w, v^*)}{\partial p_i} = \frac{\partial \mathcal{L}(x^c, n^c, l^c)}{\partial p_i} = x^c(p, w, v^*).$$

The second derivative of the expenditure function gives:

$$\frac{\partial^2 c(p, w, v^*)}{\partial p_i^2} = \frac{\partial x^c(p, w, v^*)}{\partial p_i}.$$

Because the expenditure function is concave, the matrix of compensated substitution effects for $x^c(p, w, v^*)$ is symmetric and negative semidefinite. Q.E.D.

Proposition 1.3

The matrix of compensated substitution effects for the product $n^c(p, w, v^*)q^c(p, w, v^*)$ is symmetric and negative semidefinite.

Proof of Proposition 1.3

This is a corollary of the previous result, which follows from the identity $n^c q^c \equiv x^c$. From the Envelope Theorem, we have $\partial c(p, w, v^*)/\partial p_i = \partial \mathcal{L}(x^c, n^c, l^c)/\partial p_i = n^c q^c$.

Then, taking the second derivative, we get $\partial^2 c(p,w,v^*)/\partial p_i^2 = (\partial n^c/\partial p_i)q_i + (\partial q^c/\partial p_i)n_i$. Thus, following from the concavity of the expenditure function, the matrix of the compensated substitution effect for the product $n_i^c(p,w,v^*)q_i^c(p,w,v^*)\forall i = 1,2,\ldots,M$ or $NQ_D^c = (\partial n^c/\partial p')q_i + (\partial q^c/\partial p')n_i$ is symmetric and negative semidefinite. Q.E.D.

Table A1. Compensated elasticities

		Frequency	
	Salmon	White Fish	Other Fish
Price elasticities	Estimate	Estimate	Estimate
Salmon·10	0.011	0.022	0.038
White Fish·10	0.006	0.011	0.020
Other fish·10	0.003	0.006	0.010
		Average Quantities	
	Salmon	White Fish	Other Fish
Price elasticities	Estimate	Estimate	Estimate
Salmon·10	0.009	0.009	0.010
White Fish·10	0.008	0.009	0.010
Other fish·10	0.007	0.007	0.008

Note: The t-value for each estimate in the frequency part is 62.41, as indicated by Equation (17) in A1, and the t-value for each estimate in the average quantity part is 19.59, as indicated by Equation (18) in A1.

Appendix 2. Markov Chain Monte Carlo Sampling

The sampling of b_N and b_Q starts with specifying the target densities:

$$\pi(b_N|n,\beta,D) = \prod_{k=1}^{K}\prod_{i=1}^{M}\pi(b_{Nik}|n_{ik},\beta,D) \tag{19}$$

$$\pi\left(b_Q|q,\alpha,D\right) = \prod_{k=1}^{K}\prod_{i=1}^{M}\pi\left(b_{Qik}|q_{ik},\alpha,D\right) \tag{20}$$

To sample the density of the k^{th} household of the i^{th} cluster, we have:

$$\pi(b_{Nik}|n_{ik},\beta,D) = c_{ik}\phi(b_{Nik}|0,D_N)\prod_{t=1}^{T}\frac{\exp\left(-\exp(C_{ikt}\beta_t + b_{Nikt})\right)}{[1-\exp(C_{ikt}\beta_t + b_{Nikt})]} \tag{21}$$

$$\times\,[-\exp(C_{ikt}\beta_t + b_{Nikt})]^{n_{ikt}}$$

$$\equiv c_{ik}\pi^{+}(b_{Nik}|n_{ik},\beta,D_N)$$

$$\pi\left(b_{Qik}\middle|q_{ik},\alpha,\kappa,D\right) = i_{ik}\phi\left(b_{Qik}\middle|0,D_Q\right)\prod_{t=1}^{T}\frac{q_{ikt}^{C_{ikt}\alpha_t+b_{Nikt}}}{\Gamma\left(C_{ikt}\alpha_t + b_{Qikt}\right)\kappa^{C_{ikt}\alpha_t+b_{Qikt}}} \tag{22}$$

$$\times\ \exp[-q_{ikt}/\kappa]$$

$$\equiv i\pi^{+}\left(b_{Qik}\middle|q_{ik},\alpha,\kappa,D_Q\right).$$

The likelihoods of the target distributions are a truncated Poisson log-normal mixture and a gamma log-lognormal mixture, respectively, as specified in Section 3.1. The simulation algorithm is a random-walk Metropolis algorithm. This algorithm requires little computation power in each step, but because of high autocorrelation, more Markov chain Monte Carlo steps are required. The proposal density is found by approximating the target density around the modal value by a multivariate *t*-distribution. Let $\hat{b}_{Nik} = \operatorname{argmax}\ln\pi^{+}(b_{Nik}|n_{ik},\beta,D_N)$ and $V_{b_{Nik}} = \left(H_{b_{Nik}}\right)^{-1}$ be the inverse of the Hessian of $\ln\pi^{+}(b_{Nik}|n_{ik},\beta,D_N)$ at the mode $\hat{b}_{Nik}$. To find these quantities, we use the Newton-Raphson algorithm. Then, our proposal density is $b_{Nik}^{(s)}\left|b_{Nik}^{(s-1)} \sim t\left(b_{Nik}^{(s-1)},V_{b_{Nik}},v\right)\right.$, where *v* is the degrees of freedom and *s* indicates the number of the draws. We make a draw e_{ik} from $t\left(0,V_{b_{Nik}},v\right)$, where $b_{Nik}^{(s)} = b_{Nik}^{(s-1)} + e_{ik}$, and we move from $b_{Nik}^{(s-1)}$ to $b_{Nik}^{(s)}$ with probability:

$$r = \min\left\{\frac{\pi^{+}\left(b_{Nik}^{(s)}\middle|n_{ik},\beta,D_N\right)}{\pi^{+}\left(b_{Nik}^{(s-1)}\middle|n_{ik},\beta,D_N\right)},1\right\} \tag{23}$$

Next, we sample u from a uniform distribution $\mathrm{U}(0,1)$, and if $u < r$, then $b_{Nik}^{(s)} = b_{Nik}^{*}$, otherwise $b_{Nik}^{(s-1)} = b_{Nik}^{*}$. We use the same steps for the sampling of b_Q. The sampling of β and α follow the approach above. The respective target distributions are given as follows:

$$\pi(\beta|n, b_N, D_N) = \phi(\beta|\beta_0, B_0^{-1}) \prod_{k=1}^{K} \prod_{i=1}^{M} \prod_{t=1}^{T} \frac{\exp(-\exp(C_{ikt}\beta_t + b_{Nikt}))}{[1 - \exp(C_{ikt}\beta_t + b_{Nikt})]} \times [-\exp(C_{ikt}\beta_t + b_{Nikt})]^{n_{ikt}} \tag{24}$$

$$\pi(\alpha|q, b_Q, \kappa, D_Q) = \phi(\alpha|\alpha_0, A_0^{-1}) \prod_{k=1}^{K} \prod_{i=1}^{M} \prod_{t=1}^{T} \frac{q_{ikt}^{C_{ikt}\alpha_t + b_{Nikt}}}{\Gamma(C_{ikt}\alpha_t + b_{Nikt})\kappa^{C_{ikt}\alpha_t + b_{Qikt}}} \times \exp[-q_{ikt}/\kappa]. \tag{25}$$

Sampling D_N^{-1} and D_Q^{-1} is a simpler process than the other two blocks of parameters because we specified a hyperprior, which resulted in a Wishart distribution. We sample D_o^{-1}, $o = N, Q$, from a distribution proportional to:

$$f_W(D_o^{-1}|v_{o0}, R_{o0}^{-1}) \prod_{k=1}^{K} \prod_{i=1}^{M} \phi(b_{oik}|0, D_o). \tag{26}$$

Combining terms, the expression results in the Wishart distribution:

$$D_o^{-1}|b_0 \sim \text{Wishart}\left(M + v_{o0}, \left[R_{o0}^{-1} + \sum_{k=1}^{K} \sum_{i=1}^{M} (b_{oik}' b_{oik})\right]^{-1}\right) \tag{27}$$

with degrees of freedom $M + v_{o0}$ and scale matrix $[R_{o0}^{-1} + \sum_{k=1}^{K} \sum_{i=1}^{M} (b_{oik}' b_{oik})]^{-1}$.