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Positive Invertibility of Matrices and Exponential Stability of Impulsive Systems of Itô Linear Differential Equations with Bounded Delays

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Abstract—Basing on the theory of positively invertible matrices, we study certain questions of the exponential $2p$-stability ($1 \leq p < \infty$) of systems of Itô linear differential equations with bounded delays and impulse actions on certain solution components. We apply the ideas and methods developed by N.V. Azbelev and his followers for studying the stability of deterministic functional differential equations. For the systems of equations mentioned above, we establish sufficient conditions for the exponential $2p$-stability ($1 \leq p < \infty$) stated in terms of the positive invertibility of matrices constructed from parameters of these systems. We verify the feasibility of these conditions for certain specific systems of equations.

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INTRODUCTION

Stochastic differential equations describe many actual real life problems of modern physics, biology, economics, engineering, and other applied sciences. In particular, Itô impulse differential equations with aftereffect represent an illustrative mathematical model of certain financial processes. One of the most important questions among those that occur in studying such problems is the analysis of the stability of solutions to stochastic functional differential equations.

The stability of solutions to systems with random parameters was studied by many Russian and international mathematicians. Fundamental studies in this realm find many applications, which, in turn, often give rise to new theoretical thought.

The study of the stability of systems with random parameters became widespread in 1960 due to the paper by I.Ya. Kats and N.N. Krasovskii, where they give basic definitions of the stochastic stability. Moreover, the mentioned authors were first to solve the considered equations by the second (direct) Lyapunov method based on the construction of the corresponding functions. This idea was later used for studying the Itô equations with aftereffect (the method of Lyapunov–Krasovskii–Razumikhin functionals); there are many papers devoted to these equations (see their rather complete list in monographs [1]–[4]). However, in many cases, the application of the direct Lyapunov method and its stochastic analogs encounters serious difficulties. In particular, usually one can prove effective stability criteria with the help of these methods only for relatively simple classes of stochastic functional differential equations.

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On the other hand, the method of auxiliary or “model” equations, also called the “W-method”, proposed by N.V. Azbelev [5], [6] has proved to be effective in studying the stability issues in the deterministic case. The authors of this paper (mainly, the first one) have applied the mentioned method to studying stochastic functional differential equations [7]–[14]. In principle, the W-method is universal, i. e., applicable both in the deterministic case, and in the stochastic one. Certainly, this does not mean that it always gives the best results. However, this method can be helpful in many “nontrivial” cases, when the use of the Lyapunov function is difficult. In particular, the W-method allows one to eliminate some difficulties that occur in studying (by commonly used schemes) the stability issues for equations with unbounded delays, with random coefficients and delays, and with impulse actions.

The Lyapunov stability of solutions with respect to the initial function for deterministic impulse differential equations was studied in papers [15]–[18]. For Itô impulse differential equations with aftereffect, the stability of solutions with respect to the initial function was studied for special classes of equations with the help of the Lyapunov method in [19]. Papers [10], [12], [13], and [20] are devoted to studying the stability of solutions to systems of Itô linear differential equations with aftereffect and with impulse actions on all components of solutions. The research technique used in the mentioned papers is analogous to that applied in [18]. Namely, this is the method of auxiliary or “model” equations; it is described in detail in monographs [5], [6] for the deterministic case, and in papers [7], [9], [11] for the stochastic one.

In this paper, we study the exponential $2p$-stability $(1 \le p < \infty)$ of systems of Itô linear differential equations with bounded delays and with impulse actions on some components of solutions. To this end, we apply the ideas of the method of auxiliary equations and the theory of positively invertible matrices. The distinction from the classical method of auxiliary equations consists in the fact that each equation in the system is transformed independently of others, and each component of the solution is estimated separately. This approach combined with the theory of positively invertible matrices allows us to obtain new results, including those for the deterministic case, and to effectively study the stability issues for equations with impulse actions on some components of solutions.

1. PRELIMINARY INFORMATION. THE OBJECT OF THE STUDY

We use the following denotations: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ is a stochastic basis; k^n is a linear space of n-dimensional $\mathcal{F}_0$-measurable random values; $\mathcal{B}_i$, $i = 2, \dots, m$, are independent standard Wiener processes; $1 \le p < \infty$; c_p is a positive value (depending on p) ([21], p. 65) which is used in estimate (2); E is the mean value symbol; $|.|$ is the norm in R^n; $||.||$ is the norm of an $n \times n$-matrix concordant with the norm in R^n; $||.||_X$ is the norm in a normed space X; μ is the Lebesgue measure on $[0, +\infty)$; l is a certain integer such that $0 \le l \le n$.

Let $B = (b_{ij})_{i,j=1}^m$ be some $m \times m$-matrix. The matrix B is said to be nonnegative, if $b_{ij} \ge 0$, $i, j = 1, \dots, m$, and it is said to be positive, if $b_{ij} > 0$, $i, j = 1, \dots, m$.

Definition 1 ([22])**.** A matrix $B = (b_{ij})_{i,j=1}^m$ is said to be an $\mathcal{M}$-matrix, if $b_{ij} \le 0$ for $i, j = 1, \dots, m$, $i \ne j$, and one of the following conditions takes place:

– for the matrix B there exists a positive inverse matrix B^{-1};

– principal diagonal minors of the matrix B are positive.

According to ([22], p. 338), a matrix B is an $\mathcal{M}$-matrix, if $b_{ij} \le 0$ for $i, j = 1, \dots, m$, $i \ne j$, and if there exist positive values ξ_i, $i = 1, \dots, m$, such that one of the following conditions takes place:

$$\xi_i b_{ii} > \sum_{j=1, i\ne j}^{m} \xi_j |b_{ij}|, \quad i = 1, \dots, m, \qquad \text{or} \qquad \xi_j b_{jj} > \sum_{i=1, i\ne j}^{m} \xi_i |b_{ij}|, \quad j = 1, \dots, m.$$

In particular, if in the first of these inequalities, $\xi_i = 1$, $i = 1, \dots, m$, then the class of $\mathcal{M}$-matrices includes the class of matrices with strong diagonal dominance ([22], p. 418), whose off-diagonal elements are nonpositive.

In this paper, we study the stability issues for the following system of Itô linear differential equations with bounded delays and impulse actions on certain components of solutions:

$$dx(t) = -\sum_{j=1}^{m_1} A_{1j}(t)x(h_{1j}(t))dt + \sum_{i=2}^{m}\sum_{j=1}^{m_i} A_{ij}(t)x(h_{ij}(t))d\mathcal{B}_i(t) \ \ (t \geq 0),$$
$$x(\mu_j) = B_j x(\mu_j - 0), \ j = 1, 2, \ldots, \text{almost surely (a. s.)} \tag{1}$$

with respect to the initial data

$$x(t) = \varphi(t) \quad (t < 0), \tag{1a}$$

$$x(0) = b, \tag{1b}$$

where

1. $x = \mathrm{col}(x_1, \ldots, x_n)$ is the unknown n-dimensional random process;

2. $A_{ij} = (a_{sk}^{ij})_{s,k=1}^{n}$ are $n \times n$-matrices with all $i = 1, \ldots, m, \ j = 1, \ldots, m_i$, where elements of matrices $A_{1j}, \ j = 1, \ldots, m_1$, are progressively measurable scalar random processes, whose trajectories a. s. are locally summable, while elements of matrices A_{ij}, $i = 2, \ldots, m$, $j = 1, \ldots, m_i$, are progressively measurable scalar random processes, whose trajectories are a. s. locally square summable;

3. $h_{ij}, \ i = 1, \ldots, m, \ j = 1, \ldots, m_i$, are Lebesgue measurable functions defined on $[0, \infty)$ such that $0 \leq t - h_{ij}(t) \leq \tau_{ij}$ $(t \in [0, \infty))$ μ-almost everywhere for some positive τ_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$;

4. μ_j, $j = 1, 2, \ldots$, are real values such that $0 = \mu_0 < \mu_1 < \mu_2 < \ldots$, $\lim\limits_{j\to\infty} \mu_j = \infty$;

5. B_j is a real diagonal $n \times n$-matrix, all whose diagonal elements differ from zero and $b_{ii}^{j} = 1$, $i = 1, \ldots, l$, for $j = 1, 2, \ldots$;

6. $\varphi = \mathrm{col}(\varphi_1, \ldots, \varphi_n)$ is an $\mathcal{F}_0$-measurable n-dimensional random process defined on $[-\hat{\sigma}, 0)$, where $\hat{\sigma} = \max\{\tau_{ij}, \ i = 1, \ldots, m, j = 1, \ldots, m_i\}$;

7. $b = \mathrm{col}(b_1, \ldots, b_n)$ is an $\mathcal{F}_0$-measurable n-dimensional random value, i. e., $b \in k^n$.

Note that under the above assumptions problem (1), (1a), (1b) has a unique solution [8]. Denote this solution by $x(t, b, \varphi)$, i. e., $x(t, b, \varphi) = \varphi$ with $t < 0$ and $x(0, b, \varphi) = b$.

Introduce a special denotation for the linear normed subspace of the space k^n defined as follows:

$$k_q^n = \left\{\alpha : \alpha \in k^n, \|\alpha\|_{k_q^n} = (E|\alpha|^q)^{1/q} < \infty\right\}.$$

Definition 2. We say that system (1) is exponentially *q-stable* $(1 \leq q < \infty)$ with respect to the initial data, if there exist positive values K, λ such that solutions $x(t, b, \varphi)$ to problem (1), (1a), (1b) satisfy the inequality

$$(E|x(t, b, \varphi)|^q)^{1/q} \leq K \exp\{-\lambda t\} \left(||b||_{k_q^n} + \operatorname*{vrai\,sup}_{t<0}(E|\varphi(t)|^q)^{1/q}\right) \ \ (t \geq 0).$$

Lemma 1. *Let $f(s)$ be a scalar random process integrable with respect to the Wiener process $\mathcal{B}(s)$ on the segment $[0, t]$. Then*

$$\left(E\left|\int_0^t f(s)d\mathcal{B}(s)\right|^{2p}\right)^{1/(2p)} \leq c_p \left(E\left(\int_0^t |f(s)|^2 ds\right)^p\right)^{1/(2p)}; \tag{2}$$

here c_p is some value depending on $p \geq 1$.

The validity of inequality (2) follows from inequality (4) in ([21], p. 65), where one obtains a concrete expression for c_p.

Lemma 2. *Assume that $g(s)$ is a scalar function defined on $[0,\infty)$, whose square is locally summable, and $f(s)$ is a scalar random process such that $\sup\limits_{s\geq 0}(E|f(s)|^{2p})^{1/(2p)} < \infty$. Then*

$$\sup_{t\geq 0}\left(E\left|\int_0^t g(s)f(s)ds\right|^{2p}\right)^{1/(2p)} \leq \sup_{t\geq 0}\left(\int_0^t |g(s)|ds\right)\sup_{s\geq 0}\left(E\left|f(s)\right|^{2p}\right)^{1/(2p)}, \tag{3}$$

$$\sup_{t\geq 0}\left(E\left|\int_0^t (g(s))^2(f(s))^2 ds\right|^{p}\right)^{1/(2p)} \leq \sup_{t\geq 0}\left(\int_0^t (g(s))^2 ds\right)^{1/2}\sup_{s\geq 0}\left(E\left|f(s)\right|^{2p}\right)^{1/(2p)}. \tag{4}$$

This lemma is proved in [14].

2. THE RESEARCH TECHNIQUE

As was mentioned in the Introduction, in this paper we study the stability of the trivial solution to system (1). To this end, we transform the system under consideration, namely, with the help of a simpler auxiliary (model) equation we get an integral equation, for which the conditions that ensure the stability of the trivial solution to (1) can be verified immediately.

Therefore, along with system (1) we consider the following auxiliary system of ordinary linear differential equations with impulse actions on certain components of solutions:

$$\begin{aligned} dx(t) &= [B(t)x(t) + f(t)]dt \ \ (t\geq 0),\\ x(\mu_j) &= B_j x(\mu_j - 0), \ j = 1,2,\dots\ , \end{aligned} \tag{5}$$

where $B(t)$ is an $n\times n$-matrix, whose elements are Lebesgue measurable functions, $f(t)$ is an n-dimensional Lebesgue measurable function, while $B_j, \mu_j, j = 1,2,\dots,$ are the same values as in system (1).

For system (5), the corresponding linear homogeneous system takes the form

$$\begin{aligned} dx(t) &= B(t)x(t)dt \ \ (t\geq 0),\\ x(\mu_j) &= B_j x(\mu_j - 0), \ j = 1,2,\dots . \end{aligned} \tag{6}$$

Definition 3. An $n\times n$-matrix $X(t)\,(t\geq 0)$, whose columns are solutions to system (6), while $X(0) = \bar{E}$, is called the fundamental matrix of system (5).

Since for any $x_0 \in k^n$ there exists a unique solution to system (6) that goes through it, $\det X(t) \neq 0$ with $t \geq 0$.

The following assertion is valid.

Lemma 3. *The solution to system* (5) *going through $x_0 \in k^n$ allows the representation*

$$x(t) = X(t)x_0 + \int_0^t X(t)X(s)^{-1}f(s)ds \ \ (t\geq 0). \tag{7}$$

Using system (5) and Lemma 3, we can write problem (1), (1a), (1b) in the equivalent form

$$x(t) = X(t)b + (\Theta x)(t) + (C\varphi)(t) \ \ (t\geq 0), \tag{8}$$

where

$$(\Theta x)(t) = \int_0^t X(t)X(s)^{-1}\left[B(s) - \sum_{j=1}^{m_1} A_{1j}(s)\bar{x}(h_{1j}(s))\right]ds + \int_0^t X(t)X(s)^{-1}\sum_{i=2}^{m}\sum_{j=1}^{m_i} A_{ij}(s)\bar{x}(h_{ij}(s))d\mathcal{B}_i(s),$$

$$(C\varphi)(t) = \int_0^t X(t)X(s)^{-1}\left[-\sum_{j=1}^{m_1} A_{1j}(s)\bar{\varphi}(h_{1j}(s))\right]ds + \int_0^t X(t)X(s)^{-1}\sum_{i=2}^{m}\sum_{j=1}^{m_i} A_{ij}(s)\bar{\varphi}(h_{ij}(s))d\mathcal{B}_i(s).$$

Here $\bar{x}(t)$ is the unknown n-dimensional random process on $(-\infty, \infty)$ such that $\bar{x}(t) = 0$ with $t < 0$, while $\bar{\varphi}(t)$ is a known n-dimensional random process on $(-\infty, \infty)$ such that $\bar{\varphi}(t) = \varphi(t)$ with $t \in [-\hat{\sigma}, 0)$ and $\bar{\varphi}(t) = 0$ with $t \in (-\infty, -\hat{\sigma}) \cup [0, +\infty)$.

Let us state one useful assertion which follows from more general results obtained in [12].

Theorem 1. *Let* $1 \le q < \infty$. *Assume that for some positive value* λ *and arbitrary* φ, b *such that* $\operatorname{vrai\,sup}_{t<0}(E|\varphi(t)|^q)^{1/q} < \infty$, $b \in k_q^n$, *system* (6) *allows estimates*

$$\sup_{t\ge 0}(E|\exp\{\lambda t\}X(t)b|^q)^{1/q} \le c_1\|b\|_{k_q^n},$$

$$\sup_{t\ge 0}(E|\exp\{\lambda t\}(\Theta x)(t)|^q)^{1/q} \le c_2 \sup_{t\ge 0}(E|\exp\{\lambda t\}x(t)|^q)^{1/q},$$

$$\sup_{t\ge 0}(E|\exp\{\lambda t\}(C\varphi)(t)|^q)^{1/q} \le c_3 \operatorname{vrai\,sup}_{t<0}(E|\varphi(t)|^q)^{1/q},$$

where c_1, c_2, c_3 *are some positive values,* $c_2 < 1$. *Then system* (1) *is exponentially* q*-stable with respect to the initial data.*

On the base of this theorem, in the paper [12] we establish sufficient conditions for the exponential q-stability of systems in form (1) in terms of parameters of these systems.

Denote $x(t) = \operatorname{col}(x_1(t), \dots, x_n(t))$ $(t \ge 0)$, $\bar{x}_i^\lambda = \sup_{t\ge 0}(E|\exp\{\lambda t\}x_i(t)|^q)^{1/q}$, $i = 1, \dots, n$, $\bar{x}^\lambda = \operatorname{col}(\bar{x}_1^\lambda, \dots, \bar{x}_n^\lambda)$.

Let $1 \le q < \infty$. Assume that for some positive λ by componentwise estimation of solutions to system (8) we have succeeded in obtaining the matrix inequality

$$\bar{E}\bar{x}^\lambda \le C\bar{x}^\lambda + \bar{c}\|b\|_{k_q^n}\hat{E} + \hat{c}\operatorname{vrai\,sup}_{t<0}(E|\varphi(t)|^q)^{1/q}\hat{E}, \tag{9}$$

where C is some $n \times n$-matrix, $\bar{c}, \hat{c}$ are some positive values, $\hat{E}$ is the n-dimensional vector, all whose elements equal one. The following assertion is valid.

Theorem 2. *If the matrix* $\bar{E} - C$ *is an* $\mathcal{M}$*-matrix, then system* (1) *is exponentially* q*-stable with respect to the initial data.*

Proof. Under assumptions of the theorem, the matrix $\bar{E} - C$ is positively invertible. Consequently, we can write inequality (9) as follows:

$$\bar{E}\bar{x}^\lambda \le (\bar{E} - C)^{-1}(\bar{c}\|b\|_{k_q^n}\hat{E} + \hat{c}\operatorname{vrai\,sup}_{t<0}(E|\varphi(t)|^q)^{1/q}\hat{E}).$$

The obtained inequality implies the correlation

$$|\bar{x}^\lambda| \le K(\|b\|_{k_q^n} + \operatorname{vrai\,sup}_{t<0}(E|\varphi(t)|^q)^{1/q}), \tag{10}$$

where $K = \||(\bar{E} - C)^{-1}\||\,|\hat{E}|\max\{\bar{c}, \hat{c}\}$. Since $x(t, b, \varphi) = x(t)$ and $\sup_{t\ge 0}(E|\exp\{\lambda t\}x(t, b, \varphi)|^q)^{1/q} \le |\bar{x}^\gamma|$, inequality (10) implies that the estimate

$$\sup_{t\ge 0}(E|\exp\{\lambda t\}x(t, b, \varphi)|^q)^{1/q} \le c(\|b\|_{k_q^n} + \operatorname{vrai\,sup}_{t<0}(E|\varphi(t)|^q)^{1/q}),$$

where c is some positive value, takes place with any φ, b such that $\operatorname{vrai\,sup}_{t<0}(E|\varphi(t)|^q)^{1/q} < \infty$, $b \in k_q^n$. Consequently, system (1) is exponentially q-stable with respect to the initial data. □

In the next section, on the base of Theorem 2, assuming that $q = 2p,\ 1 \le p < \infty$, we establish sufficient conditions for the exponential q-stability of system (1) in terms of the positive invertibility of the matrix calculated from parameters of this system.

3. THE MAIN RESULT

In this section, we assume that there exist subsets of numbers $I_s \subset \{1, \dots, m_1\}$, $s = 1, \dots, n$, and positive values $\rho, \sigma,\ \bar{b}_s,\ s = l+1, \dots, n,\ \bar{a}_s, \bar{a}^{ij}_{sk},\ i = 1, \dots, m,\ j = 1, \dots, m_i,\ s, k = 1, \dots, n$, such that the following estimates are valid for system (1):

$$|b^j_{ss}| \le \bar{b}_s,\ s = l+1, \dots, n,\ j = 1, 2, \dots,$$

$$\rho \le \mu_{j+1} - \mu_j \le \sigma,\ j = 1, 2, \dots,$$

$$|a^{ij}_{sk}(t)| \le \bar{a}^{ij}_{sk},\ t \in [0, +\infty),\ i = 1, \dots, m,\ j = 1, \dots, m_i,\ s, k = 1, \dots, n,$$

$P \times \mu$-almost everywhere,

$$\sum_{k \in I_s} a^{1k}_{ss}(t) \ge \bar{a}_s,\ t \in [0, +\infty),\ s = 1, \dots, n,$$

$P \times \mu$-almost everywhere, and there also exist positive values $\hat{c}_s, s = l+1, \dots, n$, such that

$$\exp\{-\bar{a}_s t\} \prod_{0 < \mu_j \le t} |b^j_{ss}| < \hat{c}_s \quad \text{with } t \in [0, +\infty), \quad s = l+1, \dots, n.$$

Denote by the symbol C the $n \times n$-matrix, whose elements are defined as follows:

$$c_{ss} = 1 - \tfrac{1}{\bar{a}_s} \left[\sum_{k \in I_s} \sum_{j=1}^{m_1} \bar{a}^{1k}_{ss} \tau_{1k} \bar{a}^{1j}_{ss} + \sum_{j=1, j \notin I_s}^{m_1} \bar{a}^{1j}_{ss} \right] -$$
$$- c_p \left(\tfrac{1}{2\bar{a}_s} \right)^{1/2} \left[\sum_{k \in I_s} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \bar{a}^{1k}_{ss} \sqrt{\tau_{1k}} \bar{a}^{ij}_{ss} + \sum_{i=2}^{m} \sum_{j=1}^{m_i} \bar{a}^{ij}_{ss} \right],\ s = 1, \dots, l,$$

$$c_{sj} = -\tfrac{1}{\bar{a}_s} \left[\sum_{k \in I_s} \sum_{\nu=1}^{m_1} \bar{a}^{1k}_{ss} \tau_{1k} \bar{a}^{1\nu}_{sj} + \sum_{\nu=1}^{m_1} \bar{a}^{1\nu}_{sj} \right] -$$
$$- c_p \left(\tfrac{1}{2\bar{a}_s} \right)^{1/2} \left[\sum_{k \in I_s} \sum_{i=2}^{m} \sum_{\nu=1}^{m_i} \bar{a}^{1k}_{ss} \sqrt{\tau_{1k}} \bar{a}^{i\nu}_{sj} + \sum_{i=2}^{m} \sum_{\nu=1}^{m_i} \bar{a}^{i\nu}_{sj} \right],\ s = 1, \dots, l,\ j = 1, \dots, n,\ s \ne j,$$

$$c_{ss} = 1 - \tfrac{\max\{1, \bar{b}_s\}(1 - \exp\{-\bar{a}_s \sigma\})}{\bar{a}_s (1 - \exp\{-\bar{a}_s \rho\} \bar{b}_s)} \left[\sum_{k \in I_s} \sum_{j=1}^{m_1} \bar{a}^{1k}_{ss} \tau_{1k} \bar{a}^{1j}_{ss} + \sum_{j=1, j \notin I_s}^{m_1} \bar{a}^{1j}_{ss} \right] -$$
$$- c_p \left(\tfrac{\max\{1, \bar{b}^2_s\}(1 - \exp\{-2\bar{a}_s \sigma\})}{2\bar{a}_s (1 - \exp\{-2\bar{a}_s \rho\} \bar{b}^2_s)} \right)^{1/2} \left[\sum_{k \in I_s} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \bar{a}^{1k}_{ss} \sqrt{\tau_{1k}} \bar{a}^{ij}_{ss} + \sum_{i=2}^{m} \sum_{j=1}^{m_i} \bar{a}^{ij}_{ss} \right],\ s = l+1, \dots, n,$$

$$c_{sj} = -\tfrac{\max\{1, \bar{b}_s\}(1 - \exp\{-\bar{a}_s \sigma\})}{\bar{a}_s (1 - \exp\{-\bar{a}_s \rho\} \bar{b}_s)} \left[\sum_{k \in I_s} \sum_{\nu=1}^{m_1} \bar{a}^{1k}_{ss} \tau_{1k} \bar{a}^{1\nu}_{sj} + \sum_{\nu=1}^{m_1} \bar{a}^{1\nu}_{sj} \right] -$$
$$- c_p \left(\tfrac{\max\{1, \bar{b}^2_s\}(1 - \exp\{-2\bar{a}_s \sigma\})}{2\bar{a}_s (1 - \exp\{-2\bar{a}_s \rho\} b^2_s)} \right)^{1/2} \left[\sum_{k \in I_s} \sum_{i=2}^{m} \sum_{\nu=1}^{m_i} \bar{a}^{1k}_{ss} \sqrt{\tau_{1k}} \bar{a}^{i\nu}_{sj} + \sum_{i=2}^{m} \sum_{\nu=1}^{m_i} \bar{a}^{i\nu}_{sj} \right],$$
$$s = l+1, \dots, n, \quad j = 1, \dots, n, \quad s \ne j.$$

Theorem 3. *If the matrix C is an $\mathcal{M}$-matrix, then system (1) is exponentially $2p$-stable with respect to the initial data.*

Proof. Let us write system (1) subject to (1a) in the form

$$
\begin{aligned}
d\bar{x}_s(t) = -\sum_{j=1}^{m_1}\sum_{k=1}^{n} a_{sk}^{1j}(t)[\bar{x}_k(h_{1j}(t)) + \bar{\varphi}_k(h_{1j}(t))]dt + \\
+\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{k=1}^{n} a_{sk}^{ij}(t)[\bar{x}_k(h_{ij}(t)) + \bar{\varphi}_k(h_{ij}(t))]d\mathcal{B}_i(t)\ (t \ge 0),\ s = 1,\dots,n, \\
\bar{x}_s(\mu_j) = b_{ss}^{j}\bar{x}_s(\mu_j - 0),\ j = 1,2,\dots,\ \text{a. s.},\ s = l+1,\dots,n,
\end{aligned}
\tag{11}
$$

where $\bar{x}_s(t)$ is the unknown scalar random process on $(-\infty,\infty)$ such that $\bar{x}_s(t) = 0$ with $t < 0$, and $\bar{\varphi}_s(t)$ is a known scalar random process on $(-\infty,\infty)$ such that $\bar{\varphi}_s(t) = \varphi_s(t)$ with $t \in [-\hat{\sigma}, 0)$ and $\bar{\varphi}_s(t) = 0$ with $t \in (-\infty, -\hat{\sigma}) \cup [0, +\infty)$ for $s = 1,\dots,n$. Let the symbol $\bar{x}(t, b, \bar{\varphi})$ denote the solution to system (11) that satisfies condition (1b). Evidently, the solution to problem (11), (1b) with $t \ge 0$ coincides with that to problem (1), (1a), (1b), i. e., $x(t, b, \varphi) = \bar{x}(t, b, \bar{\varphi})$, $t \ge 0$.

If in system (11) we put $\bar{x}_s(t) = \exp\{-\lambda t\}y_s(t)$, where $y_s(t)$ is the unknown scalar random process on $(-\infty,\infty)$ such that $y_s(t) = 0$ with $t < 0$ and all $s = 1,\dots,n$, while $0 < \lambda < \min\{\bar{a}_s, s = 1,\dots,n\}$, then we get the system

$$
\begin{aligned}
dy_s(t) = \left[\lambda y_s(t) - \sum_{j=1}^{m_1}\sum_{k=1}^{n} a_{sk}^{1j}(t)[\exp\{\lambda(t - h_{1j}(t))\}y_k(h_{1j}(t)) + \exp\{\lambda t\}\bar{\varphi}_k(h_{1j}(t))]\right] dt + \\
+\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{k=1}^{n} a_{sk}^{ij}(t)\left[\exp\{\lambda(t - h_{ij}(t))\}y_k(h_{ij}(t)) + \exp\{\lambda t\}\bar{\varphi}_k(h_{ij}(t))\right] d\mathcal{B}_i(t)\ (t \ge 0),\ s = 1,\dots,n, \\
y_s(\mu_j) = b_{ss}^{j}y_s(\mu_j - 0),\ j = 1,2,\dots,\ \text{a. s.},\ s = l+1,\dots,n.
\end{aligned}
\tag{12}
$$

Putting $\eta_s(t) = \sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda(t - h_{1k}(t))\} - \lambda$ with $s = 1,\dots,n$ and taking into account the equality $\int_{h_{1k}(t)}^{t} dy_s(\tau) = y_s(t) - y_s(h_{1k}(t))$, $k \in I_s$, we rewrite system (12) in the form

$$
\begin{aligned}
dy_s(t) = \Bigg[- \eta_s(t)y_s(t) + \sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda(t - h_{1k}(t))\}\int_{h_{1k}(t)}^{t} dy_s(\tau) + \sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda t\}\bar{\varphi}_k(h_{1k}(t)) + \\
+\sum_{j=1}^{m_1}\sum_{k=1, k\ne s \text{ with } j\in I_s}^{n} a_{sk}^{1j}(t)[\exp\{\lambda(t - h_{1j}(t))\}y_k(h_{1j}(t)) + \exp\{\lambda t\}\bar{\varphi}_k(h_{1j}(t))]\Bigg] dt + \\
+\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{k=1}^{n} a_{sk}^{ij}(t)\left[\exp\{\lambda(t - h_{ij}(t))\}y_k(h_{ij}(t)) + \exp\{\lambda t\}\bar{\varphi}_k(h_{ij}(t))\right] d\mathcal{B}_i(t)\ (t \ge 0),\ s = 1,\dots,n, \\
y_s(\mu_j) = b_{ss}^{j}y_s(\mu_j - 0),\ j = 1,2,\dots,\ \text{a. s.},\ s = l+1,\dots,n.
\end{aligned}
\tag{13}
$$

Substituting the expression for $dy_s(t)$ on the right-hand side of the sth $(s = 1,\dots,n)$ equation in system (12) to the sth equation in system (13), we get equalities

$$
\begin{aligned}
dy_s(t) = \Bigg[-\eta_s(t)y_s(t) + \sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda(t - h_{1k}(t))\}\times \\
\times \int_{h_{1k}(t)}^{t} \Bigg\{\left[\lambda y_s(\tau) - \sum_{j=1}^{m_1}\sum_{\nu=1}^{n} a_{s\nu}^{1j}(\tau)[\exp\{\lambda(\tau - h_{1j}(\tau))\}y_\nu(h_{1j}(\tau)) + \exp\{\lambda\tau\}\bar{\varphi}_\nu(h_{1j}(\tau))]\right] d\tau + \\
+\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{k=1}^{n} a_{sk}^{ij}(\tau)\left[\exp\{\lambda(\tau - h_{ij}(\tau))\}y_k(h_{ij}(\tau)) + \exp\{\lambda\tau\}\bar{\varphi}_k(h_{ij}(\tau))\right] d\mathcal{B}_i(\tau)\Bigg\} +
\end{aligned}
$$

$$+\sum_{k\in I_s} a_{ss}^{1k}(t)\exp\{\lambda t\}\bar{\varphi}_k(h_{1k}(t))+$$

$$\left.+\sum_{j=1}^{m_1}\sum_{k=1,k\neq s \text{with} j\in I_s}^{n} a_{sk}^{1j}(t)[\exp\{\lambda(t-h_{1j}(t))\}y_k(h_{1j}(t))+\exp\{\lambda t\}\bar{\varphi}_k(h_{1j}(t))]\right] dt+$$

$$+\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{k=1}^{n} a_{sk}^{ij}(t)\left[\exp\{\lambda(t-h_{ij}(t))\}y_k(h_{ij}(t))+\exp\{\lambda t\}\bar{\varphi}_k(h_{ij}(t))\right] d\mathcal{B}_i(t)\ (t\geq 0),\ s=1,\ldots,n,$$

$$y_s(\mu_j)=b_{ss}^j y_s(\mu_j-0),\ j=1,2,\ldots,\ \text{a. s.},\ s=l+1,\ldots,n. \tag{14}$$

Let $m_s(t,\varsigma)=\exp\left\{-\int\limits_{\varsigma}^{t}\eta_s(\zeta)d\zeta\right\}$, $s=1,\ldots,l$, and $m_s(t,\varsigma)=\exp\left\{-\int\limits_{\varsigma}^{t}\eta_s(\zeta)d\zeta\right\}\prod\limits_{\varsigma<\mu_j\leq t} b_{ss}^j$, $s=l+1,\ldots,n$. Using the formula for the representation of solutions to Itô linear scalar differential equations with impulse actions [12], from system (14), taking into account condition (1b), we deduce the system

$$y_s(t)=m_s(t,0)b_s+\sum_{k\in I_s}\int\limits_0^t m_s(t,\varsigma)a_{ss}^{1k}(\varsigma)\exp\{\lambda(\varsigma-h_{1k}(\varsigma))\}\int\limits_{h_{1k}(\varsigma)}^{\varsigma}\lambda y_s(\tau)d\tau d\varsigma-$$

$$-\sum_{k\in I_s}\sum_{j=1}^{m_1}\sum_{\nu=1}^{n}\int\limits_0^t m_s(t,\varsigma)a_{ss}^{1k}(\varsigma)\exp\{\lambda(\varsigma-h_{1k}(\varsigma))\}\times$$

$$\times\int\limits_{h_{1k}(\varsigma)}^{\varsigma} a_{s\nu}^{1j}(\tau)[\exp\{\lambda(\tau-h_{1j}(\tau))\}y_\nu(h_{1j}(\tau))+\exp\{\lambda\tau\}\bar{\varphi}_\nu(h_{1j}(\tau))]d\tau d\varsigma+$$

$$+\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{\nu=1}^{n}\int\limits_0^t m_s(t,\varsigma)a_{ss}^{1k}(\varsigma)\exp\{\lambda(\varsigma-h_{1k}(\varsigma))\}\times$$

$$\times\int\limits_{h_{1k}(\varsigma)}^{\varsigma} a_{s\nu}^{ij}(\tau)\left[\exp\{\lambda(\tau-h_{ij}(\tau))\}y_\nu(h_{ij}(\tau))+\exp\{\lambda\tau\}\bar{\varphi}_\nu(h_{ij}(\tau))\right] d\mathcal{B}_i(\tau)d\varsigma+$$

$$+\sum_{k\in I_s}\int\limits_0^t m_s(t,\varsigma)a_{ss}^{1k}(\varsigma)\exp\{\lambda\varsigma\}\bar{\varphi}_k(h_{1k}(\varsigma))d\varsigma+$$

$$+\sum_{j=1}^{m_1}\sum_{k=1,k\neq s \text{with} j\in I_s}^{n}\int\limits_0^t m_s(t,\varsigma)a_{sk}^{1j}(\varsigma)[\exp\{\lambda(\varsigma-h_{1j}(\varsigma))\}y_k(h_{1j}(\varsigma))+\exp\{\lambda\varsigma\}\bar{\varphi}_k(h_{1j}(\varsigma))]d\varsigma+$$

$$+\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{k=1}^{n}\int\limits_0^t m_s(t,\varsigma)a_{sk}^{ij}(\varsigma)\left[\exp\{\lambda(\varsigma-h_{ij}(\varsigma))\}y_k(h_{ij}(\varsigma))+\exp\{\lambda\varsigma\}\bar{\varphi}_k(h_{ij}(\varsigma))\right] d\mathcal{B}_i(\varsigma)$$

$$(t\geq 0),\quad s=1,\ldots,n. \tag{15}$$

For simplicity of notations, we put $\hat{y}_s=\sup\limits_{t\geq 0}\left(E|y_s(t)|^{2p}\right)^{1/(2p)}$, $\hat{\varphi}_s=\operatorname{vrai}\sup\limits_{t<0}\left(E|\varphi_s(t)|^{2p}\right)^{1/(2p)}$,

$s=1,\dots,n$, $||\varphi|| = \operatorname{vrai\,sup}_{t<0}\left(E|\varphi(t)|^{2p}\right)^{1/(2p)}$. Below we also use the following evident inequalities:

$$\operatorname{vrai\,sup}_{t\geq 0}\left(E|\exp\{\lambda t\}\bar{\varphi}_s(h_{ij}(t))|^{2p}\right)^{1/(2p)} \leq \exp\{\lambda\tau_{ij}\}\operatorname{vrai\,sup}_{t<0}\left(E|\varphi_s(t)|^{2p}\right)^{1/(2p)},$$
$$s=1,\dots,n,\ i=1,\dots,m,\ j=1,\dots,m_i;$$

$$|m_s(t,\varsigma)| \leq \exp\{-(\bar{a}_s-\lambda)(t-\varsigma)\},\ t\in[0,+\infty),\ \varsigma\in[0,t],\ P\times\mu-\text{almost everywhere},\ s=1,\dots,l,$$

$$|m_s(t,\varsigma)| \leq \exp\{-(\bar{a}_s-\lambda)(t-\varsigma)\}\prod_{\varsigma<\mu_j\leq t}|b_{ss}^j|,\ t\in[0,+\infty),\ \varsigma\in[0,t],\ P\times\mu-\text{almost everywhere},$$

$s=l+1,\dots,n$, as well as estimates

$$\int_0^t \exp\{-(\bar{a}_s-\lambda)(t-\varsigma)\}\prod_{\varsigma<\mu_j\leq t}|b_{ss}^j|d\varsigma \leq \frac{\max\{1,\bar{b}_s\}(1-\exp\{-(\bar{a}_s-\lambda)\sigma\})}{(\bar{a}_s-\lambda)(1-\exp\{-(\bar{a}_s-\lambda)\rho\}\bar{b}_s)},\ s=l+1,\dots,n,$$

proved in [18] and estimates

$$\left(\int_0^t \exp\{-2(\bar{a}_s-\lambda)(t-\varsigma)\}\prod_{\varsigma<\mu_j\leq t}(b_{ss}^j)^2d\varsigma\right)^{1/2} \leq \left(\frac{\max\{1,\bar{b}_s^2\}(1-\exp\{-2(\bar{a}_s-\lambda)\sigma\})}{2(\bar{a}_s-\lambda)(1-\exp\{-2(\bar{a}_s-\lambda)\rho\}b_s^2)}\right)^{1/2},$$
$$s=l+1,\dots,n,$$

whose validity immediately follows from above bounds.

Taking into account these denotations and inequalities, as well as correlations (2)–(4), we can easily deduce the following estimates from Eq. (15):

$$\begin{gathered}
\hat{y}_s \leq \hat{c}_s||b_s||_{k_{2p}^1} + \lambda L_{1s}\left[\sum_{k\in I_s}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\tau_{1k}\right]\hat{y}_s+ \\
+L_{1s}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1}\sum_{\nu=1}^{n}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\tau_{1k}\bar{a}_{s\nu}^{1j}\exp\{\lambda\tau_{1j}\}(\hat{y}_\nu+\hat{\varphi}_\nu)\right]+ \\
+c_pL_{2s}\left[\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{\nu=1}^{n}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\sqrt{\tau_{1k}}\bar{a}_{s\nu}^{ij}\exp\{\lambda\tau_{ij}\}(\hat{y}_\nu+\hat{\varphi}_\nu)\right]+ \\
+L_{1s}\left[\sum_{k\in I_s}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\hat{\varphi}k\right]+L_{1s}\left[\sum_{j=1}^{m_1}\sum_{k=1,k\neq s\,\text{with}\,j\in I_s}^{n}\bar{a}_{sk}^{1j}\exp\{\lambda\tau_{1j}\}(\hat{y}_k+\hat{\varphi}_k)\right]+ \\
+c_pL_{2s}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{k=1}^{n}\bar{a}_{sk}^{ij}\exp\{\lambda\tau_{ij}\}(\hat{y}_k+\hat{\varphi}_k)\right],\ s=1,\dots,n;
\end{gathered} \tag{16}$$

here $\hat{c}_s=1$ for $s=1,\dots,l$, while $\hat{c}_s$ for $s=l+1,\dots,n$ are defined above,

$$L_{1s} := \frac{1}{(\bar{a}_s-\lambda)},\ L_{2s} := \left(\frac{1}{2(\bar{a}_s-\lambda)}\right)^{1/2},\ s=1,\dots,l,\quad L_{1s} := \frac{\max\{1,\bar{b}_s\}(1-\exp\{-(\bar{a}_s-\lambda)\sigma\})}{(\bar{a}_s-\lambda)(1-\exp\{-(\bar{a}_s-\lambda)\rho\}\bar{b}_s)},$$
$$L_{2s} := \left(\frac{\max\{1,\bar{b}_s^2\}(1-\exp\{-2(\bar{a}_s-\lambda)\sigma\})}{2(\bar{a}_s-\lambda)(1-\exp\{-2(\bar{a}_s-\lambda)\rho\}b_s^2)}\right)^{1/2},\ s=l+1,\dots,n.$$

Taking into account estimates (16) and the fact that the norm in R^n is chosen so that $\hat{\varphi}_j \leq ||\varphi||$ for all $j=1,\dots,n$, we conclude that

$$\hat{y}_s \leq \hat{c}_s||b_s||_{k_{2p}^1} + \sum_{j=1}^{n}N_{sj}(\lambda)\hat{y}_j + M_s(\lambda)||\varphi||,\ s=1,\dots,n. \tag{17}$$

Here

$$N_{ss}(\lambda) := \lambda L_{1s}\left[\sum_{k\in I_s}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\tau_{1k}\right]+$$

$$+L_{1s}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\tau_{1k}\bar{a}_{ss}^{1j}\exp\{\lambda\tau_{1j}\}+\sum_{j=0,j\notin I_s}^{m_1}\bar{a}_{ss}^{1j}\exp\{\lambda\tau_{1j}\}\right]+$$

$$+c_pL_{2s}\left[\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\sqrt{\tau_{1k}}\bar{a}_{ss}^{ij}\exp\{\lambda\tau_{ij}\}+\sum_{i=2}^{m}\sum_{j=1}^{m_i}\bar{a}_{ss}^{ij}\exp\{\lambda\tau_{ij}\}\right],\ s=1,\ldots,n,$$

$$N_{sj}(\lambda) := L_{1s}\left[\sum_{k\in I_s}\sum_{\nu=0}^{m_1}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\tau_{1k}\bar{a}_{sj}^{1\nu}\exp\{\lambda\tau_{1\nu}\}+\sum_{\nu=1}^{m_1}\bar{a}_{sj}^{1\nu}\exp\{\lambda\tau_{1\nu}\}\right]+$$

$$+c_pL_{2s}\left[\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\sqrt{\tau_{1k}}\bar{a}_{sj}^{i\nu}\exp\{\lambda\tau_{i\nu}\}+\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}\bar{a}_{sj}^{i\nu}\exp\{\lambda\tau_{i\nu}\}\right], s,j=1,\ldots,n,\ s\neq j,$$

$$M_s(\lambda) := L_{1s}\left[\sum_{k\in I_s}\sum_{j=1}^{m_1}\sum_{\nu=1}^{n}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\tau_{1k}\bar{a}_{s\nu}^{1j}\exp\{\lambda\tau_{1j}\}+\right.$$

$$\left.+\sum_{k\in I_s}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}+\sum_{j=1}^{m_1}\sum_{k=1,k\neq s\,\mathrm{with}\,j\in I_s}^{n}\bar{a}_{sk}^{1j}\exp\{\lambda\tau_{1j}\}\right]+$$

$$+c_pL_{2s}\left[\sum_{k\in I_s}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{\nu=1}^{n}\bar{a}_{ss}^{1k}\exp\{\lambda\tau_{1k}\}\sqrt{\tau_{1k}}\bar{a}_{s\nu}^{ij}\exp\{\lambda\tau_{ij}\}+\sum_{i=2}^{m}\sum_{j=1}^{m_i}\sum_{\nu=1}^{n}\bar{a}_{s\nu}^{ij}\exp\{\lambda\tau_{ij}\}\right], s=1,\ldots,n.$$

Put $y(t)=\mathrm{col}(y_1(t),\ldots,y_n(t))$, $\bar{y}=\mathrm{col}(\bar{y}_1,\ldots,\bar{y}_n)$, $M(\lambda)=\mathrm{col}(M_1(\lambda),\ldots,M_n(\lambda))$ and assume that $C(\lambda)=(c_{ij}(\lambda))_{i,j=1}^{n}$ is the $n\times n$-matrix, whose elements are defined as follows:

$$c_{ss}(\lambda)=1-N_{ss}(\lambda),\ s=1,\ldots,n,\ c_{sj}(\lambda)=-N_{sj}(\lambda),\ s,j=1,\ldots,n,\ s\neq j.$$

Then estimates (17) imply the correlation

$$C(\lambda)\bar{y}\le\hat{c}||b||_{k_{2p}^n}\hat{E}+M(\lambda)||\varphi||, \tag{18}$$

where $\hat{c}=\max\{\hat{c}_s, s=1,\ldots,n\}$, $\hat{E}$ is the n-dimensional vector, all whose elements equal one. It is also evident that $C(0)=C$. According to assumptions of the theorem, the matrix C is an $\mathcal{M}$-matrix. Then with sufficiently small λ the matrix $C(\lambda)$ also is an $\mathcal{M}$-matrix, consequently, there exists $\lambda=\lambda_0$ such that the matrix $C(\lambda_0)$ is positively invertible. Therefore inequality (18) gives the correlation

$$|\bar{y}|\le K(||b||_{k_{2p}^n}+||\varphi||), \tag{19}$$

where $K=\|C(\lambda_0)^{-1}\||\hat{E}|\max\{\hat{c},|M(\lambda_0)|\}$.

Since $x(t,b,\varphi)=\exp\{-\lambda t\}y(t)$ and $\sup\limits_{t\ge0}(E|y(t)|^{2p})^{1/(2p)}\le|\bar{y}|$, inequality (19) implies that there exist positive values $\lambda=\lambda_0$, $K=\|(C(\lambda_0)^{-1}\||\hat{E}|\max\{\hat{c},|M(\lambda_0)|\}$ such that the solution $x(t,b,\varphi)$ to problem (1), (1a), (1b) satisfies the inequality

$$(E|x(t,b,\varphi)|^{2p})^{1/(2p)}\le K\exp\{-\lambda t\}\left(||b||_{k_{2p}^n}+\operatorname*{vrai\,sup}_{t<0}(E|\varphi(t)|^{2p})^{1/(2p)}\right)\ (t\ge0).$$

Consequently, system (1) is exponentially $2p$-stable with respect to the initial data. □

Remark 1. One can immediately verify whether the matrix C is an $\mathcal{M}$-matrix by evaluating its diagonal minors. Namely, if all these minors are positive, then the matrix C is an $\mathcal{M}$-matrix. Moreover, one can establish this fact by verifying sufficient conditions given in Section 1.

4. COROLLARIES OF THE MAIN RESULT

Assume that elements of the matrix A_{ij}, $i = 2, \ldots, m$, $j = 1, \ldots, m_i$, equal zero with $t \in [0, +\infty)$ $P \times \mu$-almost everywhere and there exist subsets of numbers $I_s \subset \{1, \ldots, m_1\}, s = 1, \ldots, n,$ and positive values

$$\rho, \ \ \sigma, \ \ \bar{b}_s, \ s = l+1, \ldots, n, \ \ \bar{a}_s, \bar{a}_{sk}^{ij} \ \ \ i = 1, \ldots, m, \ j = 1, \ldots, m_i, \ s, k = 1, \ldots, n,$$

such that system (1) satisfies estimates

$$|b_{ss}^j| \le \bar{b}_s, \ s = l+1, \ldots, n, \ j = 1, 2, \ldots, \ \ \rho \le \mu_{j+1} - \mu_j \le \sigma, \ j = 1, 2, \ldots,$$

$$|a_{sk}^{ij}(t)| \le \bar{a}_{sk}^{ij}, \ t \in [0, +\infty), \ i = 1, \ldots, m, \ j = 1, \ldots, m_i, \ s, k = 1, \ldots, n,$$

$P \times \mu$-almost everywhere. Assume also that

$$\sum_{k \in I_s} a_{ss}^{1k}(t) \ge \bar{a}_s, \ \ \ t \in [0, +\infty), \ s = 1, \ldots, n,$$

$P \times \mu$-almost everywhere and there exist positive values $\hat{c}_s, \ s = l+1, \ldots, n$, such that

$$\exp\{-\bar{a}_s t\} \prod_{0 < \mu_j \le t} |b_{ss}^j| < \hat{c}_s \text{ with } t \in [0, +\infty), \ \ \ s = l+1, \ldots, n.$$

Let us define an $n \times n$-matrix $C_1 = (c_{sj})_{s,j=1}^n$ as follows:

$$c_{ss} = 1 - \tfrac{1}{\bar{a}_s} \left[\sum_{k \in I_s} \sum_{j=1}^{m_1} \bar{a}_{ss}^{1k} \tau_{1k} \bar{a}_{ss}^{1j} + \sum_{j=1, j \notin I_s}^{m_1} \bar{a}_{ss}^{1j} \right], \ s = 1, \ldots, l,$$

$$c_{sj} = -\tfrac{1}{\bar{a}_s} \left[\sum_{k \in I_s} \sum_{\nu=1}^{m_1} \bar{a}_{ss}^{1k} \tau_{1k} \bar{a}_{sj}^{1\nu} + \sum_{\nu=1}^{m_1} \bar{a}_{sj}^{1\nu} \right], \ s = 1, \ldots, l, \ j = 1, \ldots, n, \ s \ne j.$$

$$c_{ss} = 1 - \tfrac{\max\{1, \bar{b}_s\}(1 - \exp\{-\bar{a}_s \sigma\})}{\bar{a}_s (1 - \exp\{-\bar{a}_s \rho\} \bar{b}_s)} \left[\sum_{k \in I_s} \sum_{j=1}^{m_1} \bar{a}_{ss}^{1k} \tau_{1k} \bar{a}_{ss}^{1j} + \sum_{j=1, j \notin I_s}^{m_1} \bar{a}_{ss}^{1j} \right], \ s = l+1, \ldots, n,$$

$$c_{sj} = -\tfrac{\max\{1, \bar{b}_s\}(1 - \exp\{-\bar{a}_s \sigma\})}{\bar{a}_s (1 - \exp\{-\bar{a}_s \rho\} \bar{b}_s)} \left[\sum_{k \in I_s} \sum_{\nu=1}^{m_1} \bar{a}_{ss}^{1k} \tau_{1k} \bar{a}_{sj}^{1\nu} + \sum_{\nu=1}^{m_1} \bar{a}_{sj}^{1\nu} \right], s = l+1, \ldots, n, \ j = 1, \ldots, n, \ s \ne j.$$

Corollary 1. If C_1 is an $\mathcal{M}$-matrix, then system (1) is exponentially $2p$-stable with respect to the initial data.

The validity of this assertion immediately follows from Theorem 3.

Remark 2. If under assumptions of Corollary 1 elements of matrices A_{1j}, $j = 1, \ldots, m_1$, are measurable locally summable functions, then system (1) is a deterministic system of linear differential equations with bounded delays; it is exponentially stable with respect to the initial data.

Assume that elements of matrices A_{1k}, $k = 2, \ldots, m_1$, A_{ij}, $i = 2, \ldots, m$, $j = 1, \ldots, m_i$, equal zero with $t \in [0, +\infty)$ $P \times \mu$-almost everywhere and there exist positive values ρ, σ, $\bar{b}_s$, $s = l+1, \ldots, n$, $\bar{a}_s$, $\bar{a}_{sk}^{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$, $s, k = 1, \ldots, n$, such that system (1) satisfies inequalities $|b_{ss}^j| \le \bar{b}_s$, $s = l+1, \ldots, n$, $j = 1, 2, \ldots$, $\rho \le \mu_{j+1} - \mu_j \le \sigma$ with $j = 1, 2, \ldots$, $|a_{sk}^{ij}(t)| \le \bar{a}_{sk}^{ij}$, $t \in [0, +\infty)$, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$, $s, k = 1, \ldots, n$, $P \times \mu$-almost everywhere, and $a_{ss}^{11}(t) \ge \bar{a}_s$, $t \in [0, +\infty)$, $s = 1, \ldots, n$, $P \times \mu$-almost everywhere and there exist positive values $\hat{c}_s$, $s = l+1, \ldots, n$, such that $\exp\{-\bar{a}_s t\} \prod_{0 < \mu_j \le t} |b_{ss}^j| < \hat{c}_s$ with $t \in [0, +\infty)$, $s = l+1, \ldots, n$. Let us define an $n \times n$-matrix $C_2 = (c_{sj})_{s,j=1}^n$ as follows:

$$c_{ss} = 1 - \tfrac{1}{\bar{a}_s} (\bar{a}_{ss}^{11})^2 \tau_{11}, s = 1, \ldots, l, \ c_{sj} = -\tfrac{1}{\bar{a}_s} \left[(\bar{a}_{ss}^{11})^2 \tau_{11} \ + \bar{a}_{sj}^{11} \right], s = 1, \ldots, l, \ j = 1, \ldots, n, \ s \ne j.$$

$$c_{ss} = 1 - \frac{\max\{1,\bar{b}_s\}(1-\exp\{-\bar{a}_s\sigma\})}{\bar{a}_s(1-\exp\{-\bar{a}_s\rho\}\bar{b}_s)}(\bar{a}^{11}_{ss})^2\tau_{11},\ s = l+1,\dots,n,$$

$$c_{sj} = -\frac{\max\{1,\bar{b}_s\}(1-\exp\{-\bar{a}_s\sigma\})}{\bar{a}_s(1-\exp\{-\bar{a}_s\rho\}\bar{b}_s)}\left[(\bar{a}^{11}_{ss})^2\tau_{11} + \bar{a}^{11}_{sj}\right],\ s = l+1,\dots,n,\ j = 1,\dots,n,\ s \neq j.$$

Then (in above denotations) Theorem 3 implies the following assertion.

Corollary 2. If C_2 is an $\mathcal{M}$-matrix, then system (1) is exponentially $2p$-stable with respect to the initial data.

Let us now assume that in system (1), $m_1 = 1$ and there exist positive values $\rho, \sigma,\ \bar{b}_s$, $s = l+1,\dots,n,\ \bar{a}_s,\ \bar{a}^{ij}_{sk},\ i = 1,\dots,m,\ j = 1,\dots,m_i,\ s,k = 1,\dots,n$, such that system (1) satisfies the following estimates: $|b^j_{ss}| \le \bar{b}_s,\ s = l+1,\dots,n,\ j = 1,2,\dots,\ \rho \le \mu_{j+1} - \mu_j \le \sigma$ for $j = 1,2,\dots$, $|a^{ij}_{sk}(t)| \le \bar{a}^{ij}_{sk},\ t \in [0,+\infty),\ i = 1,\dots,m,\ j = 1,\dots,m_i,\ s,k = 1,\dots,n,\ P \times \mu$-almost everywhere. Assume also that $a^{11}_{ss}(t) \ge \bar{a}_s,\ t \in [0,+\infty),\ s = 1,\dots,n,\ P \times \mu$-almost everywhere and there exist positive values $\hat{c}_s,\ s = l+1,\dots,n$, such that $\exp\{-\bar{a}_s t\}\prod_{0<\mu_j\le t}|b^j_{ss}| < \hat{c}_s$ with $t \in [0,+\infty),\ s = l+1,\dots,n$. Let us define an $n \times n$-matrix $C_3 = (c_{sj})^n_{s,j=1}$ as follows:

$$c_{ss} = 1 - \frac{1}{\bar{a}_s}(\bar{a}^{11}_{ss})^2\tau_{11} - c_p\left(\frac{1}{2\bar{a}_s}\right)^{1/2}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\bar{a}^{11}_{ss}\sqrt{\tau_{11}}\bar{a}^{ij}_{ss} + \sum_{i=2}^{m}\sum_{j=1}^{m_i}\bar{a}^{ij}_{ss}\right],\ s = 1,\dots,l,$$

$$c_{sj} = -\frac{1}{\bar{a}_s}\left[(\bar{a}^{11}_{ss})^2\tau_{11} + \bar{a}^{11}_{sl}\right] - c_p\left(\frac{1}{2\bar{a}_s}\right)^{1/2}\left[\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}\bar{a}^{11}_{ss}\sqrt{\tau_{11}}\bar{a}^{i\nu}_{sj} + \sum_{i=2}^{m}\sum_{\nu=1}^{m_i}\bar{a}^{i\nu}_{sj}\right],$$
$$s = 1,\dots,l,\quad j = 1,\dots,n,\quad s \neq j.$$

$$c_{ss} = 1 - \frac{\max\{1,\bar{b}_s\}(1-\exp\{-\bar{a}_s\sigma\})}{\bar{a}_s(1-\exp\{-\bar{a}_s\rho\bar{b}_s)}(\bar{a}^{11}_{ss})^2\tau_{11} -$$
$$-c_p\left(\frac{\max\{1,\bar{b}^2_s\}(1-\exp\{-2\bar{a}_s\sigma\})}{2\bar{a}_s(1-\exp\{-2\bar{a}_s\rho\}\bar{b}^2_s)}\right)^{1/2}\left[\sum_{i=2}^{m}\sum_{j=1}^{m_i}\bar{a}^{11}_{ss}\sqrt{\tau_{11}}\bar{a}^{ij}_{ss} + \sum_{i=2}^{m}\sum_{j=1}^{m_i}\bar{a}^{ij}_{ss}\right],\ s = l+1,\dots,n,$$

$$c_{sj} = -\frac{\max\{1,\bar{b}_s\}(1-\exp\{-\bar{a}_s\sigma\})}{\bar{a}_s(1-\exp\{-\bar{a}_s\rho\}\bar{b}_s)}\left[(\bar{a}^{11}_{ss})^2\tau_{11} + \bar{a}^{11}_{sj}\right] - c_p\left(\frac{\max\{1,\bar{b}^2_s\}(1-\exp\{-2\bar{a}_s\sigma\})}{2\bar{a}_s(1-\exp\{-2\bar{a}_s\rho\}\bar{b}^2_s)}\right)^{1/2}\times$$
$$\times\left[\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}\bar{a}^{11}_{ss}\sqrt{\tau_{11}}\bar{a}^{i\nu}_{sj} + \sum_{i=2}^{m}\sum_{\nu=1}^{m_i}\bar{a}^{i\nu}_{sj}\right],\ s = l+1,\dots,n,\ j = 1,\dots,n,\ s \neq j.$$

Corollary 3. If C_3 is an $\mathcal{M}$-matrix, then system (1) is exponentially $2p$-stable with respect to the initial data.

The validity of this assertion immediately follows from Theorem 3.

Corollary 4. Let system (1) satisfy all propositions that precede Corollary 3. Assume, in addition, that $n = 2$ and elements of the 2×2-matrix $C_4 = (c_{sj})^2_{s,j=1}$ satisfy inequalities $c_{11} > 0,\quad c_{11}c_{22} > c_{12}c_{21}$. Then system (1) is exponentially $2p$-stable with respect to the initial data.

The validity of Corollary 4 follows from Corollary 3 and the fact that under the above assumptions the 2×2-matrix C_4 is an $\mathcal{M}$-matrix, because its diagonal minors are positive.

Let system (1) satisfy assumptions that precede Corollary 3, and let $h_{11}(t) = t\ (t \in [0,\infty))$ μ-almost everywhere. Elements of the $n \times n$-matrix $C_5 = (c_{sj})^n_{s,j=1}$ obey the formulas

$$c_{ss} = 1 - c_p\left(\frac{1}{2\bar{a}_s}\right)^{1/2}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\bar{a}^{ij}_{ss},\ s = 1,\dots,l,\ c_{sj} = -\frac{1}{\bar{a}_s}\bar{a}^{11}_{sj} - c_p\left(\frac{1}{2\bar{a}_s}\right)^{1/2}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\bar{a}^{ij}_{sj},$$
$$s = 1,\dots,l,\quad j = 1,\dots,n,\quad s \neq j.$$

$$c_{ss} = 1 - c_p \left(\frac{\max\{1,\bar{b}_s^2\}(1-\exp\{-2\bar{a}_s\sigma\})}{2\bar{a}_s(1-\exp\{-2\bar{a}_s\rho\}\bar{b}_s^2)} \right)^{1/2} \sum_{i=2}^{m} \sum_{j=1}^{m_i} \bar{a}_{ss}^{ij}, \ s = l+1, \dots, n,$$

$$c_{sj} = -\frac{\max\{1,\bar{b}_s\}(1-\exp\{-\bar{a}_s\sigma\})}{\bar{a}_s(1-\exp\{-\bar{a}_s\rho\}\bar{b}_s)} \bar{a}_{sj}^{11} - c_p \left(\frac{\max\{1,\bar{b}_s^2\}(1-\exp\{-2\bar{a}_s\sigma\})}{2\bar{a}_s(1-\exp\{-2\bar{a}_s\rho\}\bar{b}_s^2)} \right)^{1/2} \sum_{i=2}^{m} \sum_{\nu=1}^{m_i} \bar{a}_{sj}^{i\nu},$$

$$s = l+1, \dots, n, \quad j = 1, \dots, n, \quad s \neq j.$$

In this case, Theorem 3 implies the following assertion.

Corollary 5. If C_5 is an $\mathcal{M}$-matrix, then system (1) is exponentially $2p$-stable with respect to the initial data.

5. EXAMPLES

Consider the following system of deterministic linear differential equations with constant delays and coefficients subject to impulse actions on certain variables:

$$\begin{aligned} dx(t) &= -\sum_{j=1}^{m} A_j x(t-h_j)dt \quad (t \geq 0), \\ x(\mu_j) &= B_j x(\mu_j - 0), \ j = 1, 2, \dots; \end{aligned} \tag{20}$$

here $A_j = (a_{sk}^j)_{s,k=1}^n$, $j = 1, \dots, m$, are real $n \times n$-matrices, h_j, $j = 1, \dots, m$, are nonnegative real values, μ_j, $j = 1, 2, \dots$, are real values such that $0 = \mu_0 < \mu_1 < \mu_2 < \dots$, $\lim\limits_{j\to\infty} \mu_j = \infty$, B_j is a real diagonal $n \times n$-matrix, all whose diagonal elements differ from zero, and $b_{ii}^j = 1$, $i = 1, \dots, l$, for $j = 1, 2, \dots$.

Assertion 1. *Assume that system* (20) *satisfies the inequality* $\sum\limits_{j=1}^{m} a_{ss}^j = a_s > 0$, $s = 1, \dots, n$, *and there exist positive values* ρ, σ, $\bar{b}_s$, $\hat{c}_s$, $s = l+1, \dots, n$, *such that* $\exp\{-a_{ss}^1 t\} \prod\limits_{0<\mu_j\leq t} |b_{ss}^j| < \hat{c}_s$ *with* $t \in [0, +\infty)$, $|b_{ss}^j| \leq \bar{b}_s$, $j = 1, 2, \dots$, $s = l+1, \dots, n$, $\rho \leq \mu_{j+1} - \mu_j \leq \sigma$ *for* $j = 1, 2, \dots$. *If under these assumptions the* $n \times n$*-matrix* C_6, *whose elements obey formulas*

$$c_{ss} = 1 - \frac{1}{a_s} \sum_{k=1}^{m} \sum_{j=1}^{m} |a_{ss}^k| h_k |a_{ss}^j|, \ s = 1, \dots, l,$$

$$c_{sj} = -\frac{1}{a_s} \left[\sum_{k=1}^{m} \sum_{\nu=1}^{m} |a_{ss}^k| h_k |a_{sj}^\nu| + \sum_{\nu=1}^{m} |a_{sj}^\nu| \right], \ s = 1, \dots, l, \ j = 1, \dots, n, \ s \neq j,$$

$$c_{ss} = 1 - \frac{\max\{1,\bar{b}_s\}(1-\exp\{-a_s\sigma\})}{a_s(1-\exp\{-a_s\rho\}\bar{b}_s)} \sum_{k=1}^{m} \sum_{j=1}^{m} |a_{ss}^k| h_k |a_{ss}^j|, \ s = l+1, \dots, n,$$

$$c_{sj} = -\frac{\max\{1,\bar{b}_s\}(1-\exp\{-a_s\sigma\})}{a_s(1-\exp\{-a_s\rho\}\bar{b}_s)} \left[\sum_{k=1}^{m} \sum_{\nu=1}^{m} |a_{ss}^k| h_k |a_{sj}^\nu| + \sum_{\nu=1}^{m} |a_{sj}^\nu| \right], s = l+1, \dots, n, j = 1, \dots, n, s \neq j,$$

is an $\mathcal{M}$*-matrix, then system* (20) *is exponentially stable with respect to the initial data.*

The validity of assertion 1 follows from Corollary 1 of Theorem 3.

Assume that system (20) satisfies correlations $h_1 = 0$, $a_{ss}^1 > 0$, $s = 1, \dots, n$. In this case, Corollary 1 of Theorem 3 implies the following assertion.

Assertion 2. *Assume that for system* (20) *there exist positive values* ρ, σ, $\bar{b}_s$, $\hat{c}_s$, $s = l+1, \dots, n$, *such that the following inequalities are valid:* $\exp\{-a_{ss}^1 t\} \prod\limits_{0<\mu_j\leq t} |b_{ss}^j| < \hat{c}_s$ *with* $t \in [0, +\infty)$, $|b_{ss}^j| \leq$

$\bar{b}_s$, $j = 1, 2, \ldots$, $s = l+1, \ldots, n$, $\rho \leq \mu_{j+1} - \mu_j \leq \sigma$ *for* $j = 1, 2, \ldots$. *If the* $n \times n$*-matrix* C_7*, whose elements obey formulas*

$$
\begin{gathered}
c_{ss} = 1 - \frac{1}{a_{ss}^1} \sum_{j=2}^{m} |a_{ss}^j|, \; s = 1, \ldots, l, c_{sj} = -\frac{1}{a_{ss}^1} \sum_{\nu=1}^{m} |a_{sj}^\nu|, \; s = 1, \ldots, l, \; j = 1, \ldots, n, \; s \neq j, \\
c_{ss} = 1 - \frac{\max\{1, \bar{b}_s\}(1 - \exp\{-a_{ss}^1 \sigma\})}{a_{ss}^1 (1 - \exp\{-a_{ss}^1 \rho\} \bar{b}_s)} \sum_{j=2}^{m} |a_{ss}^j|, \; s = l+1, \ldots, n, \\
c_{sj} = -\frac{\max\{1, \bar{b}_s\}(1 - \exp\{-a_{ss}^1 \sigma\})}{a_{ss}^1 (1 - \exp\{-a_{ss}^1 \rho\} \bar{b}_s)} \sum_{\nu=1}^{m} |a_{sj}^\nu|, \; s = l+1, \ldots, n, \; j = 1, \ldots, n, \; s \neq j,
\end{gathered}
\tag{21}
$$

is an $\mathcal{M}$*-matrix, then system* (20) *is exponentially stable with respect to the initial data.*

If, for example, $c_{ss} > \sum_{j=1, j \neq s}^{n} |c_{sj}|$, $s = 1, \ldots, n$, where c_{sj}, $s, j = 1, \ldots, n$, obey formulas (21), then in view of Proposition 2 system (20) is exponentially stable with respect to the initial data. Really, in this case, valid are sufficient conditions given in Section 1, which guarantee that the $n \times n$-matrix C_7 is an $\mathcal{M}$-matrix.

In particular, if all elements of matrices A_j, $j = 2, \ldots, m$, equal zero and $a_{ss}^1 > \sum_{j=1, j \neq s}^{n} |a_{sj}^1|$, $s = 1, \ldots, l$, while

$$
\frac{a_{ss}^1 (1 - \exp\{-a_{ss}^1 \rho\} \bar{b}_s)}{\max\{1, \bar{b}_s\}(1 - \exp\{-a_{ss}^1 \sigma\})} > \sum_{j=1, j \neq s}^{n} |a_{sj}^1|, \quad s = l+1, \ldots, n,
$$

then system (20) is exponentially stable with respect to the initial data.

Consider the following impulse system of Itô linear differential equations with constant delays:

$$
\begin{gathered}
dx(t) = -\sum_{j=1}^{m_1} A_{1j} x(t - h_{1j}) dt + \sum_{i=2}^{m} \sum_{j=1}^{m_i} A_{ij} x(t - h_{ij}) d\mathcal{B}_i(t) \quad (t \geq 0), \\
x(\mu_j) = B_j x(\mu_j - 0), \; j = 1, 2, \ldots, \; \text{a.s.},
\end{gathered}
\tag{22}
$$

where $A_{ij} = (a_{sk}^{ij})_{s,k=1}^n$, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$, are $n \times n$-matrices with real-valued elements, h_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$, are nonnegative real values, μ_j, $j = 1, 2, \ldots$, are real values such that $0 = \mu_0 < \mu_1 < \mu_2 < \ldots$, $\lim_{j \to \infty} \mu_j = \infty$, B_j is a real diagonal $n \times n$-matrix, all whose diagonal elements differ from zero, and $b_{ii}^j = 1$, $i = 1, \ldots, l$, for $j = 1, 2, \ldots$.

Assertion 3. *Assume that system* (22) *satisfies correlations* $\sum_{j=1}^{m_1} a_{ss}^{1j} = a_s > 0$, $s = 1, \ldots, n$, *and there exist positive values* ρ, σ, $\bar{b}_s$, $\hat{c}_s$, $s = l+1, \ldots, n$, *such that* $\exp\{-a_s t\} \prod_{0 < \mu_j \leq t} |b_{ss}^j| < \hat{c}_s$ *with* $t \in [0, +\infty)$, $|b_{ss}^j| \leq \bar{b}_s$, $j = 1, 2, \ldots$, $s = l+1, \ldots, n$, $\rho \leq \mu_{j+1} - \mu_j \leq \sigma$ *for* $j = 1, 2, \ldots$. *If the* $n \times n$*-matrix* C_8 *defined by formulas*

$$
\begin{gathered}
c_{ss} = 1 - \frac{1}{a_s} \sum_{k=1}^{m_1} \sum_{j=1}^{m_1} |a_{ss}^{1k}| h_{1k} |a_{ss}^{1j}| - c_p \left(\frac{1}{2a_s}\right)^{1/2} \left[\sum_{k=1}^{m_1} \sum_{i=2}^{m} \sum_{j=1}^{m_i} |a_{ss}^{1k}| \sqrt{h_{1k}} |a_{ss}^{ij}| + \sum_{i=2}^{m} \sum_{j=1}^{m_i} |a_{ss}^{ij}| \right], s = 1, \ldots, l, \\
c_{sj} = -\frac{1}{a_s} \left[\sum_{k=1}^{m_1} \sum_{\nu=1}^{m_1} |a_{ss}^{1k}| h_{1k} |a_{sj}^{1\nu}| + \sum_{\nu=1}^{m_1} |a_{sj}^{1\nu}| \right] - c_p \left(\frac{1}{2a_s}\right)^{1/2} \left[\sum_{k=1}^{m_1} \sum_{i=2}^{m} \sum_{\nu=1}^{m_i} |a_{ss}^{1k}| \sqrt{h_{1k}} |a_{sj}^{i\nu}| + \sum_{i=2}^{m} \sum_{\nu=1}^{m_i} |a_{sj}^{i\nu}| \right], \\
s = 1, \ldots, l, \quad j = 1, \ldots, n, \quad s \neq j,
\end{gathered}
$$

$$c_{ss}=1-\frac{\max\{1,\bar b_s\}(1-\exp\{-a_s\sigma\})}{a_s(1-\exp\{-a_s\rho\}\bar b_s)}\sum_{k=1}^{m_1}\sum_{j=1}^{m_1}|a_{ss}^{1k}|h_{1k}|a_{ss}^{1j}|-$$

$$-c_p\left(\frac{\max\{1,\bar b_s^2\}(1-\exp\{-2a_s\sigma\})}{2a_s(1-\exp\{-2a_s\rho\}\bar b_s^2)}\right)^{1/2}\left[\sum_{k=1}^{m_1}\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{ss}^{1k}|\sqrt{h_{1k}}|a_{ss}^{ij}|+\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{ss}^{ij}|\right],\ s=l+1,\dots,n,$$

$$c_{sj}=-\frac{\max\{1,\bar b_s\}(1-\exp\{-a_s\sigma\})}{a_s(1-\exp\{-a_s\rho\}\bar b_s)}\left[\sum_{k=1}^{m_1}\sum_{\nu=1}^{m_1}|a_{ss}^{1k}|h_{1k}|a_{sj}^{1\nu}|+\sum_{\nu=1}^{m_1}|a_{sj}^{1\nu}|\right]-$$

$$-c_p\left(\frac{\max\{1,\bar b_s^2\}(1-\exp\{-2a_s\sigma\})}{2a_s(1-\exp\{-2a_s\rho\}\bar b_s^2)}\right)^{1/2}\left[\sum_{k=1}^{m_1}\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{ss}^{1k}|\sqrt{h_{1k}}|a_{sj}^{i\nu}|+\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{sj}^{i\nu}|\right],$$

$$s=l+1,\dots,n,\quad j=1,\dots,n,\quad s\neq j,$$

is an $\mathcal{M}$-matrix, then system (22) is exponentially $2p$-stable with respect to the initial data.

The validity of Proposition 3 immediately follows from Theorem 3.

Assume that system (22) satisfies correlations $m_1=1$, $h_{11}=0$, and $a_{ss}^{11}>0$, $s=1,\dots,n$. Theorem 3 implies the the following assertion.

Assertion 4. *Assume that for system (22) there exist positive values ρ, σ, $\bar b_s, c_s$, $s=l+1,\dots,n$, such that $\exp\{-a_{ss}^1 t\}\prod_{0<\mu_j\le t}|b_{ss}^j|<\hat c_s$ with $t\in[0,+\infty)$, $|b_{ss}^j|\le \bar b_s$, $j=1,2,\dots,$ $s=l+1,\dots,n$, $\rho\le\mu_{j+1}-\mu_j\le\sigma$ for $j=1,2,\dots$. If the $n\times n$-matrix C_9 defined by formulas*

$$c_{ss}=1-c_p\left(\frac{1}{2a_{ss}^{11}}\right)^{1/2}\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{ss}^{ij}|,\ s=1,\dots,l,$$

$$c_{sj}=-|a_{sj}^{11}|\frac{1}{a_{ss}^{11}}-c_p\left(\frac{1}{2a_{ss}^{11}}\right)^{1/2}\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{sj}^{i\nu}|,\ s=1,\dots,l,\ j=1,\dots,n,\ s\neq j,$$

$$c_{ss}=1-c_p\left(\frac{\max\{1,\bar b_s^2\}(1-\exp\{-2a_{ss}^{11}\sigma\})}{2a_{ss}^{11}(1-\exp\{-2a_{ss}^{11}\rho\}\bar b_s^2)}\right)^{1/2}\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{ss}^{ij}|,\ s=l+1,\dots,n,$$

$$c_{sj}=-|a_{sj}^{11}|\frac{\max\{1,\bar b_s\}(1-\exp\{-a_{ss}^{11}\sigma\})}{a_{ss}^{11}(1-\exp\{-a_{ss}^{11}\rho\}\bar b_s)}-c_p\left(\frac{\max\{1,\bar b_s^2\}(1-\exp\{-2a_{ss}^{11}\sigma\})}{2a_{ss}^{11}(1-\exp\{-2a_{ss}^{11}\rho\}\bar b_s^2)}\right)^{1/2}\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{sj}^{i\nu}|,$$

$$s=l+1,\dots,n,\quad j=1,\dots,n,\quad s\neq j,$$

is an $\mathcal{M}$-matrix, then system (22) is exponentially $2p$-stable with respect to the initial data.

One can verify whether the matrix C_9 mentioned in assertion 4 is an $\mathcal{M}$-matrix, making use of sufficient conditions established in Section 1. If, for example,

$$1-c_p\left(\frac{1}{2a_{ss}^{11}}\right)^{1/2}\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{ss}^{ij}|>\sum_{j=1}^{n}\left(|a_{sj}^{11}|\frac{1}{a_{ss}^{11}}+c_p\left(\frac{1}{2a_{ss}^{11}}\right)^{1/2}\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{sj}^{i\nu}|\right),\ s=1,\dots,l,$$

$$1-c_p\left(\frac{\max\{1,\bar b_s^2\}(1-\exp\{-2a_{ss}^{11}\sigma\})}{2a_{ss}^{11}(1-\exp\{-2a_{ss}^{11}\rho\}\bar b_s^2)}\right)^{1/2}\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{ss}^{ij}|>$$

$$>\sum_{j=1,j\neq s}^{n}\left(|a_{sj}^{11}|\frac{\max\{1,\bar b_s\}(1-\exp\{-a_{ss}^{11}\sigma\})}{a_{ss}^{11}(1-\exp\{-a_{ss}^{11}\rho\}\bar b_s)}+c_p\left(\frac{\max\{1,\bar b_s^2\}(1-\exp\{-2a_{ss}^{11}\sigma\})}{2a_{ss}^{11}(1-\exp\{-2a_{ss}^{11}\rho\}\bar b_s^2)}\right)^{1/2}\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{sj}^{i\nu}|\right),$$

$$s=l+1,\dots,n,$$

then C_9 is an $\mathcal{M}$-matrix, therefore, system (22) is exponentially $2p$-stable with respect to the initial data.

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