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Espen G. Haug

Implied moments from OBX options, and a detailed analysis of moment-adjusted option models.

Implisitte momenter fra opsjoner på OBX indeksen og analyse av opsjonsmodeller justert for høyere momenter.

Geir Skirbekk & Veronica Aamodt Miljeteig
Business Administration
School of Economics and Business

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We would also like to thank friends and family who have proof read the thesis and given us feedback all along.

Abstract.

In this thesis we study the performance of skewness and kurtosis adjusted option pricing models. We estimated and analyzed volatility, skewness and kurtosis from a risk-neutral distribution from historical option prices on OBX TR index. We found that skewness and kurtosis adjusted option models based on Edgeworth and Gram - Charlier expansions are not robust when pricing out of the money options and highly sensitive to low volatility and short time to maturity due to negative probabilities from the expansion series. Compared to Black, Scholes and Merton's option model, skewness and kurtosis price adjustment resulted in lower at the money option prices and higher out of the money. For hedging purposes, delta hedging adjusted for skewness and kurtosis require far less contracts near at the money and more contracts deep out of the money for a call option with negative skewness. Our empirical analysis, based on the methodology from Gurdip Bakshi, Kapadia, and Madan (2003), found moments from short maturity options to be higher in absolute values and more sensitive to outliers compared to medium maturity. We found that implied kurtosis is highly volatile and a strong negative correlation with implied skewness. This could have big implications for traders, investment managers and risk managers trying to take into account skewness and kurtosis in their models. The conclusion is that it seems to be very challenging, something this thesis will point out and discuss.

Abstrakt.

I denne avhandlingen analyserer vi resultatene av skjevhet og kurtose justerte opsjons prismodeller. Vi estimerte og analyserte volatilitet, skjevhet og kurtose fra en risiko - nøytral fordeling fra historiske opsjonspriser på OBX TR index. Vi fant at skjevhet og kurtose justerte opsjons prismodeller basert på Edgeworth og Gram-Charlier serier er lite robuste for prising av opsjoner med innløsningspris out of the money(OTM), og er svært følsomme for lav volatilitet og kort tid til forfall på grunn av negative sannsynligheter for de ovennevnte serier. Sammenlignet med Black, Scholes og Merton's opsjons prismodell, prisjustering med skjevhet og kurtose resulterte i lavere pris på opsjoner at the money(ATM), og høyere priser på OTM opsjoner.

Vi fant at delta sikring justert for skjevhet og kurtose krever langt færre kontrakter ATM og flere kontrakter OTM for en kjøpsopsjon med negativ skjevhet. Vår empiriske analyse, basert på metodikk fra Gurdip Bakshi, Kapadia og Madan (2003), viste at for opsjoner med kort tid til forfall er kurtosen høyere og skjevhet mer negative og svært følsomme for pris avvik/hopp i forhold til opsjoner med middels (lengre) tid til forfall. Implisitt kurtose er også svært volatil og har en høy negativ korrelasjon med implisitt skjevhet. Dette kan ha stor innvirkning for tradere, forvaltere og risiko analytikere som prøver å ta hensyn til skjevhet og kurtose i sine modeller. Konklusjonen er at dette er svært utfordrende, noe denne oppgaven vil ta opp og diskutere.

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Symbols

δ - Modified Corrado – Su delta
S - Spot price
F - Future price
X - Exercise price
μ - Average
N - Number of observations
c - Call
p - Put
σ - standard deviation
r - risk - free interest
t - time
b - Cost of carry rate
γ - Convenience yield
Q_3 - Skewness element
Q_4 - Kurtosis element
μ_3 - Skewness
μ_4 - Kurtosis
d - The standardized z value in a normal distribution.
N(d) or N(.) – The cumulative normal distribution function.
n(d) or n(.) – The standard normal density function

Abbreviations.

ATM – at the money
OTM – out of the money
ITM – in the money
BSM – Black, Scholes and Merton
BKM – Bakshi, Kapadia and Madan
IV – Implied volatility
RND – Risk-neutral density
GBM – Geometric Brownian motion
SDE – Stochastic differential equation
RNG – Random number generator
MCS – Modified Corrado – Su model
TTM/ttm – Time to maturity
GLD – Generalized Lambda Distribution

1. Introduction

Black, Scholes and Merton's way of deriving their option pricing model is based on continuous dynamic delta hedging to arrive at a risk-neutral valuation. Their model has been extensively utilized and consistent with existing theory, it is however extremely sensitive to jumps in the asset price and stochastic volatility (Haug & Taleb, 2011). Empirical evidence shows that financial price data have higher peak and heavier tails compared to a normal distribution (Mandelbrot, 1997). Unexpected news can cause instant shocks, but the reversion back to mean level is typically gradual (slow) (Haug, Frydenberg, & Westgaard, 2010). Low frequency events with extreme impact is by Taleb (2007a) called black swans. These events give increased probability in the tails compared to a normal distribution and measured by the fourth moment (kurtosis). In the real world we observe asset prices in discrete time and will never be able to remove all risk by delta hedging, even if we re-hedge several times a day (Emanuel Derman, 1998). Gamma increase closer to expiration, especially around at the money, and maintaining an approximate risk-neutral position is close to impossible. Because of transaction-costs, you are actually guaranteed to lose money by continuously re-hedging your position (Haug, 2007b). Generally, option traders prefer to hedge options with options. Though, a static or approximate continuous delta hedge can remove a lot of risk, only a position in another option can hedge against Greeks, i.e. gamma and vega.

Implied volatility and volatility smile is often seen as better estimate of future volatility compared to historical volatility. The expectation of future volatility is likely only one of several factors affecting implied volatility (Haug et al., 2010). We believe the volatility smile reflects supply and demand for options, see for instance Haug and Taleb (2011) or Garleanu, Pedersen, and Poteshman (2009). Also a risk-premium for skewness and kurtosis (Gurdip Bakshi et al., 2003), as these moments are not included in the original BSM-formula. Implied volatility and skewness have been a subject in several previous studies, such as Gurdip Bakshi et al. (2003) and Conrad, Dittmar, and Ghysels (2013), but not much on kurtosis as we are aware. The aim of this thesis is to study how higher moments, with focus on kurtosis, affect option prices and delta. We also look at statistical behavior and distribution of moments implied from option prices. We apply extended models of Black-Scholes-Merton to study the effect of skewness and kurtosis in pricing securities and delta hedging, and the model's robustness to changes in volatility, time to maturity and different values of skewness and kurtosis.

Defining a suitable window of historical skewness and kurtosis can be challenging, which is why we look at moments from option prices that are forward looking by nature. An empirical analysis of historical option prices on OBX is performed where we calculated implied moments every Thursday from May 2006 to June 2017, and analyzed the distribution and behavior of kurtosis, skewness and volatility. To our knowledge, this has not been done in this particular market before.

The difference between delta-adjustment with skewness and kurtosis and original Black-Scholes-Merton, as we will show, is highly relevant for investors, option-traders, risk managers and other stakeholders in the financial market who try to hedge a position. Based on one author's professional experience, understanding the distribution of and adjusting for higher moments is especially relevant in VaR calculations, valuation of businesses and/or trading options or contracts with embedded options.

Results from earlier studies have limited comments on constraints in the skewness and kurtosis option models based on Jarrow and Rudd (1982), which is significant when studying deep out of the money options with short time to maturity and/or low volatility. However, a study by Jondeau and Rockinger (2001) find a corridor for values of skewness and kurtosis where the model yields satisfying results. We find that the model is not only sensitive for values of skewness and kurtosis, but also less robust for pricing option out of the money options with short time to maturity and low volatility. The original model, and the extensions of it, is expanded with series that yields negative probabilities in certain areas. Mainly there is a problem with negative prices when the theoretical price from Black-Scholes or Black76 is low, as is typical for options deep out of the money.

Due to the many limitations in the skewness and kurtosis models based on Jarrow and Rudd (1982), extracting the implied third and fourth moment will not be accurate. For our empirical analysis, we apply a model-free method from Gurdip Bakshi et al. (2003) to extract higher moments from option prices. Chang, Christoffersen, and Jacobs (2013) applied the same method to S&P500 options and analyzed higher moments as pricing factors in the cross section of stock returns. We relate the calculated moments to asset returns and perform a sensitivity analysis to evaluate the robustness of the method applied.
In the next section we give a brief review of previous literature and the history of option valuation. Theory and performance of modified skewness and kurtosis models found detailed

in section three and four, and description of method applied in section five. Section six describes data applied in our empirical analysis, and in section seven we present the result. In section eight, we summarize our main findings and concludes.

2. Literature review

Several economists have since the beginning of 1900s found empirical evidence of price data with high peaks, skews and heavy tails compared to the normal/Gaussian distribution. Numerous theories and formulas developed during the late 1900s are ignoring the these very important characteristics of price data. One plausible reason might be the mathematical simplicity and application of various models when assuming Gaussian distribution. Wesley C. Mitchell (1915) is probably the first to publish empirical evidence of a high peaked and heavy/fat tailed distribution. His work was updated and reprinted in both 1921 and 1938, and he is describing findings of time-varying volatility and a high frequency of rather small deviations in commodity price data from year 1891 to 1918 (Mitchell, 1938). Mandelbrot is referring to Mitchell in his well-known paper from 1963, where he also finds evidence of high peaked and fat-tailed return distributions. He found the second moment of the distribution to be very unstable when you have leptokurtic distributions and purposed a stable Paretian distribution as a better fit compared to the normal distribution. Ignoring the leptokurtic properties of financial data is very common, resulting in many well-known theories and financial models being consistent with each other but unrealistic for empirical data. For instance, Moore (1917) and Osborne (1959) found empirical evidence of high peaks and fat tails but concluded the real distribution to be approximately normally distributed (Haug, 2007b).

Louis Bachelier, a French economist, defended in 1900 his PhD thesis on option pricing, *The Theory of Speculation*. His model is very similar to the one published many years later by Black, Scholes and Merton. Bachelier assumed that the asset price was normally distributed and following an arithmetic Brownian motion (Haug, 2007a). Though assuming the asset prices follow a normal distribution is an undesirable property, his work was very innovative. He also showed in a profit and loss diagram how to create synthetic options with positions in both the underlying asset and options, and different option-strategies we know as bull-spread and call back spreads (Haug, 2007b). The put-call parity is not further described by Bachelier, but his profit and loss diagrams suggest he had some knowledge about the relationship between the two. (See section 3.2.2 for a derivation of the put-call parity)

The put-call parity has shown to be a robust property in option trading and the first known publication vaguely describing the put-call parity is from 1688 by Joseph de la Vega. Later Higgins (1902) and Nelson (1904) described the put-call parity in their books as a very robust arbitrage argument and tool to hedge options with options. Nelson (1904) also gave a description of market neutral delta hedging for at the money (ATM) options and a vague explanation of the idea behind dynamic delta hedging (Haug, 2007b). The put-call parity was rediscovered and formally described by Stoll in 1969, but his work is without any references to the earlier work of Nelson and Higgins.

In 1997 Myron Scholes and Robert C. Merton was awarded the Nobel prize for their option pricing model derived together with Fisher Black (1938-1995) in 1973. Their model is one of the most known in finance. The formula itself was not a new discovery, but rather how they derived it. The model, referred to as Black-Scholes or Black-Scholes-Merton, is by many, viewed as a good approximation for pricing European options. This model has several deficiencies, mainly that empirically we do not observe the same volatility across different strike prices or maturities, and that continuous dynamic delta hedging is not possible because we only observe discrete prices. Though dynamic delta hedging does remove a lot of the risk, Merton (1998), p. 328) was aware of the challenges with continuous dynamic delta hedging;

> "*A broader, and still open, research issue is the robustness of the pricing formulae in the absence of a dynamic portfolio strategy that exactly replicated the payoffs to the option security. Obviously, the conclusion on that issue depends on why perfect replication is not feasible as well as on the magnitude of the imperfection. Continuous trading, is, of course, only an idealized prospect, not literally obtainable; therefore with discrete trading intervals, replication is at best only approximate".*

For this reason, the Black, Scholes and Merton model cannot necessary be considered as a real risk-neutral valuation.
The volatility smile or smirk is a likely result of the leptokurtic properties of the underlying distribution. By assuming constant volatility across different strikes Black-Scholes tends to underprice out of the money (OTM) options. The implied volatilities at different strikes and the changes in the volatility smile is often used as an indication of market expectations of future volatility.

Many economic researchers have proposed extensions to improve the model. The Merton Jump diffusion model (Merton, 1976) is one, where the model accounts for jumps in the asset-price as opposed to Black-Scholes-Merton who assumed continuous price. In this thesis we will focus on models adjusting for skewness and kurtosis. In 1982, Jarrow R. and Rudd A. published a paper where they tried to improve the Black-Scholes-Merton model with a semi-parametric approach using an Edgeworth expansion to account for skewness and kurtosis in the underlying asset prices. Their model adjusted for moments in the asset price, not return, but was inconsistent with the martingale restriction. Later, adjusted and modified versions of their model have been published. Corrado and Su (1996a)) corrected Jarrow & Rudd's model to adjust for skewness and kurtosis in asset returns, and used a Gram-Charlie expansion. Brown and Robinson (2002) found a mistake in Corrado & Su's definition of Hermite polynomial and corrected the expression for the skewness coefficient. Jurczenko, Maillet, and Negréa (2004) refers to Longstaff (1995) and further improved the model to be consistent with the martingale restriction (no arbitrage) and used a modified Black-Sholes option value. As we can read from Longstaff (1995), when there exist market frictions, the martingale restriction does not need to hold. The modification only results in small deviations from the original Corrado & Su – model but could be economical significant for options deep OTM options. Interesting features about this model is the possibility to extract implied moments of skewness and kurtosis when we have information about historical prices. However, economist like Jondeau and Rockinger (2001) and Straja (2003) found that for certain levels of skewness and kurtosis the result is a negative option-price. A description of the model and the behavior of Q3, skewness, and Q4, kurtosis, is detailed in section four of this thesis.

Literature on estimation of risk-neutral densities (RND) from option prices is extensive, beginning with Breeden & Litzenberger in 1978. They introduced a state contingent security where the pay-off in perfect capital markets could be replicated by a butterfly-spread. They derived the RND-function by discounting the second derivative of the call price function, with respect to the exercise price. This method requires available option prices on a wide range of strikes, and have showed to yield unstable results (Jondeau, Poon, & Rockinger, 2007). Non-structural parametric methods to estimate RNDs allows for higher moments, and a convenient extension of BSM is RNDs as a mixture of log-normal densities. Other methods of RND estimation use different splines methods to interpolate and extrapolate the implied volatility smile when you have a narrow strike range before converting it to a density, i.e. Bliss and Panigirtzoglou (2004) and Taylor (2007).

Gurdip Bakshi and Madan (2000), (hereby referred to as BKM), found that by using explicit positioning across option strikes, any type of payoff could be priced. These findings are applied in Gurdip Bakshi et al. (2003), where they use contingent claim theory to derive a measure of volatility, skewness and kurtosis form the risk-neutral return density. By not imposing any specific structure on the underlying process, the calculated moments are more likely to be close to a risk-neutral measure and should be comparable across time. The specific position across options is feasible with only OTM options, where higher weight is assigned to options further out than options near ATM. Gurdip Bakshi et al. (2003)'s main objective was the third moment and found the index-skew to be more negative than individual skews, mainly as a result of risk aversion and fat-tailed physical distribution. This thesis employ a discretized version of this method to extract higher moments, similar to Turan G Bali and Murray (2013). A detailed description is found in section 5.

3. Theory

It is widely accepted and recognized that higher moments have a substantial impact on different pricing models and therefore models on option pricing. When referring to moments we talk about the 3rd and 4th moment of a distribution, skewness and kurtosis. The use of higher moments will most likely improve the performance of those models. Researchers and practitioners have started to use those higher moments in their models. Conventional measures are mostly the sample skewness and kurtosis.
In this section, theory of option pricing models adjusted for skewness and kurtosis will be discussed.
If the reader is familiar with skewness, kurtosis, Jarque – Bera test statistics and the BSM model, go directly to page 16, part 3.4.

3.1 Skewness and kurtosis

Skewness is defined as a measure of asymmetry of the probability distribution of a real valued random variable about its mean and is based on the third moment of the data. The interpretation is not necessary that intuitive and positive skew is indicating that the tail on the right side of the mean is fatter and longer. The opposite is when the skew is negative the tail on the left side is longer and fatter. The skewness for a normal distribution is 0, and the formula is denoted like this:

Skewness $= \frac{\sum_{i=1}^{N}(Xi - \mu)^3}{N(\sigma^3)}$ Where μ is X average and $\boldsymbol{\sigma}$ is the standard deviation

Figure showing positive and negative skew

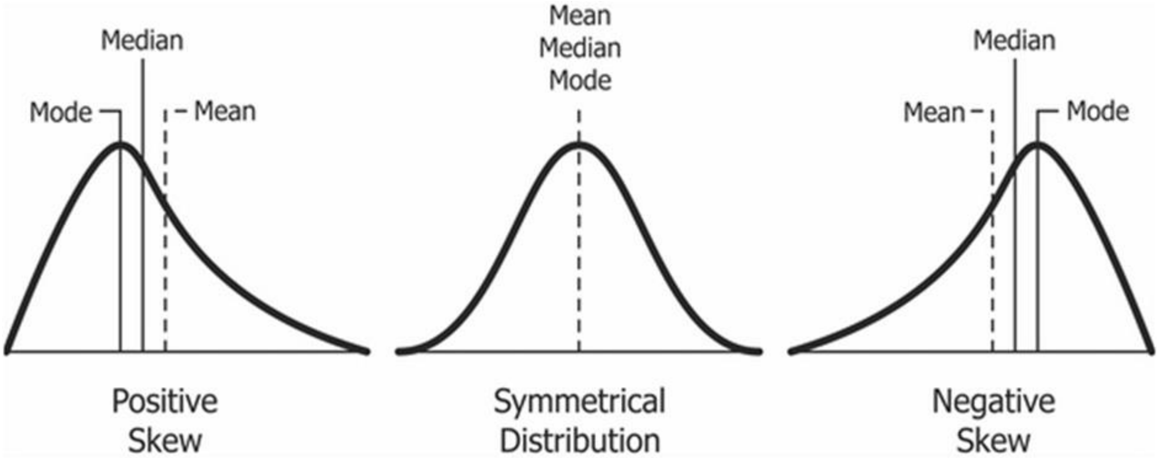

***Figure 1**. Figure of distribution with positive skew, symmetrical and negative skew.*
Source : https://www.safaribooksonline.com/library/view/clojure-for-data/9781784397180/ch01s13.html

Kurtosis defined by Pearson is related to the tails of the distribution and it is the fourth moment of the data. Especially in financial markets, most days are quiet, but we infrequently observe a few larger jumps that gives the kurtosis effect. Kurtosis is also a degree of peakedness of a distribution and the number is a standardized form of the fourth moment of those variations. Higher kurtosis is a result of more infrequent extreme outliers. A normal distribution has a kurtosis value of 3. Distributions with a kurtosis greater than 3 is defined as *leptokurtic*. Distributions equal 3 or 0 excess kurtosis is named *mesokurtic* and is equal to the normal distribution. The last one with negative excess kurtosis is named *platykurtic*.

Kurtosis $= \frac{\sum_{i=1}^{N}(Xi - \mu)^4}{N(\sigma^4)}$ where μ is X average and $\boldsymbol{\sigma}$ is the standard deviation

Kurtosis can also be written in the form where the variance is defined.

Kurtosis $= \frac{\sum_{i=1}^{N}(Xi - \mu)^4}{N(\frac{1}{N}\sum_{i=1}^{N}(Xi - \mu)^2)^2}$

In the nominator the differences are raised to the power of 4 and summarized. In the denominator the variance is squared twice, multiplied by N observations. N in denominator as in $\frac{1}{N}$ can also be N-1, due to situation if it is a population(N) or a sample (N-1), this is called the Bessel`s correction. Here you can see that there is greater effect if you raise each number and sum it up or you raise the average of a set of the same numbers.

Figure showing leptokurtic, mesokurtic and platykurtic distributions.

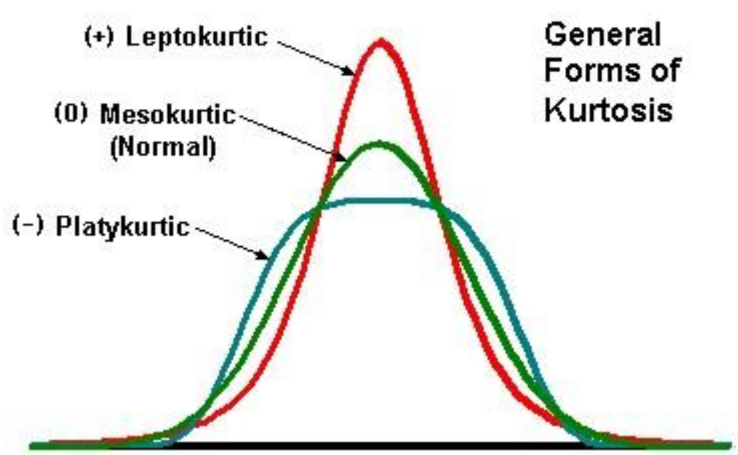

Figure 2. *Leptokurtic, mesokurtic and platykurtic distribution. Source: https://statisticsandprobability.blogspot.com/2010_01_01_archive.html*

3.1.1 Testing for normality – Jarque – Bera test statistics

The Jarque – Bera test is a goodness of fit measure of departure from normality. The test is based on the sample skewness and kurtosis, and is defined as

$$\text{JB} = \frac{n}{6}\left(S^2 + \frac{(K-3)^2}{4}\right)$$

Where

n = number of observations
S = Skewness
K = Kurtosis

The JB statistics has an asymptotic chi-square distribution with 2 degrees of freedom, see Jarque and Bera (1980).

3.2 Black, Scholes and Merton option pricing formula.

The generalized formulae include a continuous dividend-/convenience yield (cost of carry) and the model can be used to value European options on stocks, stocks with continuous dividend payout, futures and currency options. See the original paper from Black and Scholes (1973) and Merton (1973).

$$C_{BSM} = Se^{(b-r)T}\mathrm{N}(d_1) - \mathrm{X}e^{-rT}\mathrm{N}(d_2)$$

$$\mathrm{d}_1 = \frac{\ln\left(\frac{\mathrm{S}}{\mathrm{X}}\right)+\left(\mathrm{b}+\frac{\sigma^2}{2}\right)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}$$

$$\mathrm{d}_2 = \mathrm{d}_1 - \sigma\sqrt{\mathrm{T}}$$

Where

S = Asset price

X = Strike or Exercise price

σ = Volatility measured in standard deviation

T = Time to expiration

r = risk – free interest rate

b = cost of carry

N(d) = The cumulative normal distribution function.

The BSM model relies upon a set of central assumptions that does not necessarily apply to actual markets. A summary of the assumptions;

- Constant and known σ; standard deviation
- Constant and known carry rates; r, b and **γ** (convenience yield)
- No transaction costs
- Frictionless and continuous markets
- The markets follow a Geometric Brownian motion, the drift and volatility are constant.
- The stock does not pay dividend. This was one of the first assumptions, but if the dividend is known in time and size this can and is taken care of in later modifications of the formula.

3.2.1 Black76 model on futures.

In 1976 Fischer Black modified the original formula to apply for futures contracts. Which apply to this paper where the continuous 1st and 2nd pos OBX contracts are used. See also Black (1976) for details.

$$C_{B76} = e^{-rT}(F\mathrm{N}(d_1) - \mathrm{X}\mathrm{N}(d_2))$$

$$\mathrm{d}_1 = \frac{\ln\left(\frac{F}{\mathrm{x}}\right) + \left(\frac{\sigma^2}{2}\right)\mathrm{T}}{\sigma\sqrt{\mathrm{T}}}$$

$$\mathrm{d}_2 = \mathrm{d}_1 - \sigma\sqrt{\mathrm{T}}$$

Where:

F = Current future price

X = Strike or Exercise price

σ = Volatility measured in standard deviation

T = Time to expiration

N(d) = The cumulative normal distribution function.

R = risk – free interest rate

Figure 3 is a 3D plot of Black76 call values to illustrate how the option price change with time to maturity (ttm) and moneyness. Input used are; strike = 100, underlying price varying from 70 to 130 and ttm from zero to 365 days. Volatility is set at 20%, and risk – free interest rate is 2%.

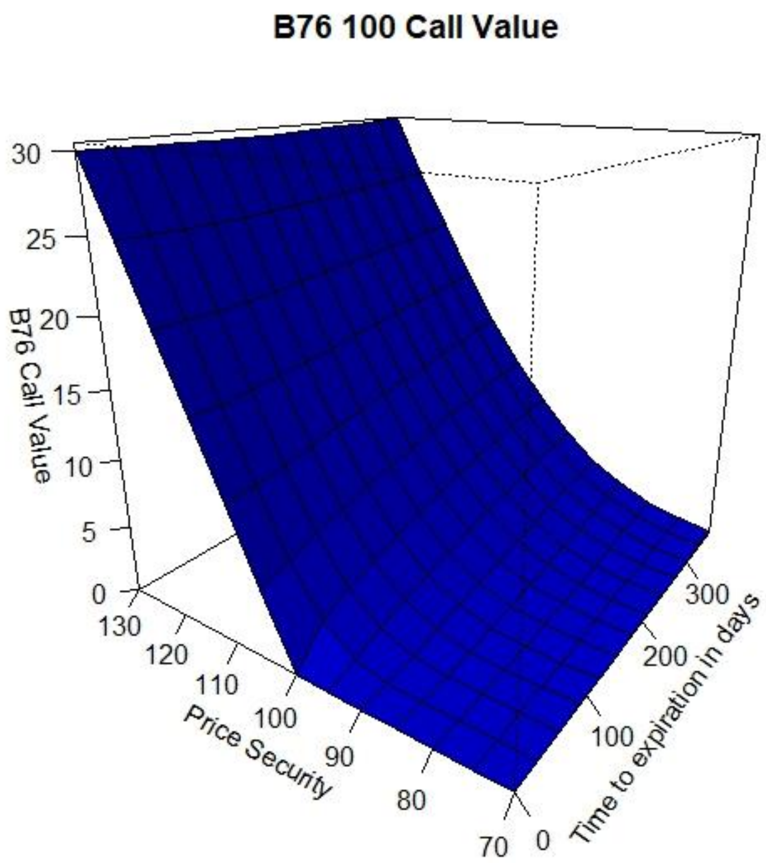

Figure 3. 3D-plot of Black76 call values with respect to underlying price and ttm. Input; Strike: 100, Asset price: 70 – 130, ttm: 0 – 365 days, volatility:20% and risk – free interest rate:2%.

3.2.2 Put – Call Parity

Put – Call parity defines the relationship between a European call option and a European put option on the same strike and same time to maturity.

Call = P + S – Xe^{-rt}
Put = C – S + Xe^{-rt}

Where:
P = Put
C = Call
S = Spot/Asset price
X = Exercise/Strike price

3.2.3 Implied volatility.

A parameter in BSM and Black76 that cannot be directly observed is volatility. Volatility implied by option price is known as implied volatility (herby referred to as IV). There is no closed form solution for IV, but numerical approximations and iterative techniques exist.

Corrado and Miller Jr (1996)'s extended approximation for moneyness is a numerical approximation where IV for a call is;

$$\sigma \approx \frac{\sqrt{2\pi}}{Se^{(b-r)T} + Xe^{-rT}}\left\{c_m - \frac{Se^{(b-r)T} - Xe^{-rT}}{2} + \left[\left(c_m - \frac{Se^{(b-r)T} - Xe^{-rT}}{2}\right)^2 - \frac{\left(Se^{(b-r)T} - Xe^{-rT}\right)^2}{\pi}\right]^{\frac{1}{2}}\right\}/\sqrt{T}$$

Where c_m is the observed market-price for the call option, S the underlying asset price, X the strike price, b dividends, r the risk-free interest rate and T time to maturity.
See Haug (2007a) for a full description of the approximation for put-options.

This approximation is not able to find implied volatility when $\frac{\left(Se^{(b-r)T}-Xe^{-rT}\right)^2}{\pi} > \left(c_m - \frac{Se^{(b-r)T}-Xe^{-rT}}{2}\right)^2$.
For these options we find IV by iteration in excel.

3.3 Testing a Monte Carlo simulated Geometric Brownian Motion (GBM) for skew and kurtosis.

One of the assumptions in the BSM model is that the stock prices follow a GBM. See for instance Benth (2003) and Øksendal (2010) for an in depth explanation and derivation of the GBM.
A GBM is a continuous time stochastic process, in which the natural logarithm of the randomly varying quantity follows a Brownian motion, that is also called a Wiener process with a drift, μ. A drift is normally an interest rate adjusted for continuous time effect. The GBM is a stochastic process satisfying a stochastic differential equation (SDE).
If the process, S_t follows a GBM then it has to follow this stochastic differential equation.

$DS_t = \mu S_t dt + \boldsymbol{\sigma} S_t d W_t$

Where W_t is the Wiener process and the μ as mentioned above is the drift and σ is the volatility or standard deviation. The first part in this equation tells us about a drift that is deterministic or fixed. The last part is random and unpredictable increments that occur during the process.
Solving for S_t one gets this expression for the stock price;

$$S_t = S_0 \, e^{((\mu - 0.5 \, \sigma^2)t + \sigma W_t)}$$

There are several different processes for generating random number samples, and some have got critique for not generating adequately random samples or repeating itself. Excel generator has got some critique for not generating satisfying pseudo-random numbers (Haug, 2007b). In R, we are using the default generator "Inversion". This generator is regarded as one of the better pseudo random number generators and based on the "Mersenne – Twister" algorithm. See the original paper Matsumoto and Nishimura (1998).
Home page http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt.html,
and articles, http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/ARTICLES/earticles.html
See also https://www.rdocumentation.org/packages/base/versions/3.5.0/topics/Random
for other RNG alternatives in R.
The package "randtoolbox" provides R functions for the two methods pseudo and quasi RNGs, as well as statistical tests to quantify the quality of generated random numbers.
See the note on RNG with R from Dutang and Wuertz (2009).
We want to study how the skew and kurtosis evolves when simulating a GBM using the default RNG in R. A 2-year price process was simulated with 504 daily returns 200000 times, the skew and kurtosis of the returns for each simulation was calculated. See figure 4, 5 and 6 and table 1 for histograms and descriptive statistics.

The parameters used are 2% interest rate(r), volatility($\boldsymbol{\sigma}$) of 20% and the starting point is S_0 = 100. First, a sample run of 100 simulations to see how it looks like, the time t on the x axis is in years

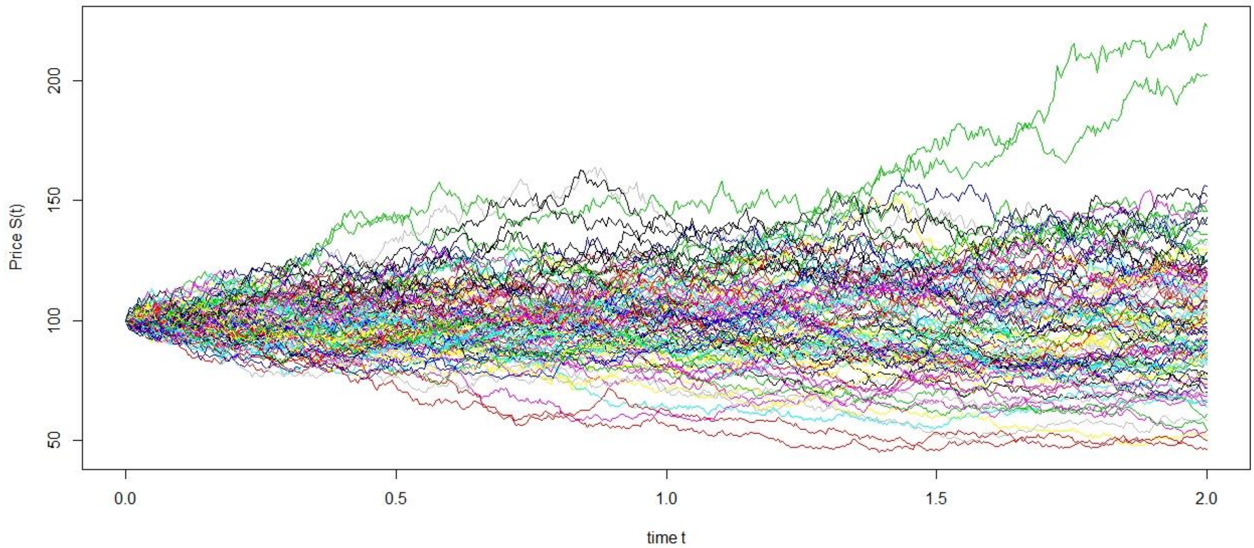

Figure 4. *100 test simulations of a GBM with r = 2%, volatility(σ) of 20% and S_0 = 100. Time t on the x axis is in years.*

The result from the 200K simulation gives these histograms of kurtosis and skewness.

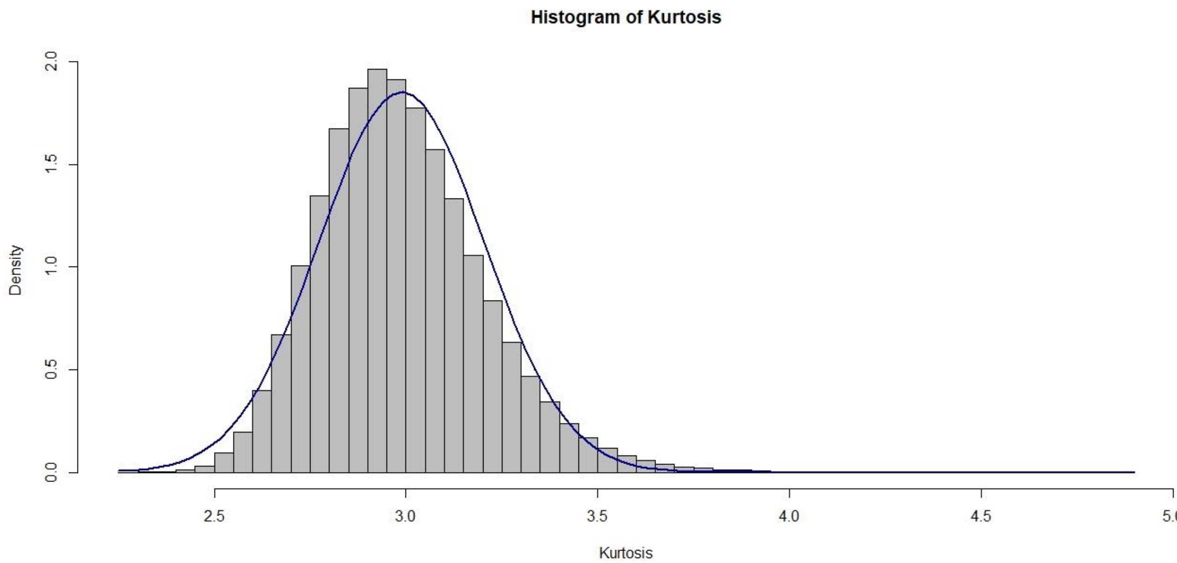

Figure 5. *Histogram of the calculated Pearson kurtosis for each simulation. A normal distribution has a Pearson kurtosis equal to 3.*

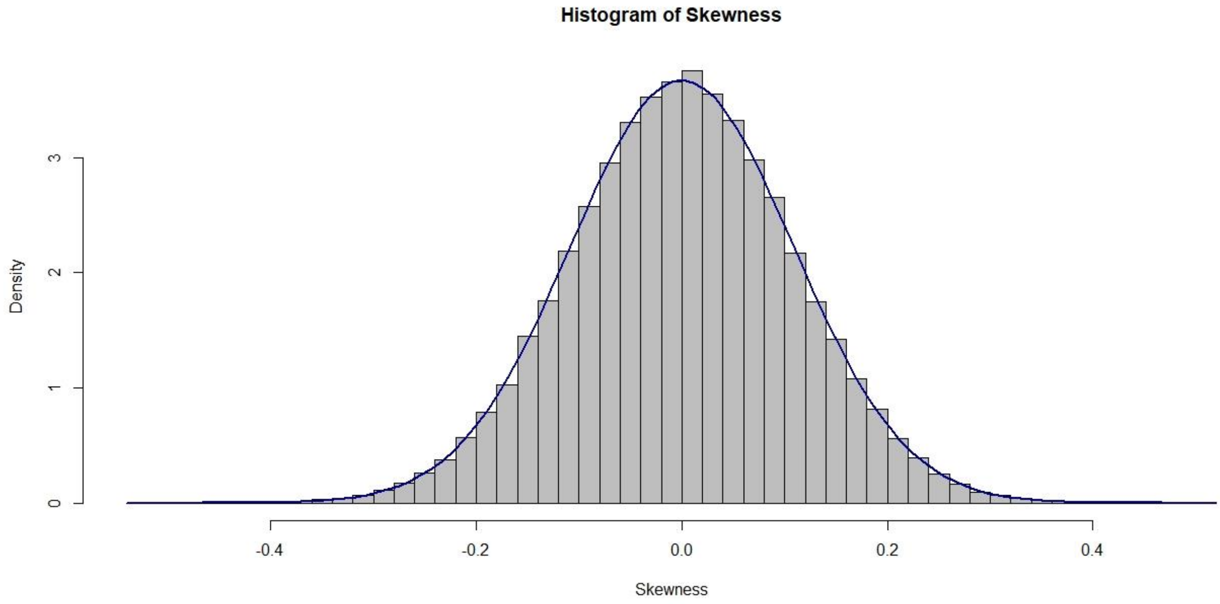

Figure 6. Histogram of the calculated skewness for each simulation. A normal distribution has a skewness value equal to 0.

Summary Monte Carlo simulation

	Skew		Kurtosis	
	Low	High	Low	High
2 year 200K sim	-0,53	0,52	2,28	4,88

Table 1. A short summary of the High – Low skew and kurtosis numbers. The Pearson kurtosis vary from 2,28 to almost 5 at the most.

Due to sampling error and/or the simulated numbers are discrete and not continuous these distributions for skew and kurtosis are far from normal. In continuous time all the skew and Pearson kurtosis's are per definition 0 and 3 respectively.

It is interesting to see the skews form a normal distribution, but as expected, due to discrete time and sampling, all the kurtosis data forms positive skew and kurtosis numbers, and is actually closer to a log normal distribution. The effect on the kurtosis is significant when the generator gives big jumps. See descriptive statistics. This means that the kurtosis is very sensitive to big jumps in the market and very volatile, in fact twice the skewness. Regarding kurtosis and discretized GBM, search is done for equivalent work but so far there is very little we have found on the subject.

	Skewness	Kurtosis
Nobs	200000	200000
Min	-0,53229	2,28830
Max	0,51798	4,88000
1.Quart	-0,07253	2,83800
3.Quart	0,07297	3,11200
Mean	0,00018	2,98800
Median	0,00063	2,96900
SE Mean	0,00024	0,00048
Stdev	0,10862	0,21539
Skew	-0,00706	0,64650
Kurtosis	0,07449	1,06200

Table 2. *Descriptive statistics for simulated skewness and kurtosis from GBM, Interval: 2 years, 200 000 simulations.*

3.4 Expansion series.

3.4.1 Taylor Series

Taylor series is named after the English mathematician Brooke Taylor (1685-1731). The first time he was credited this discovery was probably after a paper in 1786 by a Swiss mathematician with the name Simon Antoine Jean Lhuilier (1750–1840), where he referred to "Taylor series".

The approaching technique is based on a Taylor approximation or Taylor series. A Taylor series is series of polynomials and expansions of a function about a point and is very powerful for approximations. A one-dimensional Taylor series is an expansion of a real function like f(x) about a point x = a is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots.$$

If a = 0, the expansion is known as a Maclaurin series which is a Taylor expansion about 0.

For more information about Taylor and Maclaurin series, we refer to textbooks such as Lindstrøm (2016), any edition of Calculus from Edwards, Penney, and David (2013), Weir, Hass, and Thomas (2010) or Stewart (2008).

3.4.2 Edgeworth Series

Edgeworth series is named after Francis Ysidro Edgeworth (1845 – 1926). He was a political economist from Ireland and very interested in statistics during the 1880s.
Edgeworth series are series that will approximate a probability distribution by or in terms of its cumulants. The advantage is that the series errors are controlled, and it is more accurate. It is called an asymptotic expansion. The expansion is based on the normal distribution and added additional moments.

$$F(x) = \boldsymbol{\phi}(x) - \frac{1}{n^{0,5}}\frac{1}{3!}\lambda_3\,\phi^{(3)}(x) + \frac{1}{n}\left[\frac{1}{4!}\lambda_4\,\phi^{(4)}(x) + \frac{10}{6!}\lambda_3^2\,\phi^{(6)}(x)\right] - \frac{1}{n^{\frac{3}{2}}}\left[\frac{1}{5!}\lambda_5\,\phi^{(5)}(x)\,\frac{35}{7!}\lambda_3\lambda_4\,\phi^{(7)}(x) + \frac{280}{9!}\lambda_3^3\,\phi^{(9)}(x)\right] + \ldots.$$

Where $\phi(x)$ is the normal distribution
$\lambda_3 = 3^{rd}$ moment = Skewness
$\lambda_4 = 4^{th}$ moment = Kurtosis
Edgeworth series suffer from some disadvantages,

- The integral of the density is not necessarily equal to 1
- Probabilities can be negative
- They are made up of Tayler series around mean.
- They do not have a relative error only an absolute one.

Defined in Johnson, Kotz, and Balakrishnan (1994)
b(x) = standardized binomial density
μ_3 = Skewness
μ_4 = Kurtosis
$x = [\ln(\frac{S}{X}) + \frac{1}{2}\sigma^2 T] / (\boldsymbol{\sigma}\sqrt{T})$

In a standardized form the density can be written as:
f(x) = standardized “Edgeworth density”
f(x) = b(x) * E(x)
$E(x) = 1 + \frac{1}{6}\mu_3 (x^3 - 3x) + \frac{1}{24}(\mu_4 - 3)(x^4 - 6x^2 + 3) + \frac{1}{72}\mu_3^2 (x^6 - 15x^4 + 45x^2 - 15)$

3.4.3 Gram – Charlier Series

Gram – Charlier series is named after Jørgen Pedersen Gram (1850 -1916) and Carl Vilhelm Ludwig Charlier(1862 -1934). Gram was a Danish Actuary and mathematician, Charlier was a Swedish astronomer.

The Gram – Charlier "A type" series is based on the normal distribution and expanded with the 3rd and 4th moments of the normal distribution just like the Edgeworth series. Instead of cumulants in the Edgeworth series, Gram – Charlier uses moments.

$$f_A(x) = f(x) + \sum_{k=3}^{n} a_k \ f^{(k)}(x)$$

Where f(x) is the normal distribution and

$f^{(k)}$ = the k^{th} derivative of the function f and

$f^{(k)}\ (x) = -1^k\ H_k(x)f(x)$

Where H_k(x) are the Chebyshev – Hermite polynomials.

The first four polynomials are therefore

$H_0 = 1$

$H_1 = x$

$H_2 = x^2 - 1$

$H_3 = x^3 - 3x$

$H_4 = x^4 - 6x^2 + 3$

Where $x = [\ln(\frac{S}{X}) + \frac{1}{2}\sigma^2 T] / (\boldsymbol{\sigma}\sqrt{T})$

See Stuart, Ord, and Kendall (1994) pp. 226-233 for proper in depth discussion of the differences between Edgeworth and Gram – Charlier series.

Because those series consist of polynomials they have more than one root and define an area of feasible solution(s). This, as we will show in section 4, give negative probabilities and therefor negative option values. The series of Edgeworth and Gram – Charlier are almost similar but for computation the Gram – Charlier series seems to have a better performance than the Edgeworth series, see Johnson et al. (1994).

3.4 Evolution of methods, theory and models.

Jarrow and Rudd (1982) were the first to come up with a model that corrected for the third and fourth moment, skewness and kurtosis. Jarrow and Rudd's model uses an Edgeworth expansion, and is based on the BSM model adjusted for skewness and kurtosis that are different from the lognormal distribution. Therefore, the model adjusts for skewness and kurtosis in the prices instead of the return distribution.
See also Stuart et al. (1994) pp. 226 -233.

14 years later, in 1996, Corrado and Su (1996a) and Corrado and Su (1996) come up with their contribution to a non-normal option valuation and the model is dealing with excess skewness and kurtosis based on the so called true implied risk – neutral density. The model is based on an expansion of the Black-Scholes-Merton formula in order to incorporate non-normal distributions, where kurtosis and skewness of the assets' returns are considered. They used the expansion of the Gram – Charlier normal probability density series (series A) to model the distribution of the asset's return. The formula Corrado and Su came up with contained two typographical errors, Brown and Robinson (2002), observed that and corrected it in their paper. In 2002/04, Jurczenko et al. (2004) modified the model in order to prove consistency with a Martingale restriction. The differences between those models are not very significant but can be economically significant when options are far out of the money and for long maturities, especially when the market is volatile and turbulent. The Corrado – Su model modified by Jurczenko et al. is more accurate because the former model did not hold under the Martingale restriction, and this model will be used for calculations. While writing this thesis we came upon a possible mistake in Haugs book, Haug (2007a) "The Complete Guide to option pricing Formulas" on page 250, 6.7.5. The modified formula is not adjusted for Jurczenko et al. (2004)'s findings. In their paper they discounted the strike price and adjusted for w (see the formula below) in the original d in N(d) and used the corrected d in the original BSM model as well.
The difference between the formula of the Black76 model and the modified Corrado – Su model is the addition of two terms to the first model, which added the analyzed series skew and kurtosis values. The equation of the model defines the option price of a call option on a stock index future and presented below.

Call Price = $e^{-rt}(FN(d_1) - XN(d_2)) + \mu_3 Q_3 + (\mu_4 - 3) Q_4$

Where

$$Q_3 = \frac{1}{6(1+w)} F \sigma \sqrt{T} (2 \sigma \sqrt{T} - d_1) n(d_1)$$

$$Q_4 = \frac{1}{24(1+w)} F \sigma \sqrt{T} (d_1^2 - 3d_1 \boldsymbol{\sigma}\sqrt{T} + 3\sigma^2 T - 1) n(d_1)$$

$$w = \frac{\mu_3}{6} \sigma^3 T^{\frac{3}{2}} + \frac{\mu_4}{24} \sigma^4 T^2$$

$$d_1 = [\ln(\frac{F}{Xe^{-rT}}) + \frac{1}{2} \sigma^2 T - \ln(1+w)] / (\boldsymbol{\sigma}\sqrt{T})$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

N(d) = The cumulative normal distribution function.

N(d) = The standardized normal density function

F = Future price

Price of a put option is found by put-call parity.

Figure 7 and 8 show the effect of -Q3 and Q4 on a call option with different volatilities with respect to moneyness or strikes. Risk – free rate = 2%, Time to maturity = 1/12 year. Skewness and Pearson kurtosis is 0 and 3 respectively. This graph is probably shown in Heston (1993a) and (Heston, 1993b), for the first time.

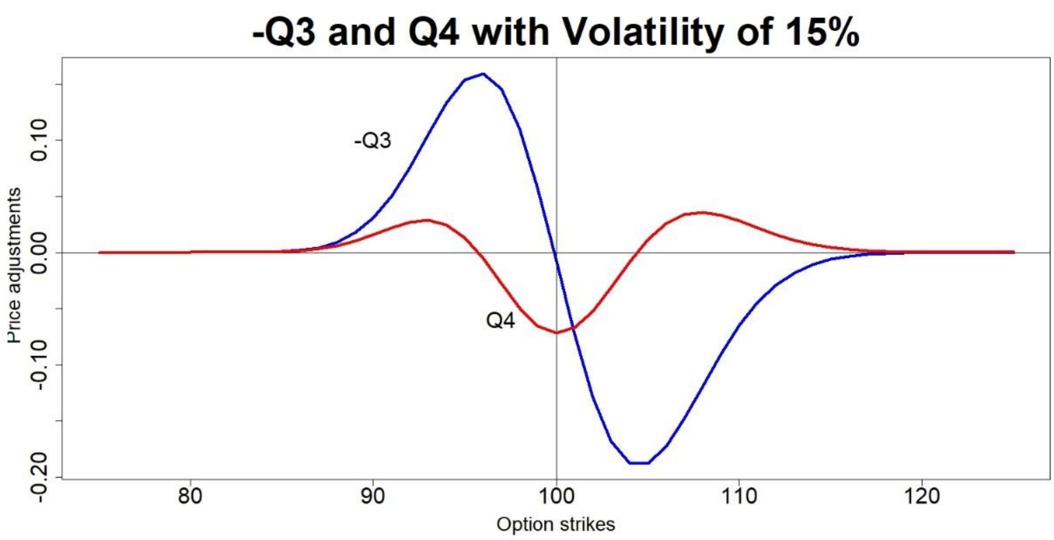

Figure 7. The effect of -Q3 and Q4 on a call option on different strikes. Input; volatility: 15%, risk – free rate: 2%, ttm: 1 month, skewness: 0 and Pearson kurtosis: 3.

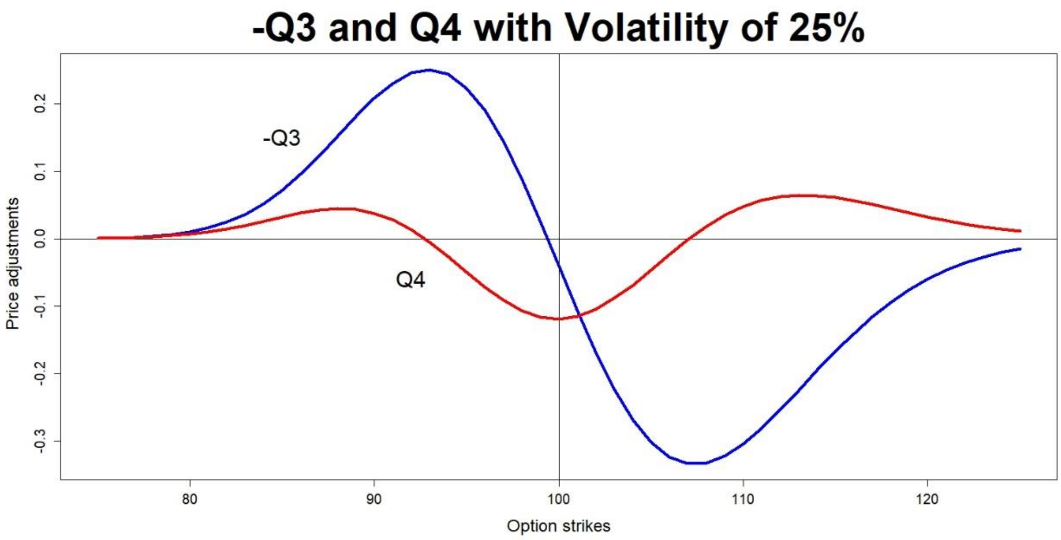

Figure 8. *The effect of -Q3 and Q4 on a call option on different strikes. Input; volatility: 25%, risk – free rate: 2%, ttm: 1 month, skewness: 0 and Pearson kurtosis: 3.*

Looking at Q4 in figure 7 and 8, an ATM call option is overpriced in the BSM world and an OTM call option is underpriced. Higher volatility, 25% (figure 8), has a larger effect on the price, and the effect widens out. The effect has a larger interval.

Figure 9 is an example of the real price difference between Adjusted Corrado and Su model and the B76 model. The difference between those models is actually the sum of $\mu_3 Q_3 + (\mu_4 - 3) Q_4$. Parameters used are S = 100, Exercise prices 75-125, r = 2%, volatility = 20%, T=1/12, skewness=-0.5 and Pearson kurtosis = 6. Under those assumptions the ATM call price in the B76 model is 0.3 overpriced and ITM and OTM options are underpriced and reach local moneyness maxima at approximately 90 and 110. If you change the parameters, the graph will change but the main conclusion is the same.

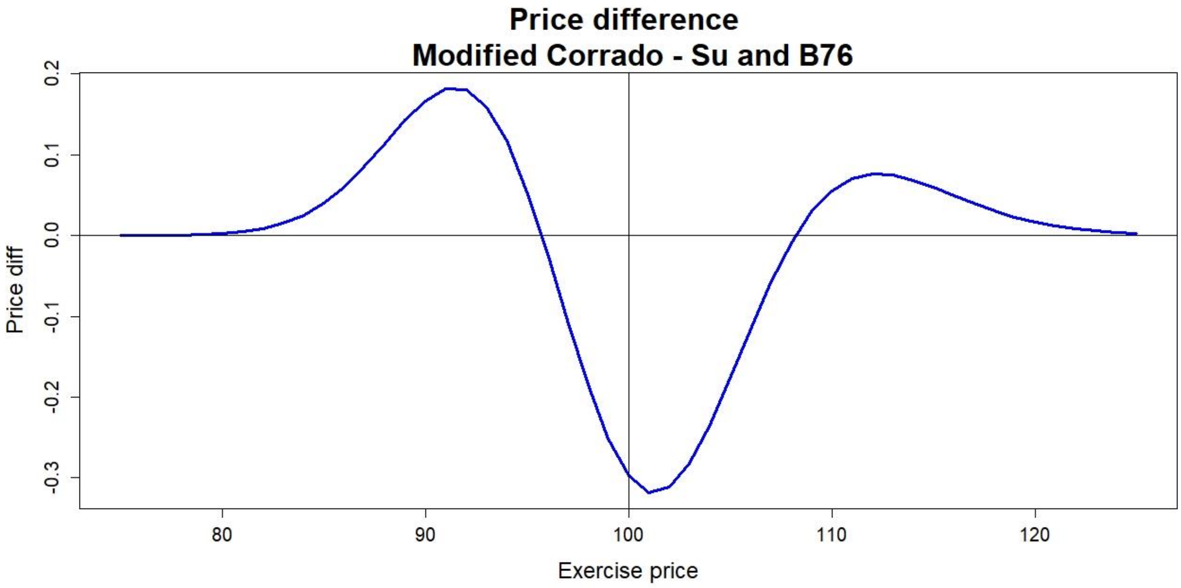

Figure 9. *Graph of the price difference, Modified Corrado & Su vs B76. Input; skew = -0,5 and Pearson kurtosis of 6. S=100 Volatility 20%.*

Gram – Charlier and similar expansions allows for excess flexibility over the normal probability density function due to introduction of the skewness and kurtosis of the empirical distribution as parameters as mentioned above. This expansion will for certain parameters of skewness and kurtosis give negative values, due to its polynomial structure and approximation. Rubinstein (1998) provides approximate skewness – kurtosis values where the Edgeworth expansion gives positive values. Jondeau and Rockinger (1999) and Jondeau and Rockinger (2001), analyzed the Gram – Charlier expansion and came up with feasible solutions where the expansion is positive. The positivity constraints give values of the kurtosis in the area of (0 – 4) and for each kurtosis that is acceptable there exists a symmetric interval for the skewness and the opposite. The skewness values are between -1.05 and 1.05. Therefore, the positivity constraints require that the Gram – Charlier methods can only be used for moderate deviations from normality. Shown later in this thesis, the price of a European option can be used to explore and calculate skewness and kurtosis for a certain day or time period similar to the procedure of Jondeau and Rockinger regarding taking or not, the positivity constraint into consideration. They suggest that when the unconstrained identification yields skewness-kurtosis values for which the positivity constraints are violated we do have a model misspecification. In theory, one can add more terms in the expansion to overcome this misspecification but there are arguments and reasons why this probably will fail.

The more terms the more roots in the polynomial and this will only move or decrease the domain where approximation is positive, especially the Edgeworth series and the parameters will mostly be more unstable, and it will also induce multicollinear parameters if you add 5^{th} and 6^{th} moment to the skewness and kurtosis model.

From Bowman and Shenton (1973) paper they say in their last conclusion that;

"The present study shows that it may not be easy to find a good fitting distribution to a theoretical distribution using five or more moments. In fact in connexon with the normal mixture, there is the paradox that, the nearer to normality the theoretical distribution is, the less likely it is that a normal mixture fit can be found."

From Jondeau and Rockingers own research they suggest that Bowman and Shentons conclusion can be changed to;

"The nearer kurtosis is to the one of the normal distribution, the less likely it is that a parametric approximation can be found."

That means the estimates of μ and $\boldsymbol{\sigma}$ vary very little but skewness and especially kurtosis dispersion/patterns and variation vary a lot, which can induce multicollinearity. They found that for a given kurtosis the larger the skewness gives better result, and the Gram – Charlier density estimation is better the more the tails deviate from the normal. Therefore, it is very important first to check for departures from normality. For instance with a Jarque Bera test. Jondeau and Rockinger (2001)'s paper is recommended for a deeper analysis of the positivity constraints and feasible solutions of the Gram – Charlier expansion with positivity constraints in such a manner that you can call it a distribution. See also Straja (2003) comments on this subject.

Since these expansions are only approximations, $\sum_j f(X_j) \neq 1$, the moments are slightly in error. Correcting for this error after the expansion, either Gram – Charlier or Edgeworth, will rescale the probabilities so that they will sum up to 1, by replacing f(X_j) with f(X_j)/ $\sum_j f(X_j)$. Using the rescaled density, calculate the mean $\mu \cong \sum_j f(X_j)\, X_j$ and then the variance around the new mean; var $\cong \sum_j f(X_j)(X_j - \mu)^2$. Last, you have to replace the X_j with the standardized zero mean, standard deviation and one random variable $(X_j - \mu)/V$. This modification will give approximate values but in return they will all be positive. See Rubinstein (1998).

4. Modified Corrado & Su sensitivity analysis and delta hedging.

In this section we analyze the robustness of modified Corrado & Su model with respect to volatility and kurtosis and show how negative probabilities from Gram-Charlier expansion affect the option price. In 4.2 we analyze how skewness and kurtosis affect delta hedging compared to BSM.

4.1 Sensitivity analysis

Sensitivity analysis of prices with respect to volatility and kurtosis are calculated. Fixed historical skew value of – 0.5 is used and the result is shown in 3D's and a contour plot with the same parameters. Here one can see what influence the kurtosis and volatility have on the call price.

Parameters used; S = 100, r = 0.02, T = 1/12, Skew = - 0.5 see figure 10.

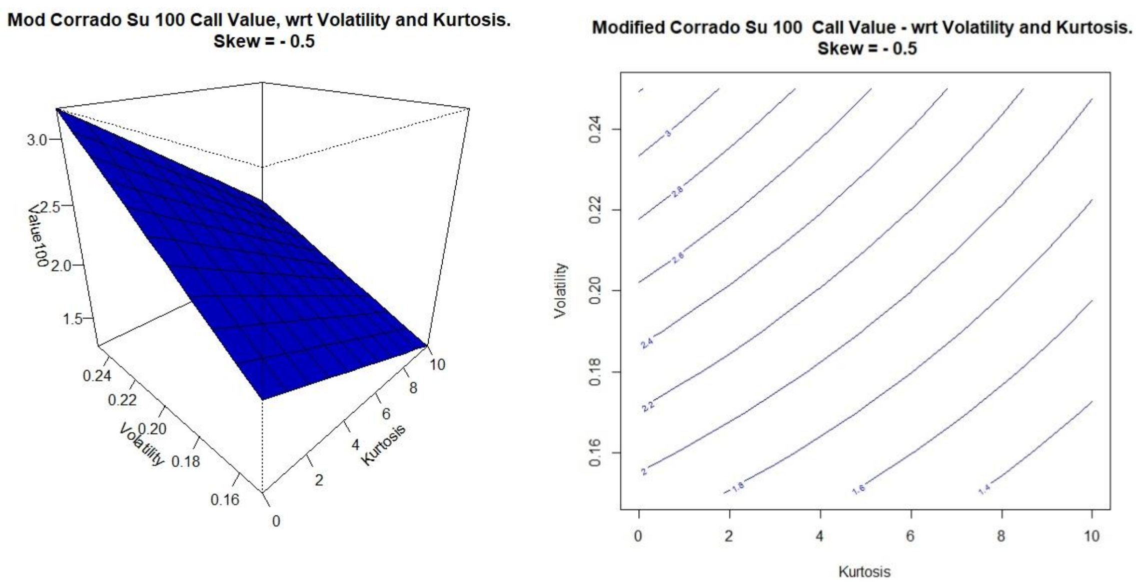

***Figure 10**. Impact of kurtosis and volatility with skew fixed at -0,5. Input; S=X= 100, ttm: 1 month, r: 2%. Left figure is a 3D plot, right figure is a contour plot with same input. Values used to create the 3D – and contour plot is found in appendix, table 11.*

The next strike is 105 and the same exercise is done, now the lowest point is at (15%,0) and the highest point is at (25%, 0), see figure 11, compared to the 100 strike where the lowest point was found to be at (15%, 10) and the highest value at (25%, 0).

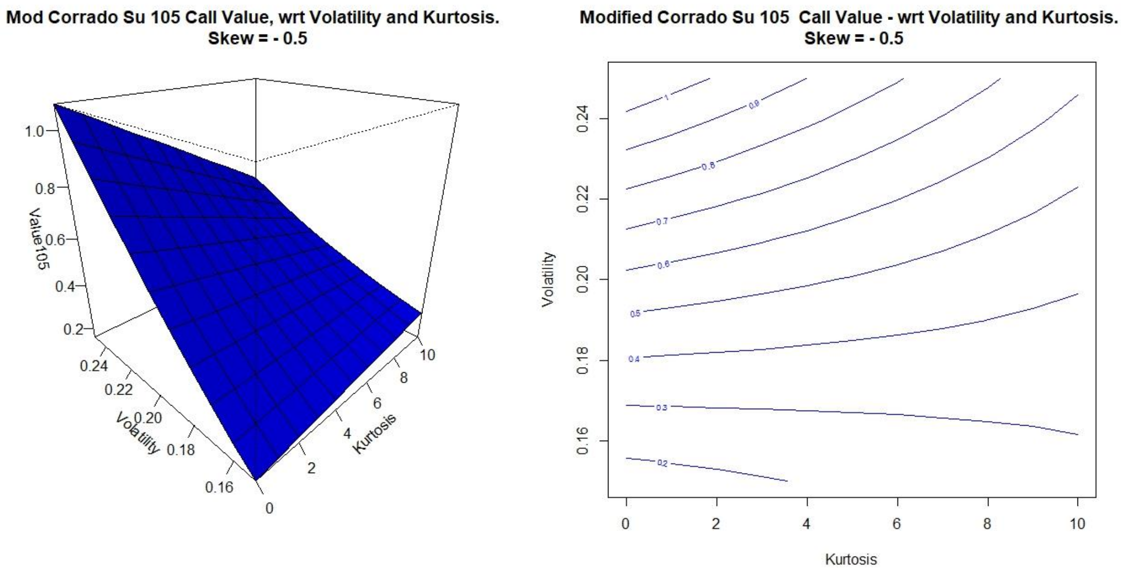

__Figure 11__. Impact of kurtosis and volatility with skew fixed at -0,5. Input; S=X= 105, ttm: 1 month, r: 2%. Left figure is a 3D plot, right figure is a contour plot with same input. Values used to create the 3D – and contour plot is found in appendix, table 12

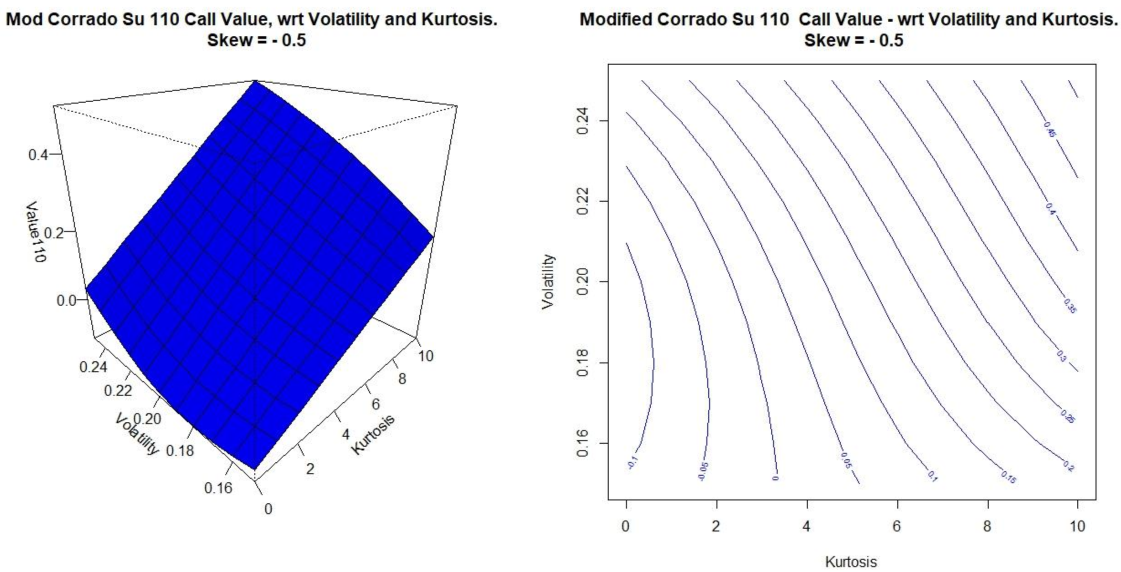

__Figure 12.__ Impact of kurtosis and volatility with skew fixed at -0,5. Input; S=X= 110, ttm: 1 month, r: 2%. Left figure is a 3D plot, right figure is a contour plot with same input. Values used to create the 3D – and contour plot is found in appendix, table 13

At the 110 strike, a quite different pattern is appearing and the lowest point is at (18%,0) and it is negative. And the highest point is at (25%,10.)

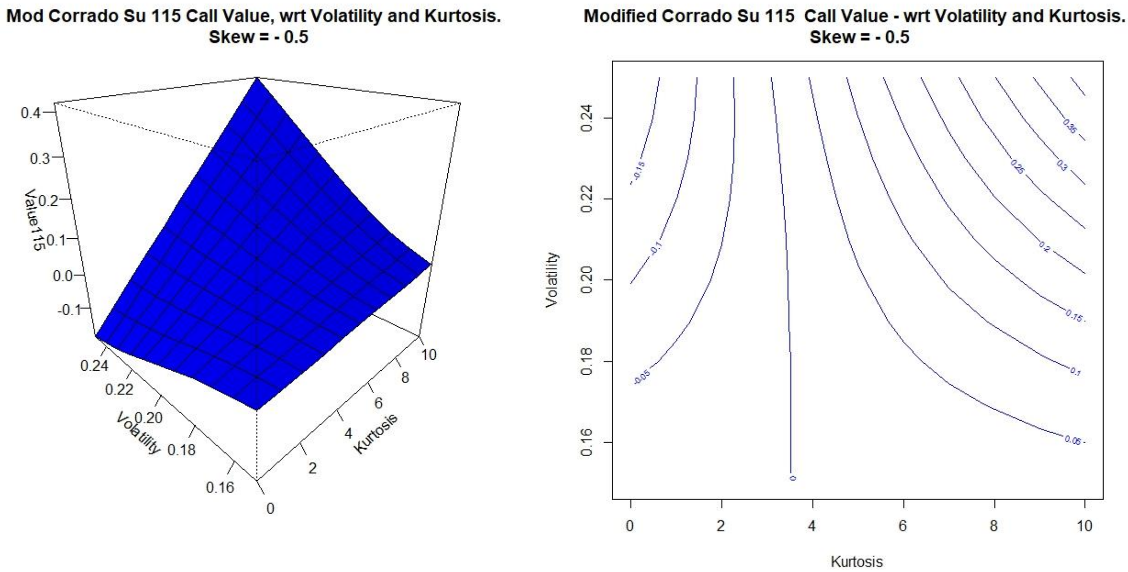

Figure 13. *Impact of kurtosis and volatility with skew fixed at -0,5. Input; S=X= 115, ttm: 1 month, r: 2%. Left figure is a 3D plot, right figure is a contour plot with same input. Values used to create the 3D – and contour plot is found in appendix, table 14*

At the 115 strike, the lowest point is at (25%,0) and it is negative. And the highest point is at (25%,10.)

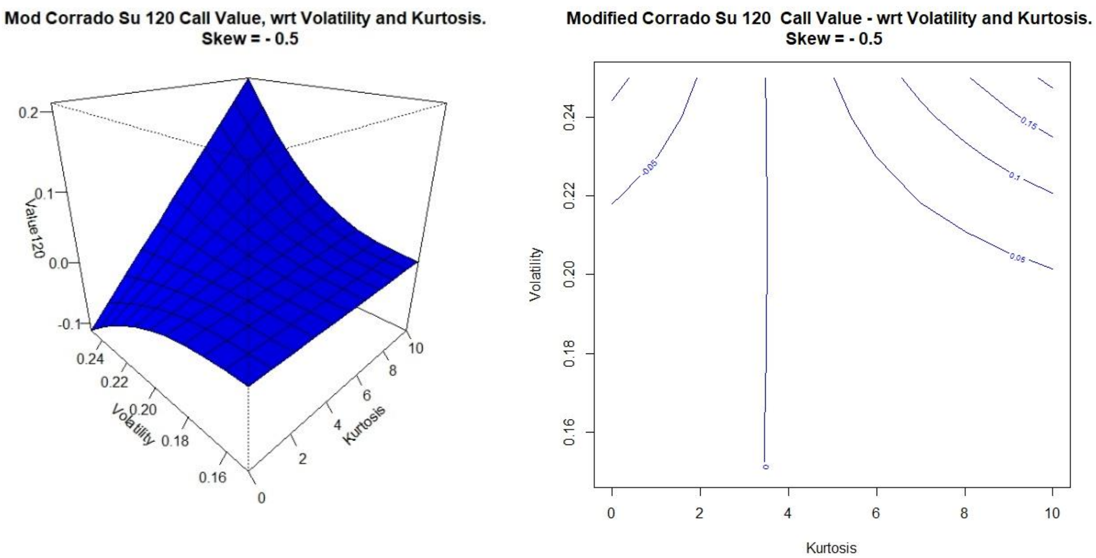

Figure 14. *Impact of kurtosis and volatility with skew fixed at -0,5. Input; S=X= 120, ttm: 1 month, r: 2%. Left figure is a 3D plot, right figure is a contour plot with same input. Values used to create the 3D – and contour plot is found in appendix, table 15*

The 120 strike, the lowest point is at (25%,0) and it is negative also here. The highest point is (25%,10.) as well.

As a curiosity a 3D plot of Price ATM call with respect to time to expiration and Pearson kurtosis is shown. The price is a decreasing function of the kurtosis.

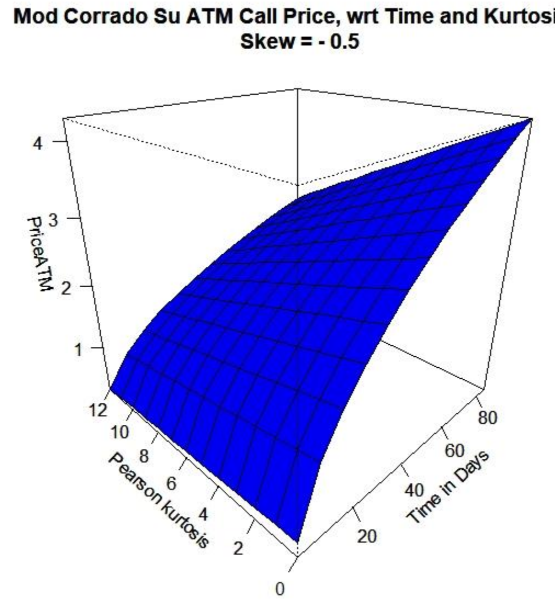

__Figure 15.__ 3D plot of the Price of an ATM call on the Z axis, time to expiration on the X axis and kurtosis on the Y axis. Skew is -0.5, r = 2%.

4.2 Delta hedging adjusted for skewness and kurtosis.

From a hedging point of view the Jurzenko et al adjusted Corrado and Su model is a closed form solution and gives therefore a closed form solution when taking the first partial derivative with respect to the underlying price S. One can write the skewness and kurtosis adjusted delta on this form. See also Vähämaa (2003), Corrado and Su (1997) and Backus, Foresi, and Wu (2004).

The delta consists of the original BSM delta plus the addition of the non-normal skewness and kurtosis elements.

$$\delta = \frac{\partial c}{\partial S} = \mathrm{N}(d_1) + \mu_3 q_3 + (\mu_4 - 3)\, q_4$$

Where

$$q_3 = \frac{\partial q_3}{\partial S} = \frac{1}{3!\,(1+w)}\left[\sigma^3 T^{3/2} N(d_1) + \left\{\frac{\phi_1 d_1}{\sigma\sqrt{T}} + \sigma^2 T - 1 - \phi_1\right\} n(d_1)\right]$$

$$q_4 = \frac{\partial q_4}{\partial S} = \frac{1}{4!\,(1+w)}\left[\sigma^4 T^2 N(d_1) + \sigma^3 T^{3/2} n(d_1) + \frac{n(d_1)}{\sigma\sqrt{T}}\{\phi_2 - 2\sigma^2 T + 2rT + 2\ln(S/K^{-rT})\} - \frac{n(d_1) d_1 \phi_2}{\sigma^2 T}\right]$$

ϕ_1 = rT – (3/2)σ^2T + ln(S/K^{-rT})

ϕ_2 = r^2T^2 – 2r σ^2T^2 + (7/4) σ^4T^2 – σ^2T + ln(S/K^{-rT}) (2rT – 2σ^2T + ln(S/K^{-rT})

w = $\frac{\mu_3}{6}\sigma^3 T^{\frac{3}{2}} + \frac{\mu_4}{24}\sigma^4 T^2$

4.2.1 Delta adjustments

In this chapter we take a look at how delta adjustments evolve during different volatilities and moneyness. As before the parameters used is; t = 1/12, r =2% and S =100

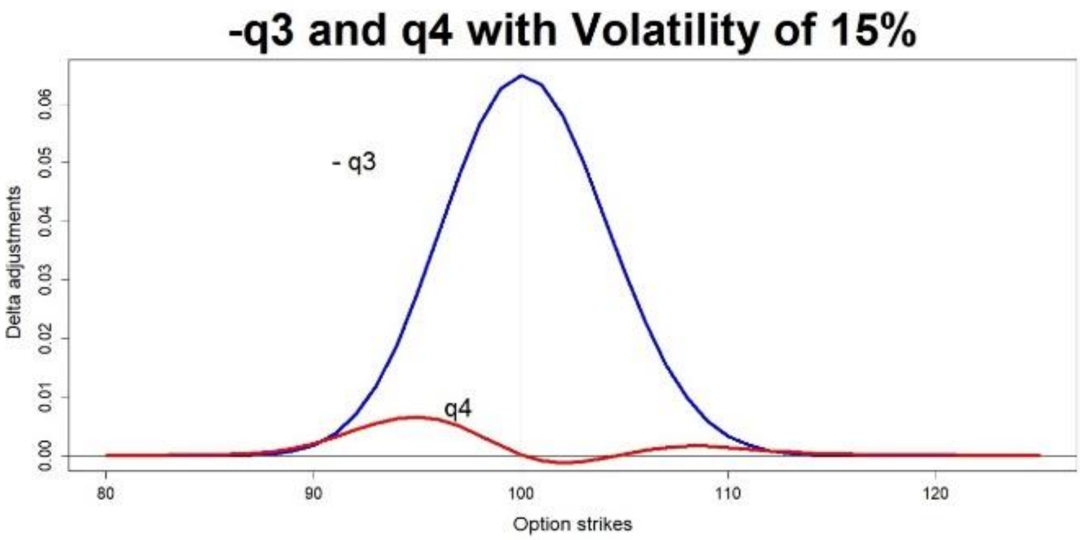

Figure 16. *Effect of -Q3 and Q4 on delta adjustment with volatility 15%. Input; S: 100, r: 2%, ttm: 1 month.*

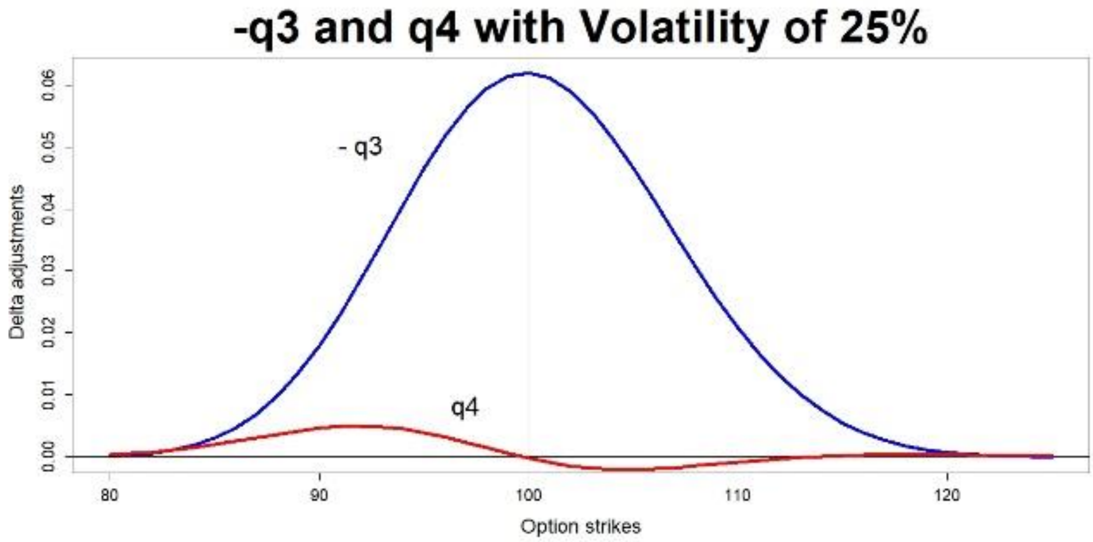

Figure 17. Effect of -Q3 and Q4 on delta adjustment with volatility 25%. Input; S: 100, r: 2%, ttm: 1 month.

The only difference between the figures with delta adjustments and moneyness is with an increased volatility the "bell" curve widens out but the top is the same, approximately 0,06 units at the most and appears when the strike is equal to the underlying security.

Figure 18 shows the combined effect of -Q3 and Q4 on delta adjustments. The adjustment is the sum of these elements $\mu_3 q_3 + (\mu_4 - 3)\, q_4$.
As before the parameters used are S = 100, Exercise prices 75-125, r = 2%, volatility = 20%, T=1/12, skewness=-0.5 and Pearson kurtosis = 6.

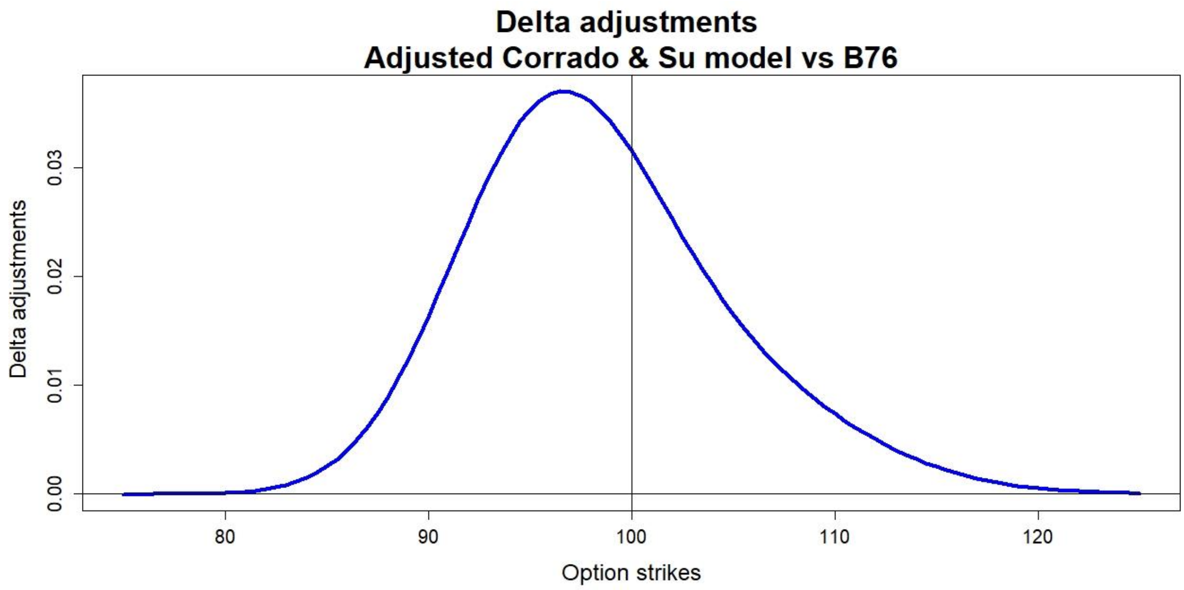

Figure 18. Delta adjustment, modified Corrado & Su vs Black 76. Input; S: 100, X: 75-125, r: 2%, Volatility: 20%, ttm: 1 month, skewness: -0,5, kurtosis: 6

4.2.2 Delta ratio

The delta ratio is defined as the skewness and kurtosis adjusted delta divided by the BSM delta

Delta ratio = $\delta / \text{N}(d_1)$ δ as in chapter 4.2

Figure 19 and 20 show plots of the delta ratio with 2 different volatilities. Parameters used in this case are: skewness of – 0.5 and kurtosis of 3 (normal). The delta ratio is a function of volatility, moneyness, skewness, kurtosis, time to maturity and interest rates.

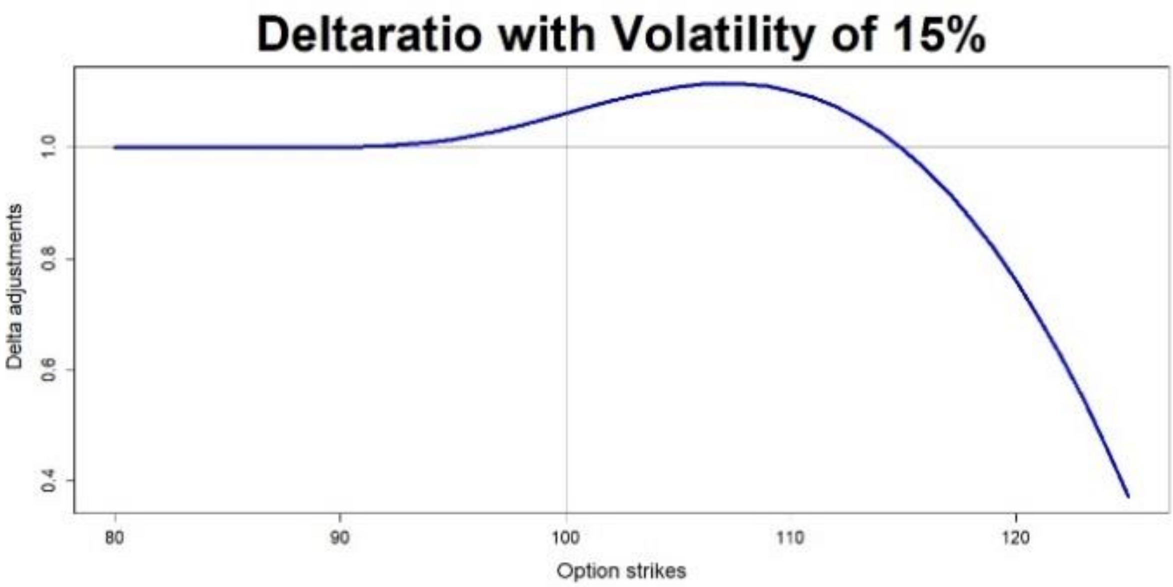

Figure 19. Deltaratio, Modified Corrado & Su delta divided by BSM, with parameters, skew= -0.5, kurtosis = 3, t=1/12, r=2% and vol=15%.

The graph reaches a maximum at approximately 1.1 or 10% less contracts. The graph crosses the 1 line at strike 115, meaning you need more contracts for hedging for strikes > 115 versus the BSM model.

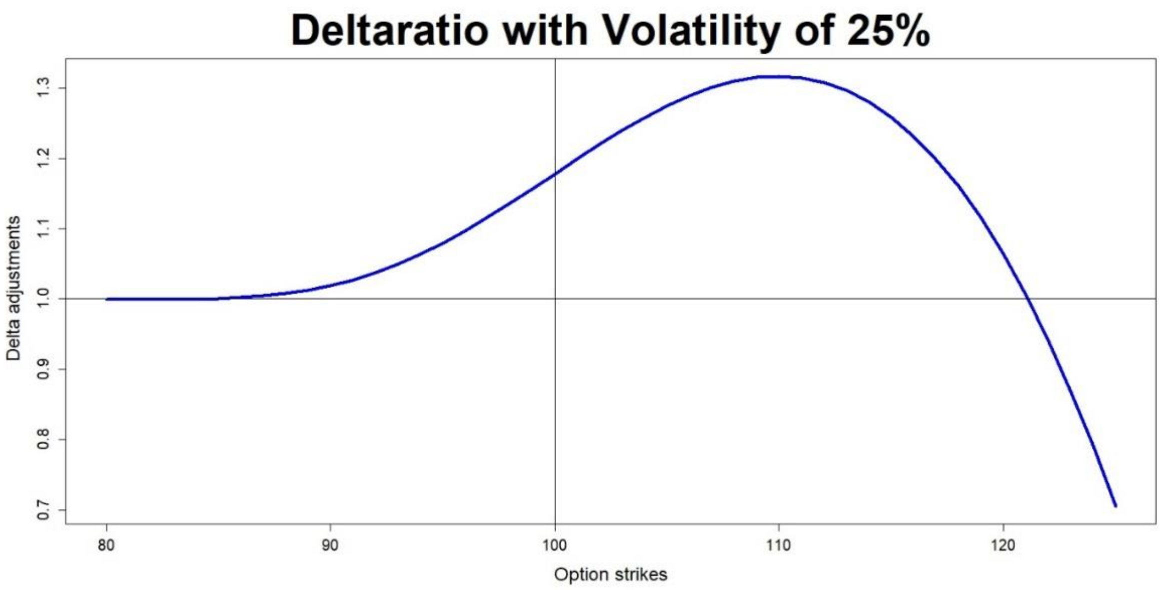

***Figure 20**. Deltaratio, Modified Corrado & Su delta divided by BSM, with parameters, skew= -0.5, kurtosis = 3, t=1/12, r=2% and vol=25%.*

At 25% volatility (figure 20), the largest positive effect on delta ratio is an approximately 110 call. The derivative of the function where its equal to 0 is a function of volatility, both level and moneyness.

An interpretation; when the graph is above 1 then you need less contracts to hedge a position in stock and vice versa when the graph is below 1 then you need more contracts to hedge the same position adjusted for the ratio the graph shows. 1.3 means 30% less contracts, and 0.5 you need to double the amount of contracts compared to the BSM model.

Kurtosis has little effect on the ATM delta despite having a relatively huge impact on the price, compared to Skewness that has little effect on the price but a huge impact on the delta. The delta equation may function as a reminder to market players and risk management that options can vary from the BSM world when the underlying market has a significant skewness and kurtosis. Especially one can experience this with short lived options where hedging options is a problem in short gamma positions.

Using excess kurtosis of 1, skewness of -1.5 and volatility of 25%, figure 21 is different from the others. Maximum adjusting for delta is almost 30% less at the 110 strike this is the same as the figure above but the curvature at the end is quite different. Compared to original BSM model you need less contracts or the delta is higher, in the range 85 to almost 130.

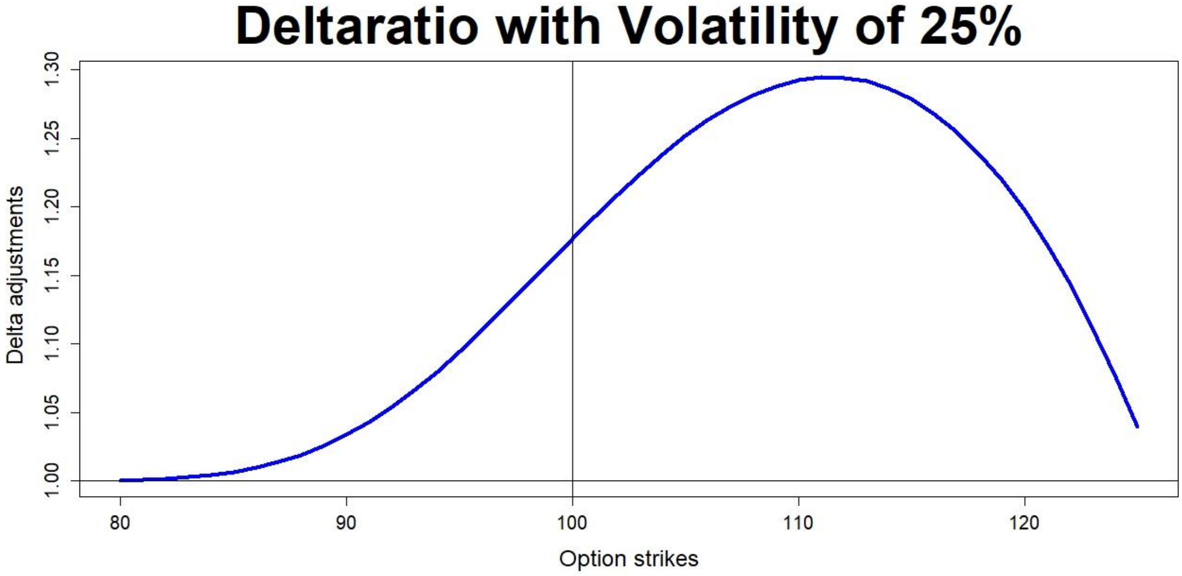

__Figure 21__. Deltaratio, Modified Corrado & Su delta divided by BSM, with parameters, skew= -1.5, kurtosis = 4, t=1/12, r=2% and vol=25%.

The figure 22 is a 3D plot of the ratio of the skewness and kurtosis adjusted delta compared to Black and Scholes delta of an ATM 100 call option for different levels of skewness and kurtosis and illustrates the impact on the delta ratio. It can be observed that delta ratio is a decreasing function of both skewness and kurtosis.

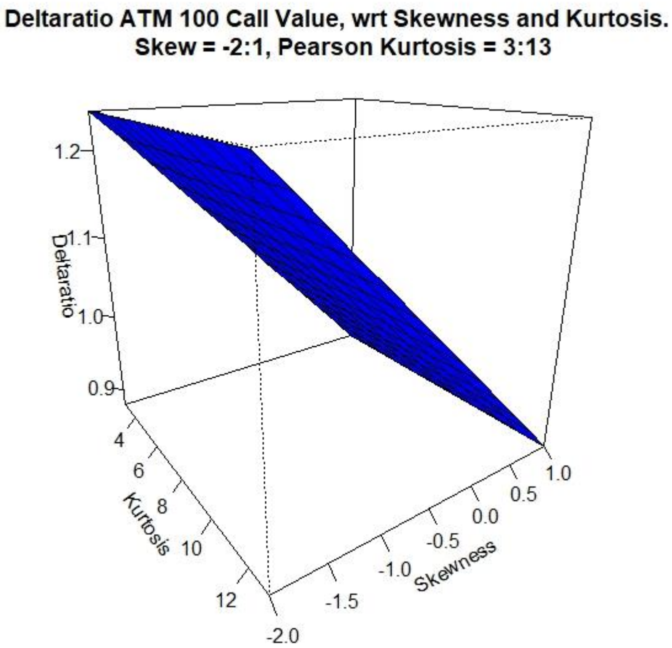

__Figure 22__. 3D of deltaratio impact of skewness and kurtosis on an ATM call.

4.2.3 Mini-case

Imagine you have 100000 stocks in a company and want to hedge your position against price risk and you want a delta neutral position. You can do it several ways, e.g. sell options, buy options or make a synthetic short position.

1. Selling calls against your position will give you extra premium in your account and no capital outlay but you have to rebalance your position because of short gamma.
2. Buying puts against your position will involve a capital outlay and a possible loss if the stock doesn't move in the hedging period but unlimited profit potential if the stock moves.
3. The last option is to do a conversion. Buy puts and sell calls against your position with minimal or zero capital outlay. 1:1, buy 1 put and sell 1 call same strike and same time to maturity.

Assuming a market regime with IV of 25%, skew = -0.5 and kurtosis of 6 the following number of contracts are needed compared to BSM.

Delta ATM options

	BSM	MCS
Delta call ATM	0.52	0.55
Delta put ATM	0.48	0.45

Strategy one

BSM sell calls 100000/0.52 = 192 307

MCS sell calls 100000/0.55 = 181 818

Diff contracts sold 10 489

Strategy two

BSM buy puts 100000/0.48 = 208 333

MCS buy puts 100000/0.45 = 222 222

Diff contracts bought = 13 889

Strategy three

Buy and sell one to one. Buy 100000 puts and sell 100000 calls against the 100000 long stock position. Same strike and same time to maturity.

What strategy to choose depends on the risk you accept, present volatility in the market, extra funds for speculation or just want to safe and secure a position (profit) or is e.g. not allowed to sell right now, employment stocks. Regardless of risk preferences, if a call contract cost for instance NOK 2,9 the difference in number of contracts amount to NOK 30 418,-.

5. Methodology – Extracting skewness and kurtosis from option prices.

5.1 Parametric risk neutral moments approach

This section is primarily based on the work done by Gurdip Bakshi et al. (2003) (BKM for short), Turan G Bali and Murray (2013), Turan G. Bali, Hu, and Murray (2017). They have also based their work on Dennis and Mayhew (2002), Duan and Wei (2009) and Conrad et al. (2013).

Moments from option prices can be extracted with several different methods, see for instance Turan G. Bali et al. (2017), here one parametric method is shown. An alternative is a nonparametric approach, based on taking differences in the implied volatilities of all the options with different strikes or moneyness.

The parametric method is based on the methodology from Gurdip Bakshi and Madan (2000) and Gurdip Bakshi et al. (2003). Turan G Bali and Murray (2013)'s paper is also based on BKM to estimate the V, W and X (representing the price of volatility, cubic and quartic contract) from real option prices and that means it is basically based on discrete prices.

The BKM moments are recovered in a model-free fashion without imposing any structure on the underlying process and based on a series of computations. Their objective is to represent moments of the risk-neutral distribution in terms of traded option and BKM show that the annualized variance (VAR^{BKM}), skewness ($Skew^{BKM}$) and excess kurtosis ($Kurt^{BKM}$) of the risk neutral distribution of a index's log return from present (t) to a time into the future (τ) can be calculated like this:

$$VAR^{BKM} = \frac{e^{r\tau}V_{i,t} - \mu^2}{\tau}$$

$$Skew^{BKM} = \frac{e^{r\tau}W - 3\mu e^{r\tau}\,V + 2\mu^3}{[e^{r\tau}\,V - \mu^2]^{\frac{3}{2}}}$$

$$Kurt^{BKM} = \frac{e^{r\tau}X + e^{r\tau}W + e^{r\tau}V + 3\mu^4}{(e^{r\tau}V - \mu^2)^2}$$

$$\mu = e^{r\tau} - 1 - \frac{e^{r\tau}}{2}\mathrm{V} - \frac{e^{r\tau}}{6}\mathrm{W} - \frac{e^{r\tau}}{24}\mathrm{X},$$

r represents the continuously compounded risk – free rate for the time t to time t + τ. V, W and X represent the risk neutral expectation of the squared, cubed and fourth power of the log of returns in the same period.

The risk based BKM volatility (Vol^{BKM}) is to be annualized standard deviation of the distribution of the log return and is defined as

$$(Vol^{BKM}) = \sqrt{Var^{BKM}}$$

The BKM vol is to be used in e.g. risk measures and VAR computations.

Calculation of V, W and X using the BKM approach.

The approach is based on two assumptions. First the spot has to be adjusted for dividends. Using the future this is already taken care of, and second the formulas in BKM assume that option prices are available with strikes that are equally spaced above and below the current spot price. And allowing for all option data one can a descrete approach defined in Turan G Bali and Murray (2013).

$$\mathrm{V} = \sum_{i=1}^{n^C} \frac{2(1-\ln\left(\frac{K_i^C}{Spot}\right))}{(K_i^C)^2}\,\mathrm{Call}(K_i^C)\Delta K_i^C + \sum_{i=1}^{n^P} \frac{2(1+\ln\left(\frac{Spot}{K_i^P}\right))}{(K_i^P)^2}\,\mathrm{Put}(K_i^P)\Delta K_i^P$$

$$\mathrm{W} = \sum_{i=1}^{n^C} \frac{6ln\left(\frac{K_i^C}{Spot}\right) - 3\ln\left(\frac{K_i^C}{Spot}\right)^2}{(K_i^C)^2}\,\mathrm{Call}(K_i^C)\Delta K_i^C - \sum_{i=1}^{n^P} \frac{6ln\left(\frac{Spot}{K_i^P}\right) + 3\ln\left(\frac{Spot}{K_i^P}\right)^2}{(K_i^P)^2}\,\mathrm{Put}(K_i^P)\Delta K_i^P$$

$$X = \sum_{i=1}^{n^C} \frac{12ln\left(\frac{K_i^C}{Spot}\right) - 4\ln\left(\frac{K_i^C}{Spot}\right)^3}{(K_i^C)^2} \text{Call}(K_i^C)\Delta K_i^C + \sum_{i=1}^{n^P} \frac{12ln\left(\frac{Spot}{K_i^P}\right)^2 + 4\ln\left(\frac{Spot}{K_i^P}\right)^3}{(K_i^P)^2} \text{Put}(K_i^P)\Delta K_i^P$$

Where i indexes the OTM call and put options with available prices.
Spot is the closing price of stock adjusted for dividend or futures in our case.
K_i^C, K_i^P is the strike of the ith OTM put or call option when the strikes are ordered in decreasing and increasing order.
Put(K_i^P), Call(K_i^C) is the price of the put or calls option with strike K_i^C, K_i^P.
The n^P, n^C is the notation of the number of OTM puts and calls that are available.
Last we set

$\Delta K_i^P = K_{i-1}^P - K_i^P$ for $2 \leq i \leq n^P$,

$\Delta K_i^P = \text{Spot} - K_i^P$,

$\Delta K_i^C = K_i^C - K_{i-1}^C$ for $2 \leq i \leq n^C$,

and

$\Delta K_i^C = K_i^C - \text{Spot}$.

Allowing ΔK to vary for each option, it relaxes the assumption in the BKM formulas that the prices are available for options with fixed intervals between strikes. The weighting structure in V, W and X assign higher weight to put options with low – and call options with high moneyness versus options near ATM. A negative skew is present when the cost of combined position in OTM put exceeds the position in calls.

All the risk neutral moments for the OBX future for week m is then calculated using weekly data calculated on every Thursday each week for the last 574 weeks for options that expires in approximately 0,5 months for the 1st pos and 1,5 months for the 2nd pos. Our data is then contemporaneous to the price used as denominator in the moment calculations.

6. Data

The underlying asset in our study is OBX Total Return (TR) 1st and 2nd position futures prices. Because information in the option prices are forward looking, it's better to use the futures price instead of OBX-index to extract implied volatilities, skewness and kurtosis. The futures price also accounts for expected dividends, so we don't need to further adjust the option prices. From DataStream, provided by Thomson Reuters, we have downloaded OBX TRc1 and OBX TRc2, from the introduction of OBX futures in March 1993, to March 2018. OBX TRc1 is a 1st position continuous series of the nearest futures prices and rolls to the next contract on the last trading day and OBX TRc2 the second position (Datastream, August 2010). First five years there are several days with no price-observations the, for this reason we continue with data from 1997 and forward. On days with missing price-observation we use the previous day's price.

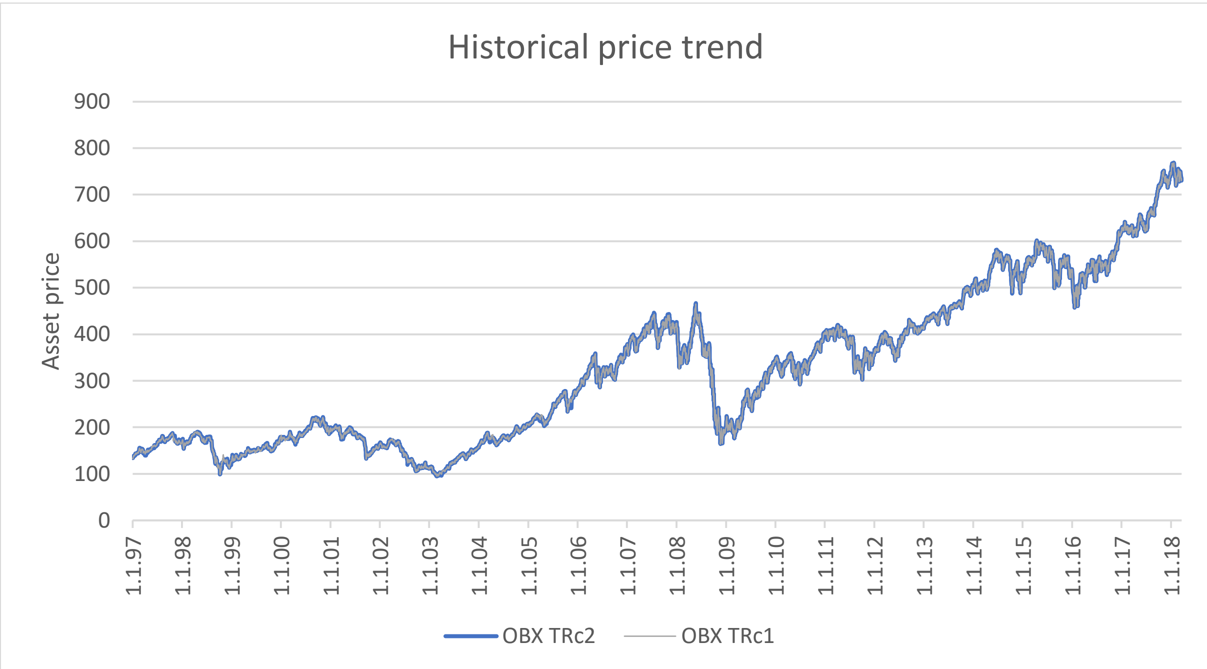

***Figure 23**. OBX, Historical futures price from January 1997 to March 20118. Data collected from Thompson Reuters Datastream.*

Figure 23 shows the historical futures price from January 1997 to March 2018. Overall, the index has been in a positive trend, but with some turbulent periods, especially in 2008 due to the global financial crisis. The largest fall is found during September 2008, which is about the time Lehman Brothers filed for bankruptcy. Before the financial crisis, the largest crash in the financial market is what known as Black Monday on October 19,1987 where stocks and indices fell over 20% in one single day (DJIA fell 22%).

The OBX is a tradable index and dividends are mostly payed during the first two quarters each year. During these months, 1st position contract could trade at a higher price compared to 2nd position before ex-dividend date.

Descriptive statistics of daily returns	
Mean	0,032 %
Standard error	0,021 %
Median	0,078 %
Standard deviation	1,520 %
Variance	0,023 %
Kurtosis	6,281
Skewness	-0,485
Minimum	-11,282 %
Maksimum	10,986 %
Observations	5299

***Table 3**. Descriptive statistics OBX TRc1 1997-2018.*

Table 3 show descriptive statistics for the 1st position continuous futures series. We find negative skewness for the whole sample-period and excess kurtosis of 6,28. In table 4, skewness and kurtosis are calculated for each year and tested for normality with Jarque – Bera test. We statistically reject normality for all years except 2009 and 2017. We find the largest yearly kurtosis in 2005, and the total (years 1997 – 2017) to be higher than any of the single years. These results are in accordance with our expectations of non-normality, even with over 5000 observations.

Year	Skew	Kurtosis	Jarque Bera	JB-p values	Level Signific
1997	-0,81	6,85	183,10	2,2*(10^-16)	100,00 %
1998	-0,18	5,08	47,01	6,2*(10^-11)	100,00 %
1999	0,03	4,23	15,94	0,0003464	99,97 %
2000	-0,50	4,29	27,82	9,084*(10^-7)	100,00 %
2001	-0,32	4,25	20,63	3,32*(10^-5)	100,00 %
2002	-0,26	4,06	14,76	0,0006223	99,94 %
2003	-0,26	4,12	16,13	0,0003141	99,97 %
2004	-0,76	4,67	53,72	2,163*(10^-12)	100,00 %
2005	-0,72	7,02	191,77	2,2*(10^-16)	100,00 %
2006	-0,57	5,54	81,39	2,2*(10^-16)	100,00 %
2007	-0,35	4,34	24,04	6,028*(10^-6)	100,00 %
2008	-0,37	4,77	38,67	4,004*(10^-9)	100,00 %
2009	-0,15	3,43	2,88**	0,237	76,26 %
2010	0,01	4,62	27,50	1,07*(10^-6)	100,00 %
2011	-0,20	3,90	10,17	0,00600	99,40 %
2012	-0,21	3,77	8,07	0,01765	98,24 %
2013	-0,25	3,65	7,13	0,02827	97,17 %
2014	0,13	4,57	26,62	0,006174	99,38 %
2015	-0,10	5,12	47,78	1,66*(10^-6)	100,00 %
2016	0,08	4,12	13,35	0,001264	99,87 %
2017	-0,20	3,29	2,61**	0,271300	72,87 %
Total	-0,48	9,27	8898	0,000	100,00 %
Max	0,13	7,02	192,00		
Min	-0,81	3,29	2,61		

Table 4. *OBX TRc1, skewness and kurtosis pr year 1997 – 2017, Jarque – Bera – test, p-values and significance level. By probability, returns in year 2009 and 2017 are close to normally distributed, market with **.*

From table 4, we find three years with marginally positive skewness, but a negative skew for the whole sample-period. The overall kurtosis is higher than the highest individual-year kurtosis, and for comparison we have done a similar analysis of other indexes that have a longer trading history. Table 5 report our findings from the different indexes. The S&P500 index has a long history of tracking large companies through different economic policies and shifts, as well as minor and major events affecting the financial market. Because of the crash in 1987, we calculate historical skewness and kurtosis before and after the event, year 1987 alone, and the whole sample period. Except from S&P 500 and NIKKEI 225, the other indexes were established during the 1980's and the crash in 1987 seems to have a large impact on their historical kurtosis. All the indexes are negatively skewed, most negative in 1987 and for the whole sample period. We find excessive kurtosis for all the indexes in 1987 and for HIS 1987-2018 and AORD 1984-2018. In comparison, the OBX seems to carry features recognizable in other indices.

KURTOSIS TYPE: PEARSON. Last trading day 2018-04-19			
Index	**Period**	**Kurtosis**	**Skew**
GSPC – S&P 500	1950 – 1987	7,23	-0,11
GSPC – S&P 500	1988 – 2018	12	-0,31
GSPC – S&P 500	1987	56,84	-5
GSPC – S&P 500	1950 – 2018	30,15	-1,017
N225 – NIKKEI 225	1987	35	-3
N225 – NIKKEI 225	1965 – 1987	13	-0,9
N225 – NIKKEI 225	1988 – 2018	8,78	-0,15
N225 – NIKKEI 225	1965 – 2018	12,57	-0,42
GDAXI – DAX	1988 – 2018	8,8	-0,24
FCHI – CAC	1990 – 2018	7,63	-0,069
HSI - HANG SENG INDEX	1987	105,5	-8,4
HSI - HANG SENG INDEX	1988 – 2018	19,63	-0,55
HSI - HANG SENG INDEX	1987 – 2018	61,48	-2,36
AORD – AUSTRALIAN ALL ORDINARIES	1987	83	-7,12
AORD – AUSTRALIAN ALL ORDINARIES	1984 – 1987	4,85	-0,2
AORD – AUSTRALIAN ALL ORDINARIES	1988 – 2018	9,48	-0,6
AORD – AUSTRALIAN ALL ORDINARIES	1984 – 2018	95,95	-3,58

Table 5. *Skewness and kurtosis on other indices, with different time-periods.*

Collecting data on historical option prices proved to be a challenge. From DataStream, we were able to find historical option-prices for ATM and implied volatility for 80% to 120% moneyness with 5% intervals from January 2010. For our analysis we need prices for OTM put and call options on the OBX, preferably from before 2008 up to today. These historical data are downloaded from Titlon.no. This database provides daily historical, high, low, last and closing prices on all options, put and call, on the OBX from 1990 to June 2017. It also holds information about open interest and expiration date, but data on traded volume is not available. We continue with the observations from 04th of May 2006 and forward. The dataset is sorted by first deleting all observations with missing closing price and bid-ask spread. Not included in our analysis is observations with only a last trading price, because we don't have any information at which time the option was last traded. We use the mid-value of bid and ask prices as closing price. All option with prices less than 0,1 NOK is also deleted from our dataset.

We do calculations of risk neutral implied moments on every Thursday from 2006 to 2017. Close prices are applied for both future and options, but there still is a possibility for errors in time quote. The consequence of this is them not being exactly comparable. The moneyness of the option is found by dividing strike price on the asset price. For each day we match the options with the shortest time to maturity with the 1st position futures contract and the second shortest with the 2nd position futures contract. In March 2015 we find some errors in the data, that is all historical option data is stored with the same expiration date. This period is therefore excluded from our empirical analysis.

Three-month NIBOR is applied as risk free rate and downloaded from Norges Bank (Bank, 2018). Though NIBOR is not risk free we use it as a proxy for treasury bills. At Oslo Børs we have found information about number of traded contacts by year on the OBX-index. Number of traded contracts peeked during the years 2006 – 2008, and at its maximum in 2007 with over 1,7 million contracts. From 2009 and forward the number of traded contracts range from 667 406 to 977 244 a year.

7. Results

In this section we present the results from our empirical analysis based on the methodology described in section 5. Gurdip Bakshi et al. (2003) employed an average of daily data to construct a weekly estimate, but due to limited time, volatility, skewness and kurtosis from a risk-neutral distribution are calculated weekly. The moments are calculated every Thursday from options expiring at the nearest and second nearest maturity. For weeks where Thursday is a holiday/non-trading day, implied moments is calculated on the previous day or nearest trading day.

When we only observe daily asset-prices in discrete time, every change from day to day is a jump. But even with intraday-data, we still would be observing jumps of different size because of discrete prices. Stochastic volatility is likely to cause jumps in the asset price independent of discrete or continuous time. Option-prices are largely driven by supply and demand, as a result of investors/stakeholder’s perception of future price movements and new information (Garleanu et al., 2009). News arrive in discrete time and at different frequency, and the effect on supply and demand is highly variating. Volatility tend to increase with both positive and negative news. Positive shocks are often more gradually, and negative shocks seems to yield an more instant effect (Haug et al., 2010). The information can be seen as a mixture of many Gauss-distributions which aggregated lead to a distribution with high peak and fat tails (Haug & Hoff, 2018), but not all information or events are of a frequent mode. Taleb (2007a) emphasizes that Gaussian is not an approximation of real randomness, and with no lower or upper bound, probabilities are not observable or measurable. Thus significance testing of problems based on probabilities we in reality cannot observe or measure, will not be possible (Taleb, 2007b). It is not our intention to draw a conclusion about the causes of jumps in financial price data, but we believe a discussion on the subject about likely causes are important for a better understanding of kurtosis. BSM assumes a constant volatility, no arbitrage and that you can remove all the risk with continuous dynamic hedging. However, this principle is very sensitive to market frictions, jumps and stochastic volatility, thus we cannot necessarily view the BSM-option price as a truly risk-neutral valuation (Taleb, 2015). If options are not risk neutral, are risk-neutral densities truly risk-neutral? Gurdip Bakshi et al. (2003)'s method is a model-free measure of risk-neutral moments without any assumptions about the form of the distribution, but there is uncertainty as to whether the calculated moments can be viewed as true risk-neutral moments when the underlying assumption behind risk-neutral valuation does not hold for empirical financial price data. We will view these moments as moments from a risk-neutral distribution, keeping in mind the arguments against the risk-neutral valuation principle.

7.1 Historical kurtosis

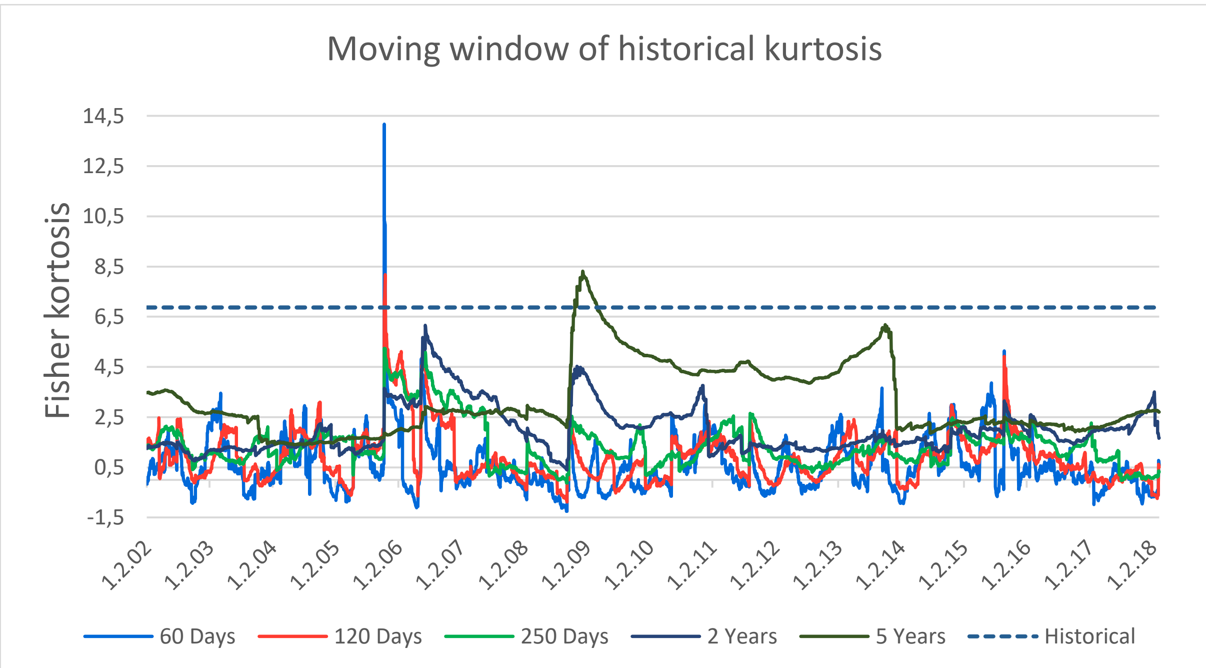

Figure 24. *Moving windows of OBX historical kurtosis and historical kurtosis from 02.01.2002 – 02.01.2018.*

Moving window of historical excess kurtosis is highly volatile and deviate from those from a normal distribution with excess kurtosis equal to zero, even with a wide window. This is a likely consistency with the result from GBM-simulations in section 3.3, where discrete time seems to cause timeseries to deviate from a normal distribution. Moving 2 and 5-year windows, from 2002 – 2018, is never less than zero and mostly higher than the shorter windows. The shorter windows are more volatile, and the historical kurtosis is negative for shorter periods but mostly >0. From figure 24 we see several spikes, occurring when there is one or a few observations with a large deviation from the mean. The large spike 12th October 2005 is mostly due to 16 previous days with missing price-observations in the underlying asset, which has a great impact on the shorter windows. The dashed line is the static historical kurtosis from January 2002 to January 2018 and for the whole period higher than the moving windows, except from the spikes in year 2005 and 2009. If the ratio between small deviations in return compared to large deviations is stable over time, the historical kurtosis will be above 0 (or 3 with Pearson kurtosis) even with long time series. Taleb (2007a) wrote;

> "*a true fat-tailed distribution can camouflage as thin-tailed in small samples; the opposite is not true*".

If the amount of small deviations increases relative to large deviations when including more observations, the larger deviations will have a greater impact on the kurtosis. Larger samples are more likely to be fat-tailed than small samples.

7.2 Moments calculated from option prices.

The number of out of the money options is highly variating throughout our sample, especially with time to maturity less than 14 days we find very few OTM options. Since the method we apply requires a minimum of two OTM puts and OTM calls to calculate implied moments, we have 9 missing calculations. One possible solution to this problem would be to use the put-call parity. But because of large market-maker spreads, and mid-price under lower bound for in the money options, this leads to negative prices out of the money out if mid-value of the spread is used at option price. Using only ask-prices, negative prices by put-call parity is not a problem but would potentially yield biased results.

	VOL 1.pos	*SKEW 1.pos*	*KURT 1.pos*
Mean	21,78%	-1,135	3,07
Median	19,50%	-1,087	2,502
Minimum	7,72%	-4,156	-1,544
Maksimum	79,36%	1,022	25,073
Observations	573	573	573
Standard deviation	9,94 %	56,34 %	265,80 %
Kurtosis	7,527	2,587	10,617
Skewness	2,207	-0,477	2,152

Table 6. *Descriptive statistics of weekly moments calculated from short maturity options.*

	VOL 2. pos	*SKEW 2.pos*	*KURT 2.pos*
Mean	21,96 %	-0,989	1,128
Median	20,14 %	-1,016	0,858
Minimum	9,15 %	-2,381	-2,4
Maksimum	63,82 %	1,115	8,472
Observations	574	574	574
Standard deviation	9,01 %	50,18 %	174,66 %
Kurtosis	4,643	0,568	0,592
Skewness	1,792	0,368	0,752

Table 7 *Descriptive statistics of weekly moments calculated from medium maturity options.*

From our time series of implied moments, 18 calculations with positive skewness is found from options with medium long maturity and 180 calculations where kurtosis is less than zero. Similar to Gurdip Bakshi et al. (2003) we find the short maturity options to be more skewed and have a higher kurtosis than medium maturity options, with 10 Thursdays having a positive implied skewness and only 32 Thursdays with excess kurtosis less than zero.

We have not managed to find previous research studying the statistical behavior and distribution of implied kurtosis. Haug et al. (2010) presented a similar study on implied volatility with a unique dataset. However, we recognize that our analysis is subject to noise and estimation errors, and with non-constant time to maturity the timeseries of moments are not exactly comparable and ideally should have the same time to maturity for all calculations.

Descriptive statistics from table 6 and 7 is not annualized and based on weekly estimates. The volatility of implied kurtosis is, as expected, extremely volatile with a standard deviation of 265,8% (174,66%) for short (medium) maturity options. Short maturity implied skewness is the only time-series with negative skewness, with the mean more negative than the median. All series have excess kurtosis, but its more evident for the short maturity moments.

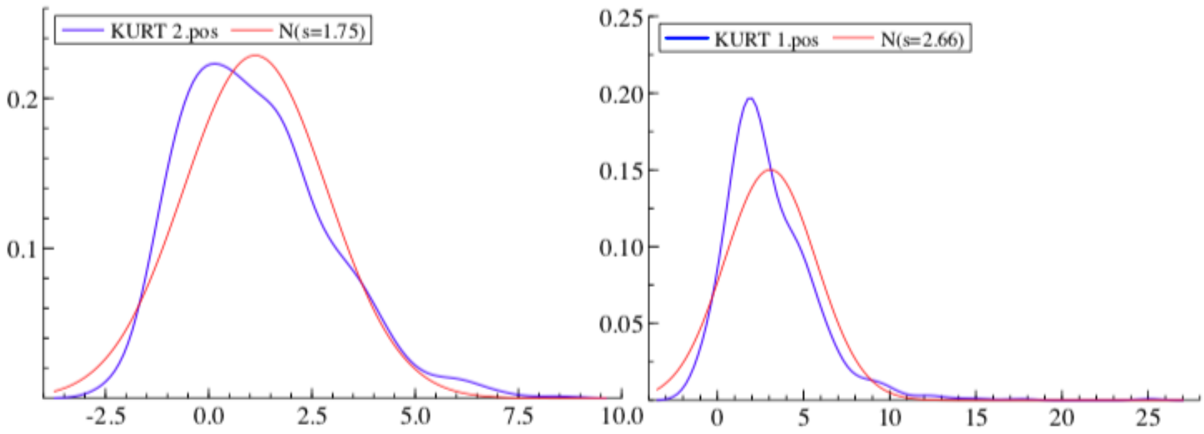

Figure 25. *Density of kurtosis from a risk-neutral distribution, short and medium maturity. For 1. Pos; Skewness: 2,152 Kurtosis: 10,617. For 2. Pos; Skewness: 0,752 Kurtosis: 0,592*

Figure 25 shows a density plot of kurtosis from a risk neutral distribution against the normal distribution. The positive skewness in the distribution is confirmed in the plot with a longer right tail, implying a greater probability of observing large values of excess kurtosis as opposed to negative values. Short maturity kurtosis is peaked, implying more observations with small deviations from the mean compared to a normal distribution, and high excess kurtosis. The shape of the density-plot for medium maturity implied kurtosis is similar to one of a log-normal distribution, with moderate positive skew and excess kurtosis, but have a wider distribution and longer right tail. Moments from a normal distribution will not itself have a distribution, only a point, and these results supports our assumption of about non-normality.

Correlation					
	VOL 1.pos	***SKEW 1.pos***		***VOL 2. pos***	***SKEW 2.pos***
SKEW 1.pos	0,31	1	**SKEW 2.pos**	0,161	1
KURT 1.pos	-0,424	-0,834	**KURT 2.pos**	-0,238	-0,848

Table 8. *Correlation between volatility measure and risk-neutral skewness and kurtosis.*

BKM did not find any consistent pattern for kurtosis, however we find a strong negative correlation between implied skewness and kurtosis. This seems to be consistent with Chang et al. (2013)'s analysis of S&P500 option, where they found a correlation between implied skewness and kurtosis of -0,83. Small values of implied skewness is found on days with small values of implied kurtosis, and vice versa a highly negative skewness on days with high kurtosis. While volatility is not highly correlated with skewness or kurtosis, the sign of the correlation-coefficient imply that implied skewness is less negative and implied kurtosis is less positive with a high volatility.

The following graphs show historical futures price and log return, calculated BKM volatility and skewness and kurtosis from a risk-neutral distribution for short and medium maturity options May 2006 to June 2017.

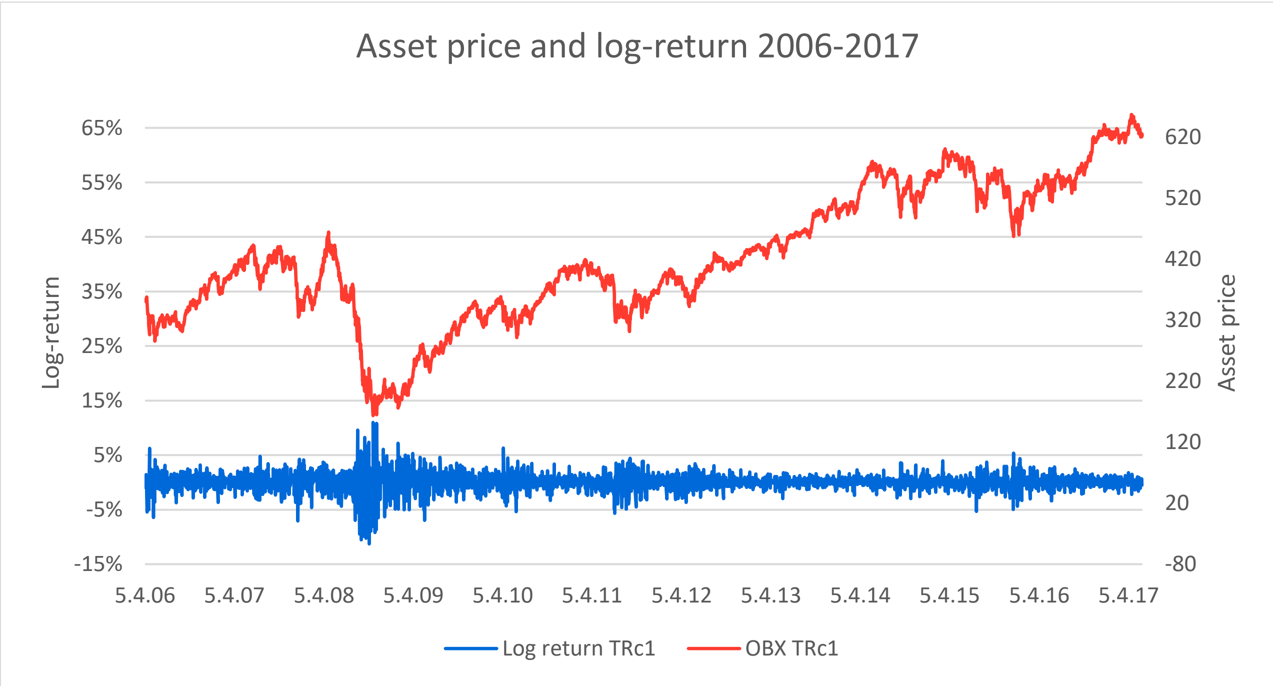

Figure 26. *Left y -axis: OBX log-return, right y-axis: OBX TRc1 asset price, May 2006 – June 2017.*

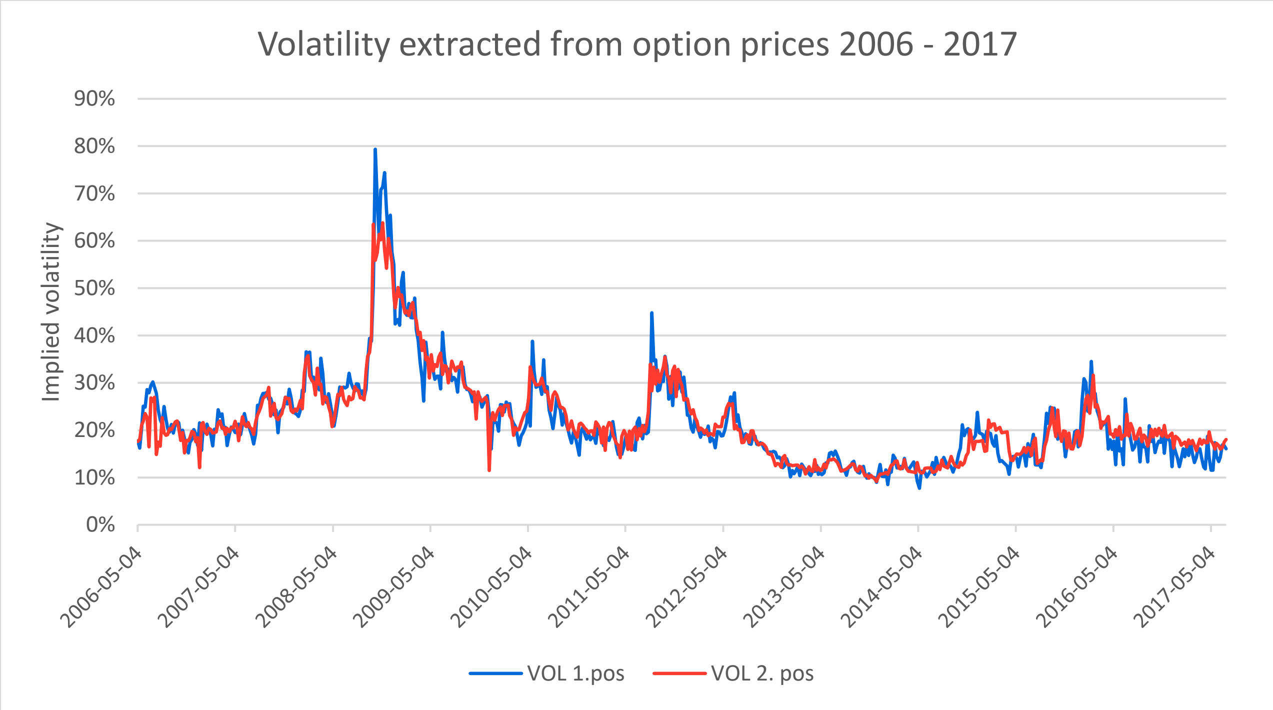

Figure 27. *Volatility from a risk-neutral distribution, calculated from short and medium maturity option prices, May 2006 – June 2017*

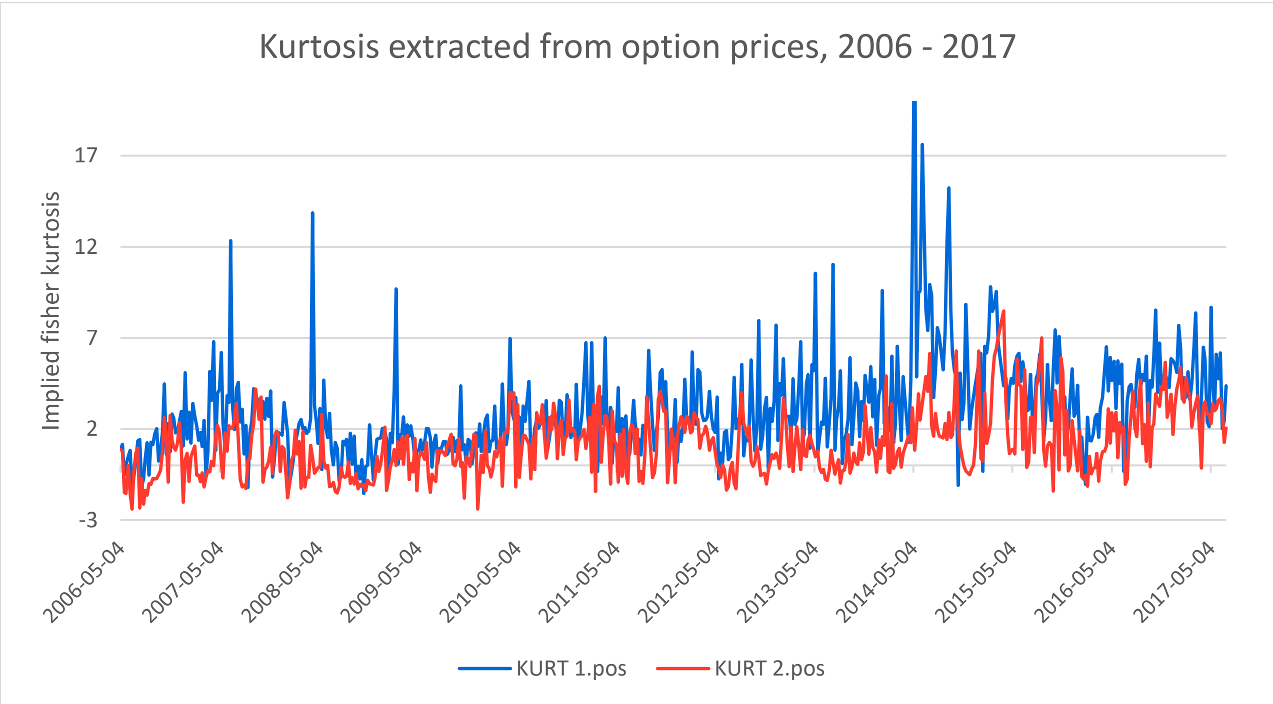

Figure 28. *Kurtosis from a risk-neutral distribution, calculated from short and medium maturity option prices, May 2006- June 2017.*

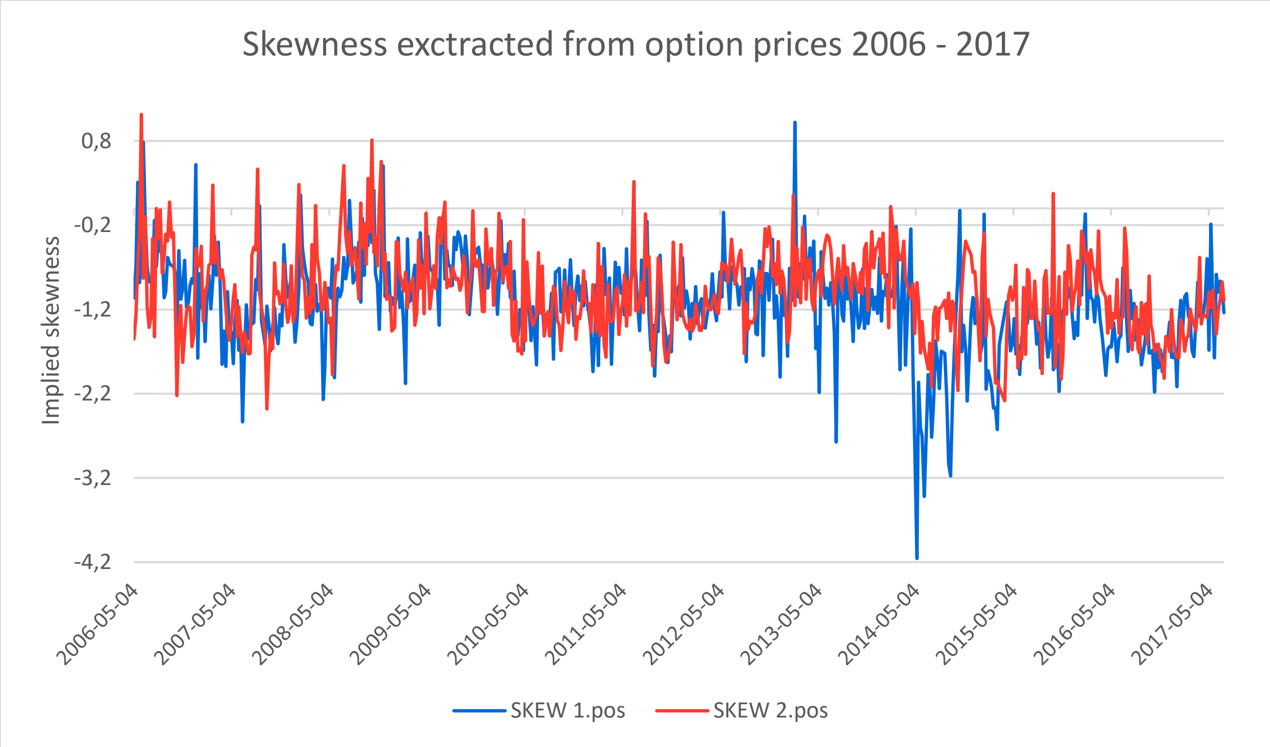

Figure 29. *Skewness from a risk-neutral distribution, calculated from short and medium maturity option prices, May 2006-June 2017.*

Volatility extracted from option prices peeks during the financial crisis in 2008 and has a maximum at 79,36% for the 1st position contract. After year 2012 and onwards this volatility measure is less volatile and at a lower level than previous period. These results seem to be in accordance with implied volatility and historical volatility observed the last years. Implied skewness and kurtosis are highly variating and more volatile compared to volatility. The short-term maturity implied kurtosis is higher and skewness more negative than the medium maturity, but they follow a similar pattern, indicating they are related. Moments decrease with long maturity options and increase with short maturity options. This is consistent with findings in previous studies like Gurdip Bakshi et al. (2003) and likely consistent with Haug et al. (2010) who found short-maturity implied volatility to be more volatile and very sensitive to shocks.

Gurdip Bakshi et al. (2003) found in their study of OEX, the index skewness never to be positive, and more negatively skewed than individual assets. As did Chang et al. (2013). Figure 29 show the implied skewness from our empirical analysis of the OBX, and unlike Gurdip Bakshi et al. (2003) and Chang et al. (2013), we find observations with positive skew. Interestingly, most of the days with positive implied skewness occurs in a highly volatile period and could imply investors and traders expecting the asset price to increase after a large fall and therefor buying OTM call options. Skewness in the underlying asset is found to be – 0,48, but the implied skewness is overall more negative with an average of -1,13 for nearest maturity and -0,98 for the next maturity. A possible explanation, might be a tendency of investors using put-options as insurance, causing OTM put option price to increase relative to OTM calls. Gearing is imposing a greater risk to investors when the market falls, and as most investors are long in stocks they can hedge some of that risk by buying OTM puts.

When volatility is abnormally high, as in year 2008, we find a tendency for calculated moments of skewness and kurtosis to be lower. High volatility implies an increased probability of large price movements and increase the option price. As a possible interpretation is that when large movements are accounted for through a high volatility, this moment dominates and suppressing/override the effect of the third and fourth moment. And vice versa, when volatility is at a normal to low level, the possibility of large price-movements seems to be accounted for and priced through skewness and kurtosis.

Similar to BKM and Conrad et al, from figure 26-29 we can see the moments are related to movements in the underlying asset. The volatility measure increases simultaneous at times with clustering of large log returns. Implied skewness and kurtosis tends to be higher, in absolute terms, when asset returns are less volatile. The moments are highly volatile through the whole sample period and, we believe, very important in valuation and for traders and stakeholders exposed to tail risk.

7.3 Sensitivity analysis.

Low volatility	Moneyness range	OTM put IV – OTM call IV	Vol	Skew	Kurt
28.11.2013	94% - 106%	2,16 %	9,14 %	-0,705	2,106
21.11.2013	95%-107,1%	0,89 %	9,70 %	-0,275	0,612
07.11.2013	95,7% - 106,8%	1,34 %	9,83 %	-0,417	0,756
High volatility					
13.11.2008	83,6% - 138,4%	17,35 %	63,82 %	0,064	-1,013
09.10.2008	89,1% - 121,7%	33,61 %	63,52 %	-0,277	-1,022
30.10.2008	79,7% - 128,1%	17,37 %	61,39 %	-0,344	-1,395
04.12.2008	74,7% - 124,4%	9,14 %	60,47 %	-0,537	-0,665
Highly negative skewness					
20.09.2007	78,3% - 102,1%	12,75 %	22,97 %	-2,381	4,192
12.02.2015	75,4%-107,7%	25,10 %	19,64 %	-2,285	8,472
19.10.2006	84,72%-101,07%	3,73 %	17,84 %	-2,222	2,627
11.09.2014	89,3% - 105,3%	14,31 %	13,24 %	-2,163	6,280
Positive skewness					
08.06.2006	97,05% - 118,81%	3,60 %	23,48 %	1,115	-0,011
16.10.2008	90,8% - 166,4%	39,13 %	55,92 %	0,813	-1,157
20.11.2008	87,1% - 151,1%	15,06 %	58,03 %	0,555	-1,047
03.07.2008	95,3% - 122%	8,06 %	25,21 %	0,507	-0,944
Low kurtosis					
22.06.2006	97,7% - 102,65%	2,61 %	16,51 %	-0,100	-2,400
17.12.2009	98,4% - 101,4%	-0,41 %	11,45 %	-0,243	-2,393
20.07.2006	97,5% - 102,2%	4,50 %	14,82 %	-0,357	-2,337
03.08.2006	97,4% - 103,6%	2,51 %	16,61 %	0,000	-2,138
28.12.2006	97,8% - 101,8%	0,97 %	12,06 %	-0,477	-2,025
High kurtosis					
12.02.2015	75,4%-107,7%	25,10 %	19,64 %	-2,285	8,472
05.02.2015	75,1%-109%	22,80 %	19,42 %	-2,129	7,059
13.08.2015	80,6%-107,4%	18,43 %	16,52 %	-1,961	7,004
29.01.2015	73,5%-110,3%	23,72 %	20,58 %	-2,043	6,395

Table 9. *Characteristics of highest and lowest implied moments of volatility, skewness and kurtosis from medium maturity options.*

Gurdip Bakshi et al. (2003) found a high implied kurtosis to flatten the implied volatility smile when implied skewness very negative. Since the negative correlation between skewness and kurtosis is so strong, the days with the most negative implied skewness also have a high implied kurtosis. The shape of the volatility smile on days with highly negative skewness without a high kurtosis is therefore difficult to test. Table 9 show characteristics of highest and lowest values of the implied moments, and for low volatility and low kurtosis both the moneyness range and difference between implied volatility for the smallest put and largest call strike is little. There is very few out of the money options for all the Thursdays with low kurtosis (excess kurtosis less than zero), and the result may be due to sample size on these days. Four out of the five is found in year 2006.

Thursdays with the highest kurtosis is observed in 2015. These days also have very negative risk-neutral skewness and modest volatility. The moneyness range is skewed with puts further out of the money compared to calls.

Average change	Volatility	Skewness	Kurtosis
Rf +2%	0,000	0,108	-0,144
Rf 0%/-2%	0,000	-0,097	0,104
TTM -20	-0,095*	-0,038	0,000
TTM +20	0,053*	0,064	-0,085
Minus one call	0,000	0,133	-0,105
Minus two calls	0,007	0,339	-0,129
Minus one put	0,013	-0,225	-0,073
Minus two puts	0,025	-0,331	0,634
Minus one call & one put	0,018	-0,096	-0,051
Minus two calls & two puts	0,032	-0,160	0,660
Minus four calls	0,002	0,097	0,373
Minus ten calls	0,015	0,418*	1,038*
Minus four puts	0,002	-0,410*	2,628*
Minus ten puts	0,014	-0,830*	4,839*

Table 10. *Sensitivity analysis of BKM's method. Average change in the implied moments when changing input parameters. Dates used in the analysis; 08.06.2006, 03.08.2006, 20.09.2007, 09.10.2008, 16.10.2008, 28.11.2013, 05.02.2015, 12.02.2015. * representing the largest change.*

Table 10 is a sensitivity analysis performed on days also represented in table 9. Options farthest from ATM is removed, and minus two calls means we have removed the two calls with the highest moneyness. Not all eight days have enough OTM puts or calls to analyze the effect of moving more than one or two observations, and thus the average change in the last five rows is based on fewer calculations. Changing time to maturity has the largest effect on the volatility-measure and removing ≥4 calls or puts has the largest effect on skewness and kurtosis (marked with * in the table). Minor changes to the input in the model does not seem to have a large effect, but with many calculations on days with few OTM options, it is difficult to draw a valid conclusion about the robustness of this method. One solution to further test the robustness of the model on days with few out of the money options could be to interpolate and extrapolate the implied volatility smile using a spline-method and compare the results.

We have run a linear regression for a simple test to see if there is a relationship between the implied moments and asset return. The time-series of implied moments are tested for unit root with a Dickey-Fuller test and found to be stationary. Following regressions is done on short and medium maturity implied moments;

$$LogReturn_t = \alpha_0 + \beta_1 Vol + \beta_2 Skew + \beta_3 Kurt$$
$$LogReturn_{t+1} = \alpha_0 + \beta_1 Vol + \beta_2 Skew + \beta_3 Kurt$$
$$LogReturn_{t+2} = \alpha_0 + \beta_1 Vol + \beta_2 Skew + \beta_3 Kurt$$
$$LogReturn_{t+4} = \alpha_0 + \beta_1 Vol + \beta_2 Skew + \beta_3 Kurt$$

Since implied moments are on a weekly basis, asset prices are sorted to correspond with the same dates before calculating log returns. The purpose of these regressions is to find out if moments have a significant impact on current and/or future returns. Referring to our discussion initially, we will not emphasize coefficient p-values too much, rather the size and sign of the coefficients. Regression results are to find in the appendix, table 17 – 18. All moments (and intercept for medium maturity) are significant with negative sign in the first regression. R-squared show that the moments have some power in explaining changes in return. Regression on moments from short maturity options, BKM vol is the largest of the coefficients, and Skew the largest for medium maturity. When regressing on return t+1, the coefficients for each of the variables are very small, and the signs change depending on horizon (t). BKM Skew from second regression on short maturity options show significant t-statistics, and coefficient with positive sign. F-statistics is only significant for regression on log returns, and we cannot statistically prove to have found any relationship between subsequent returns and BKM Vol, Skew or Kurtosis. This is not too surprising and if we assume semi efficient markets, it natural to believe that new information is reflected in the price within short time. To find out if moments from option prices hold information about future price movements, a more sophisticated method and frequent time-series of daily, or event intraday, we believe is required.

8. Conclusion

When returns in the underlying assets do not follow a normal distribution, higher moments are very important for pricing securities and managing risk. We find that Jurczenko et al. (2004)'s extensions of BSM, including parameters of higher moments, seems to have a large effect on delta. When skewness is negative, delta near ATM and up to 110% moneyness on a call option with volatility from 15% - 25% deviate notably from BSM delta. In this area delta ratio is over one, indicating one requires less contracts to hedge a position in stocks with options compared to regular BSM delta. For OTM options over 110% moneyness, delta ratio is below one and you need more contracts to hedge the position. For positive skewness, the delta adjustment will be inverted. We find that kurtosis has a great effect on price and less effect on delta, while skewness has a greater effect on both price and delta in this model. For further research, we suggest a study on delta hedging on historical price data, or Monte Carlo simulation of GBM, with different option models adjusted for skewness and kurtosis and different rebalancing intervals to find out which model minimize risk and hedging error.

Charles Corrado has written 2 very interesting papers about option pricing we recommend reading for further study, "The hidden martingale restriction in Gram – Charlier option prices" from 2007 and "Option pricing based on the generalized lambda distribution" from 2010. Both papers will increase the knowledge and will give us a broader base for further research and investigation about option pricing theory. The paper from 2001 introduced the Generalized Lambda Distribution (GLD) as a flexible tool for modelling non-lognormal security price distributions. Major advantages of the GLD include the flexibility to assign almost any combinations of skewness and kurtosis values. The paper from 2007 describes a martingale restriction "hidden" from view in the option price. This type of restriction in Gram – Charlier option prices appears to have gone unnoticed and is worth further investigation.

Skewness and kurtosis adjusted option pricing models based on Jarrow and Rudd (1982) use extensions which in certain areas give negative probabilities and also negative prices. The problem with negative prices is mostly evident for OTM options when BSM value is low, but also very sensitive to changes in volatility and short time to maturity. There are several other studies confirming this, and thus have found that these models can only be used for moderate deviations from normality (Jondeau & Rockinger, 2001).

We find that moments calculated from a risk-neutral density from short maturity options to be highly volatile and very sensitive to outliers, with higher implied kurtosis and more negative implied skewness compared to medium maturity. This is consistent with similar studies on other market indices, for instance Gurdip Bakshi et al. (2003). Moments seem to be related to movements in asset returns, this is also confirmed in regression of moments on log-returns, and we find a strong negative correlation between skewness and kurtosis. This looks to be a likely consistency with Conrad et al. (2013), Chang et al. (2013) and Turan G Bali and Murray (2013) who found a strong relationship between moments, and a negative relationship between implied skewness and return.

Though volatility and skewness might have a greater effect on security's price, both historical and implied kurtosis is significantly more volatile than second and third moment, which is why we believe kurtosis is so important for option traders and other stakeholders. The strong negative correlation between implied skewness and kurtosis is likely to be very important when adjusting for moments in derivative valuation. Distribution of kurtosis implied from medium and short maturity option prices are positively skewed, implying a higher probability of fat tails and outliers. Distribution of short maturity implied kurtosis seems to be very sensitive to outliers and show high peak, positive skew and excess kurtosis. Implied kurtosis from medium maturity seems to be closer to a log-normal distribution, with moderate positive skew and excess kurtosis. Implied volatility smile, or smirk, is recognized as a consequence of supply and demand, but likely skewness and kurtosis also, see for instance Gurdip Bakshi et al. (2003) or Garleanu et al. (2009).

Appendix.

Tables 11-15 show the underlying values used to create 3D figures (10 - 14) on impact of kurtosis and volatility with skewness fixed at -0,5 and ttm one month. Notice especially tables 13 - 15. The problem with negative prices is present when BSM price is low, low kurtosis and low to moderate volatility for short maturity option.

Modified Corrado Su Call values											
Parameters: Underlying Future OBXCF1 F=100,X=100,r=0.02,T=1/12,Vol=15 - 25%, Skew= - 0.5 Kurtosis type = Pearson											
Volatility/ Kurtosis	15%	16%	17%	18%	19%	20%	21%	22%	23%	24%	25%
0	1,9346	2,0627	2,1906	2,3184	2,4461	2,5736	2,7011	2,8284	2,9556	3,0827	3,2096
1	1,8629	1,9862	2,1093	2,2324	2,3553	2,4781	2,6009	2,7235	2,8460	2,9683	3,0906
2	1,7911	1,9096	2,0280	2,1463	2,2645	2,3826	2,5006	2,6185	2,7363	2,8540	2,9716
3	1,7194	1,8331	1,9468	2,0603	2,1738	2,2872	2,4004	2,5136	2,6267	2,7397	2,8526
4	1,6476	1,7566	1,8655	1,9743	2,0830	2,1917	2,3002	2,4087	2,5171	2,6254	2,7336
5	1,5759	1,6801	1,7842	1,8883	1,9923	2,0962	2,2000	2,3037	2,4074	2,5110	2,6146
6	1,5042	1,6036	1,7030	1,8023	1,9015	2,0007	2,0998	2,1988	2,2978	2,3967	2,4956
7	1,4324	1,5271	1,6217	1,7162	1,8107	1,9052	1,9996	2,0939	2,1882	2,2824	2,3766
8	1,3607	1,4506	1,5404	1,6302	1,7200	1,8097	1,8993	1,9890	2,0785	2,1681	2,2576
9	1,2889	1,3741	1,4592	1,5442	1,6292	1,7142	1,7991	1,8840	1,9689	2,0538	2,1386
10	1,2172	1,2975	1,3779	1,4582	1,5385	1,6187	1,6989	1,7791	1,8593	1,9394	2,0196
B76	1,7245	1,8394	1,9544	2,0693	2,1842	2,2991	2,4141	2,5290	2,6439	2,7588	2,8737

***Table 11**. Underlying values used to create 3D figure 10 on impact of kurtosis and volatility with skewness fixed at -0,5 and ttm 1 month.*

Modified Corrado Su Call values

Parameters Underlying Future OBXCF1 F=100,X=105,r=0.02,T=1/12,Vol=15 - 25%, Skew= - 0.5 Kurtosis type = Pearson

Volatility/ Kurtosis	15%	16%	17%	18%	19%	20%	21%	22%	23%	24%	25%
1	0,1590	0,2312	0,3100	0,3944	0,4836	0,5768	0,6735	0,7732	0,8755	0,9800	1,0864
2	0,1705	0,2377	0,3111	0,3898	0,4731	0,5603	0,6510	0,7446	0,8408	0,9392	1,0397
3	0,1820	0,2442	0,3122	0,3852	0,4626	0,5438	0,6284	0,7160	0,8061	0,8985	0,9930
4	0,1935	0,2507	0,3132	0,3805	0,4521	0,5273	0,6059	0,6874	0,7714	0,8578	0,9463
5	0,2049	0,2571	0,3143	0,3759	0,4416	0,5108	0,5833	0,6587	0,7368	0,8171	0,8996
6	0,2164	0,2636	0,3153	0,3713	0,4311	0,4944	0,5608	0,6301	0,7021	0,7764	0,8529
7	0,2279	0,2701	0,3164	0,3667	0,4206	0,4779	0,5383	0,6016	0,6674	0,7357	0,8062
8	0,2394	0,2766	0,3175	0,3620	0,4101	0,4614	0,5158	0,5730	0,6328	0,6950	0,7595
9	0,2508	0,2830	0,3185	0,3574	0,3996	0,4449	0,4932	0,5444	0,5981	0,6543	0,7128
10	0,2623	0,2895	0,3196	0,3528	0,3891	0,4284	0,4707	0,5158	0,5635	0,6136	0,6661
11	0,2738	0,2960	0,3207	0,3482	0,3786	0,4120	0,4482	0,4872	0,5288	0,5730	0,6194
B76	0,28817	0,35315	0,42280	0,49668	0,57425	0,65510	0,73894	0,82539	0,91418	1,0051	1,09788

Table 12. *Underlying values used to create 3D figure 11 on impact of kurtosis and volatility with skewness fixed at -0,5 and ttm 1 month.*

Modified Corrado Su Call values											
Parameters Underlying Future OBXCF1				F=100,X=110,r=0.02,T=1/12,Vol=15 - 25%, Skew= - 0.5				Kurtosis type = Pearson			
Volatility/ Kurtosis	15%	16%	17%	18%	19%	20%	21%	22%	23%	24%	25%
0	-0,0949	-0,1108	-0,1218	-0,1267	-0,1247	-0,1156	-0,0993	-0,0760	-0,0458	-0,0091	0,0336
1	-0,0667	-0,0768	-0,0826	-0,0831	-0,0779	-0,0664	-0,0488	-0,0249	0,0048	0,0403	0,0812
2	-0,0386	-0,0428	-0,0434	-0,0396	-0,0310	-0,0172	0,0018	0,0261	0,0554	0,0898	0,1288
3	-0,0105	-0,0088	-0,0042	0,0039	0,0159	0,0320	0,0524	0,0771	0,1061	0,1392	0,1764
4	0,0177	0,0252	0,0350	0,0474	0,0627	0,0812	0,1029	0,1281	0,1567	0,1887	0,2241
5	0,0458	0,0592	0,0742	0,0909	0,1096	0,1304	0,1535	0,1791	0,2073	0,2381	0,2717
6	0,0740	0,0932	0,1134	0,1345	0,1565	0,1796	0,2041	0,2301	0,2579	0,2876	0,3193
7	0,1021	0,1273	0,1526	0,1780	0,2033	0,2288	0,2547	0,2811	0,3085	0,3371	0,3669
8	0,1302	0,1613	0,1918	0,2215	0,2502	0,2780	0,3052	0,3322	0,3591	0,3865	0,4145
9	0,1584	0,1953	0,2310	0,2650	0,2971	0,3272	0,3558	0,3832	0,4098	0,4360	0,4622
10	0,1865	0,2293	0,2702	0,3085	0,3439	0,3764	0,4064	0,4342	0,4604	0,4854	0,5098
B76	0,0221	0,0346	0,0509	0,0712	0,0957	0,1245	0,1574	0,1945	0,2355	0,2805	0,3291

Table 13 *.Underlying values used to create 3D figure 12 on impact of kurtosis and volatility with skewness fixed at -0,5 and ttm 1 month.*

Modified Corrado Su Call values											
Parameters Underlying Future OBXCF1			F=100,X=115,r=0.02,T=1/12,Vol=15 - 25%, Skew= -0.5 Kurtosis type = Pearson								
Volatility/ Kurtosis	15%	16%	17%	18%	19%	20%	21%	22%	23%	24%	25%
0	-0,0159	-0,0273	-0,0423	-0,0603	-0,0805	-0,1018	-0,1231	-0,1432	-0,1613	-0,1764	-0,1879
1	-0,0114	-0,0195	-0,0303	-0,0431	-0,0574	-0,0723	-0,0870	-0,1006	-0,1122	-0,1213	-0,1272
2	-0,0068	-0,0118	-0,0183	-0,0259	-0,0343	-0,0429	-0,0510	-0,0579	-0,0631	-0,0661	-0,0666
3	-0,0023	-0,0041	-0,0063	-0,0088	-0,0113	-0,0134	-0,0149	-0,0152	-0,0140	-0,0110	-0,0059
4	0,0022	0,0037	0,0058	0,0084	0,0118	0,0160	0,0212	0,0274	0,0350	0,0441	0,0548
5	0,0067	0,0114	0,0178	0,0256	0,0349	0,0454	0,0572	0,0701	0,0841	0,0992	0,1155
6	0,0112	0,0192	0,0298	0,0428	0,0579	0,0749	0,0933	0,1128	0,1332	0,1543	0,1762
7	0,0157	0,0269	0,0418	0,0600	0,0810	0,1043	0,1293	0,1554	0,1822	0,2095	0,2369
8	0,0202	0,0347	0,0538	0,0771	0,1041	0,1338	0,1654	0,1981	0,2313	0,2646	0,2976
9	0,0247	0,0424	0,0658	0,0943	0,1272	0,1632	0,2014	0,2407	0,2804	0,3197	0,3583
10	0,0292	0,0501	0,0778	0,1115	0,1502	0,1927	0,2375	0,2834	0,3294	0,3748	0,4190
B76	0,0008	0,0017	0,0030	0,0060	0,0101	0,0160	0,0230	0,0330	0,0450	0,0600	0,0776

***Table 14**. Underlying values used to create 3D figure 13 on impact of kurtosis and volatility with skewness fixed at -0,5 and ttm 1 month.*

Modified Corrado Su Call values											
Parameters Underlying Future OBXCF1	F=100,X=120,r=0.02,T=1/12,Vol=15 - 25%, Skew= - 0.5							Kurtosis type = Pearson			
Volatility/ Kurtosis	15%	16%	17%	18%	19%	20%	21%	22%	23%	24%	25%
0	-0,0008	-0,0020	-0,0046	-0,0090	-0,0158	-0,0253	-0,0377	-0,0530	-0,0708	-0,0907	-0,1121
1	-0,0005	-0,0014	-0,0033	-0,0064	-0,0113	-0,0181	-0,0270	-0,0379	-0,0506	-0,0648	-0,0799
2	-0,0003	-0,0009	-0,0020	-0,0038	-0,0068	-0,0109	-0,0163	-0,0228	-0,0304	-0,0388	-0,0476
3	-0,0001	-0,0003	-0,0006	-0,0013	-0,0023	-0,0037	-0,0055	-0,0077	-0,0102	-0,0128	-0,0154
4	0,0001	0,0003	0,0007	0,0013	0,0022	0,0035	0,0052	0,0073	0,0100	0,0131	0,0169
5	0,0003	0,0009	0,0020	0,0039	0,0067	0,0107	0,0159	0,0224	0,0302	0,0391	0,0491
6	0,0006	0,0015	0,0033	0,0064	0,0112	0,0179	0,0267	0,0375	0,0503	0,0650	0,0814
7	0,0008	0,0021	0,0046	0,0090	0,0157	0,0251	0,0374	0,0526	0,0705	0,0910	0,1136
8	0,0010	0,0027	0,0059	0,0116	0,0202	0,0323	0,0481	0,0677	0,0907	0,1170	0,1459
9	0,0012	0,0032	0,0073	0,0141	0,0247	0,0395	0,0588	0,0827	0,1109	0,1429	0,1781
10	0,0014	0,0038	0,0086	0,0167	0,0292	0,0467	0,0696	0,0978	0,1311	0,1689	0,2104
B76	0,0000	0,0000	0,0001	0,0003	0,0007	0,0014	0,0025	0,0042	0,0066	0,0100	0,0146

***Table 15**. Underlying values used to create 3D figure 14 on impact of kurtosis and volatility with skewness fixed at -0,5 and ttm 1 month.*

					Modified Corrado Su Call Delta ratio values								
Parameters	F=100,X=100,r=0.02,T=1/12,Vol=20, Kurtosis = Pearson =3:13, Skewness = -2:1												
Skew/Kurtosis	-2	-1,75	-1,5	-1,25	-1	-0,75	-0,5	-0,25	0	0,25	0,5	0,75	1
3	1,24303	1,21267	1,18231	1,15194	1,12156	1,09118	1,06080	1,03040	1,00000	0,96959	0,93918	0,90876	0,87833
4	1,24291	1,21255	1,18219	1,15182	1,12145	1,09106	1,06068	1,03028	0,99988	0,96947	0,93906	0,90864	0,87821
5	1,24279	1,21243	1,18207	1,15170	1,12133	1,09094	1,06056	1,03016	0,99976	0,96935	0,93894	0,90852	0,87809
6	1,24267	1,21232	1,18195	1,15158	1,12121	1,09083	1,06044	1,03004	0,99964	0,96923	0,93882	0,90840	0,87797
7	1,24255	1,21220	1,18183	1,15146	1,12109	1,09071	1,06032	1,02992	0,99952	0,96911	0,93870	0,90828	0,87785
8	1,24244	1,21208	1,18172	1,15135	1,12097	1,09059	1,06020	1,02980	0,99940	0,96899	0,93858	0,90815	0,87773
9	1,24232	1,21196	1,18160	1,15123	1,12085	1,09047	1,06008	1,02968	0,99928	0,96887	0,93845	0,90803	0,87761
10	1,24220	1,21184	1,18148	1,15111	1,12073	1,09035	1,05996	1,02956	0,99916	0,96875	0,93833	0,90791	0,87748
11	1,24208	1,21172	1,18136	1,15099	1,12061	1,09023	1,05984	1,02944	0,99904	0,96863	0,93821	0,90779	0,87736
12	1,24196	1,21161	1,18124	1,15087	1,12049	1,09011	1,05972	1,02932	0,99892	0,96851	0,93809	0,90767	0,87724
13	1,24185	1,21149	1,18112	1,15075	1,12037	1,08999	1,05960	1,02920	0,99880	0,96839	0,93797	0,90755	0,87712

Table 16. *Underlying values to create figure 22 showing impact of skewness and kurtosis on deltaratio for ATM call.*

Y=LogReturn	Coefficients	Standard error	t-Stat	P-value	R^2	F
Intercept	-0,0054	0,0051	-1,0653*	0,2872	0,1071	22,7124
VOL 1.pos	-0,0608	0,0153	-3,9822*	0,0001		
SKEW 1.pos	-0,0259	0,0044	-5,8544*	0,0000		
KURT 1.pos	-0,0031	0,0010	-3,1879*	0,0015		
Y=LogReturn t+1					*R^2*	*F*
Intercept	0,0111	0,0054	2,0667*	0,0392	0,0093	1,7843
VOL 1.pos	-0,0138	0,0161	-0,8566	0,3920		
SKEW 1.pos	0,0093	0,0046	1,9951*	0,0465		
KURT 1.pos	0,0011	0,0010	1,0853	0,2783		
Y=LogReturn t+2					*R^2*	*F*
Intercept	-0,0024	0,0054	-0,4376	0,6618	0,0022	0,4085
VOL 1.pos	0,0029	0,0161	0,1816	0,8559		
SKEW 1.pos	-0,0013	0,0047	-0,2860	0,7750		
KURT 1.pos	0,0004	0,0010	0,3675	0,7134		
Y=LogReturn t+4					*R^2*	*F*
Intercept	0,0018	0,0053	0,3355	0,7374	0,0016	0,3093
VOL 1.pos	0,0059	0,0160	0,3674	0,7135		
SKEW 1.pos	0,0042	0,0046	0,8988	0,3692		
KURT 1.pos	0,0009	0,0010	0,9033	0,3668		

Table 17. *Regression results from 1. Position implied moments. Model specified on page 56. Significant t-statistics, at 95% level, marked with *.*

Y=LogReturn	*Coefficients*	*Standard error*	*t-Stat*	*P-value*	*R^2*	*F*
Intercept	-0,0314	0,0050	-6,3050*	0,0000	0,2129	51,3154
VOL 2. pos	-0,0430	0,0149	-2,8890*	0,0040		
SKEW 2.pos	-0,0526	0,0049	-10,7417*	0,0000		
KURT 2.pos	-0,0090	0,0014	-6,2787*	0,0000		
Y=LogReturn t+1					*R^2*	*F*
Intercept	0,0084	0,0056	1,4963	0,1351	0,0044	0,8360
VOL 2. pos	-0,0129	0,0168	-0,7693	0,4421		
SKEW 2.pos	0,0048	0,0055	0,8688	0,3853		
KURT 2.pos	0,0002	0,0016	0,0942	0,9250		
Y=LogReturn t+2					*R^2*	*F*
Intercept	-0,0024	0,0056	-0,4303	0,6671	0,0015	0,2912
VOL 2. pos	0,0102	0,0168	0,6088	0,5429		
SKEW 2.pos	-0,0005	0,0055	-0,0853	0,9320		
KURT 2.pos	0,0006	0,0016	0,3690	0,7123		
Y=LogReturn t+4					*R^2*	*F*
Intercept	0,0069	0,0055	1,2526	0,2109	0,0044	0,8351
VOL 2. pos	0,0016	0,0166	0,0943	0,9249		
SKEW 2.pos	0,0076	0,0055	1,3917	0,1646		
KURT 2.pos	0,0012	0,0016	0,7811	0,4350		

Table 18. *Regression results from 2. Position implied moments. Model specified on page 56. Significant t-statistics, at 95% level, marked with *.*

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Norges miljø- og biovitenskapelige universitet
Noregs miljø- og biovitskapelege universitet
Norwegian University of Life Sciences

Postboks 5003
NO-1432 Ås
Norway