ANALYSIS OF THE MCKENDRICK-VON FOERSTER EQUATION WITH WEIGHTED WHITE NOISE BY MEANS OF STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

ARCADY PONOSOV

Abstract. The existence and uniqueness of solutions of the boundary value problem for the recently introduced McKendrick-Von Foerster equation with continuous stochastic noise is proven. Motivated by the applications in mathematical biology, the boundary conditions are assumed to depend on the aggregated age variables, which makes the problem both non-local and nonlinear. The techniques used in the paper are based on a special transformation method converting the stochastic McKendrick-Von Foerster equation into a pair of finite dimensional systems of stochastic functional differential equations.

Keywords: Stochastic PDE, age-structured populations, stochastic functional differential equations.

AMS Subject Classification: 34K50, 60H30, 92Bxx.

1. Introduction.

The following stochastic partial differential equation is considered in the paper:

$$\frac{\partial u(t,a)}{\partial t}+\frac{\partial u(t,a)}{\partial a}=-[m(t,a,J(t),A(t))-\dot{\nu}(t,J(t),A(t))]u(t,a)\quad (t,a\geq 0) \tag{1}$$

subject to the non-local and nonlinear boundary conditions

$$u(0,a)=\chi(a),\qquad u(t,0)=\int_0^\infty \beta(t,a,J(t),A(t))u(t,a)da\quad (t\geq 0). \tag{2}$$

Here $\dot{\nu}(t)$ represents the stochastic noise, while

$$J(t)=\int_0^\tau u(t,a)da\quad\text{and}\quad A(t)=\int_\tau^\infty u(t,a)da \tag{3}$$

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Arcady Ponosov, Department of Sciences and Technology, Norwegian University of Life Sciences, P. O. Box 5003, NO-1432 Ås, Norway, E-mail: arkadi@nmbu.no.

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are the so-called 'aggregated variables' and $\tau \geq 0$ is some constant.

Remark 1. *Convergence of the improper integrals in (2) and (3) is not a priori assumed. It is a part of the solvability of problem (1)-(2), and it will be justified in Section 3.*

The exact conditions to be set on m, ν, χ and β will be specified later in Assumption sets **(A)** (the simplified boundary value problem) and **(B)** (the full boundary value problem), see Sections 2 and 3, respectively. The main results of the paper are Theorem 1 in Section 2 and Theorem 4 in Section 3, which justify the existence and uniqueness property for the boundary value problem (1)-(2).

If no stochastic noise is present, then Eq. (1) becomes the well-known deterministic McKendrick-Von Foerster equation (DMF), which is widely used in applications:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -mu \quad (t, a \geq 0). \tag{4}$$

This explains why Eq. (1) is called below *the stochastic McKendrick-Von Foerster equation (SMF)*.

The motivation to study DMF and SMF stems first of all from the different fields of mathematical biology such as population dynamics [10], [13], epidemiology [8], cell biology [14] as well as from some other areas such as economics [3] and geophysics [20]. In population dynamics, the function $u(t, a)$ stands for the size of a certain population of age $a > 0$ at time $t \geq 0$, $m(t, a) \geq 0$ is the population's per capita extinction rate, while the size $u(t, 0)$ of newborn individuals is calculated as $u(t, 0) = \int_0^\infty \beta(t, a)u(t, a)da$, where $\beta(t, a)$ is the per capita birth rate. The given initial age distribution function $\chi(a)$ is included in the initial condition $u(0, a) = \chi(a) \geq 0$. This population model as such has, unfortunately, similar disadvantages as the Malthusian law of the population growth [8]. To make the model more realistic, one often assumes that the birth and extinction rates depend on the size of the population, e. g. on the aggregated variables (3), where J and A stand for the juvenile and adult population, respectively, and $\tau \geq 0$ is the maturation time [22]. In a simper setting this dependence is restricted to the total size of the population $P(t) = J(t) + A(t) = \int_0^\infty u(t, a)da$ [13], [14], but according to the review paper [22], about half of all publications on the subject of age-structured populations in the period of 2000-2016 were based on models including the maturation time.

The present paper partly complements the studies started in the paper [19] by offering a proof of the fundamental existence and uniqueness theorems for the SMF equation with non-local and nonlinear boundary conditions (2). This property is known to be crucial in the deterministic case for justifying many popular biological models, like the logistic, Nicholson's blowflies [5], marine protection [6], cannibalism [9], [11], [24], compartment-delayed [12] and many other models, which can be derived from the master equation (4). Likewise, this property is of similar importance for the stochastic master equation (1), which is a source of stochastic versions of the above population models [19].

It is well-known from the theory of the DMF equation (4) that proving the existence and uniqueness results for it requires a nontrivial technique because of the complex nature of the boundary conditions and the assumptions on m and β (see e.g. [4], [13] and [14]). The problem becomes only even more complicated in the stochastic case. Therefore, we suggest here a new approach to the existence and uniqueness analysis, which is based on a special transformation of the boundary value problem (1)-(2) into the initial value problem for a connected pair of systems of stochastic functional differential equations (SFDE). The idea of this approach is, thus, to go over to a finite dimensional setting and to prove the existence and uniqueness property for SFDE using more or less standard techniques. Note that this framework helps to prove positivity of the solutions as well.

Unlike the paper [19], we study here only the case of stochastic perturbations of the type 'weighted white noise', which, in particular, means continuity of the resulting stochastic processes. Similar choice, but in a different setting, was made in the publications [1], [2]. More complicated noises have been introduced in the papers [7], [17], [19] and [23], but these cases will be addressed in the forthcoming publications of the author.

The paper is organized as follows. In Section 2 we study the simplified boundary value problem for SMF (1) under Assumption set **(A)**. In addition to this, we derive two systems of SFDE, which in Section 3 will play a crucial role in the study of the full boundary value problem (1)-(2) under Assumption set **(B)**.

The definitions from stochastic calculus to be used in this paper can be found in [21].

All stochastic processes will be defined on the fixed filtered probability space

$$(\Omega,\, \mathcal{F},\, (\mathcal{F}_t)_{t\geq 0},\, \mathbf{P}) \tag{5}$$

with a probability measure $\mathbf{P}$ on a σ-algebra $\mathcal{F}$ of subsets of Ω and an increasing, right-continuous sequence of σ-subalgebras $\mathcal{F}_t$ of $\mathcal{F}$, where all the introduced σ-algebras are complete with respect to the measure $\mathbf{P}$. The expectation on this probability space is denoted by $\mathbf{E}$.

Let $\mathcal{B}$ be the σ-algebra of all Borel subsets of the interval $[0,\infty)$. A stochastic process $v(t) = v(t,\omega)$, $t \in [0,\infty)$, $\omega \in \Omega$ is called $\mathcal{F}_t$-adapted if $v(\cdot,\cdot)$ is $\mathcal{B} \otimes \mathcal{F}$-measurable and the random variable $v(t,\cdot)$ is $\mathcal{F}_t$-measurable for each $t \in [0,\infty)$. A stochastic process $v(t) = v(t,\omega)$ is called continuous (more precisely, path-continuous) if its trajectories $v(\cdot,\omega)$ are continuous functions with probability 1, i.e. almost surely (a.s.). We assume that the above filtered probability space is rich enough to host the Brownian motion (also known as the standard scalar Wiener process) $W(t)$, which is an example of an $\mathcal{F}_t$-adapted and continuous stochastic process. One of its important features is existence of a well-defined stochastic integral (Itô's integral) $\nu(t) = \int_0^t \gamma(s)dW(s)$ with respect to $W(t)$ for $\mathcal{F}_t$-adapted processes $\gamma(t)$ with locally square-integrable on $[0,\infty)$ paths. Note that in this case $\nu(t)$ is an $\mathcal{F}_t$-adapted and continuous stochastic process.

2. The simplified boundary value problem.

In this section, we consider a linear and local version of the boundary value problem (1)-(2). In particular, the boundary condition (2) will be replaced by the following linear conditions

$$u(0,a) = \chi(a), \qquad u(t,0) = b(t) \quad (t \geq 0), \tag{6}$$

where $b(t)$ is known in the literature as the birth function [22]. The entries χ, b, m and ν in the boundary value problem (1), (6) are supposed to satisfy

Assumption set (A):

- *The extinction rate $m(t,a)$ is defined as*

$$m(t,a) = \begin{cases} m_J(t), & 0 \leq a \leq \tau, \\ m_A(t), & a > \tau, \end{cases} \tag{7}$$

  where $m_J(t) \geq 0$ and $m_A(t) \geq 0$ are $\mathcal{F}_t$-adapted continuous stochastic processes and $\tau \geq 0$.
- *the birth function $b(t) = b(t,\omega) \geq 0$ a.s. is an $\mathcal{F}_t$-adapted continuous stochastic process on $[0,\infty)$.*
- *The stochastic process $\nu(t)$, $t \geq 0$ is defined by the formula $\nu(t) = \int_0^t \gamma(s)dW(s)$, where $\gamma(t)$ is a $\mathcal{F}_t$-adapted processes with locally square-integrable on $[0,\infty)$ paths.*
- *The initial age distribution function $\chi(a) \geq 0$ is continuous and satisfies*

$$\int_0^\infty \sup_{s \geq a} \chi(s)da < \infty.$$

Remark 2.

- *The assumptions on m are similar to those used in the deterministic case [22].*

- *The assumption on χ reflects the fact that the total size of the population at time $t = 0$ must be finite (in fact, a little more by technical reasons).*

Definition 1. *A solution $u(t,a) = u(t,a,\omega)$ of the boundary value problem (1), (6) ($t \in [0,\infty)$, $a \in [0,\infty)$, $\omega \in \Omega$) is a $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{F}$-measurable function, such that the stochastic process $u(\cdot,a)$ is $\mathcal{F}_t$-adapted and continuous for all $a \geq 0$ and almost surely satisfies (6) and the integral equation*

$$\begin{aligned} &\textstyle\int_0^a (u(t,s) - \chi(s))ds + \int_0^t (u(\sigma,a) - b(\sigma))d\sigma = \\ &-\textstyle\int_0^t \left(\int_0^a m(\sigma,s)u(\sigma,s)ds\right) d\sigma + \int_0^t \left(\int_0^a u(\sigma,s)ds\right)\gamma(\sigma)dW(\sigma), \end{aligned} \tag{8}$$

for all $t, a \in [0,\infty)$.

Remark 3. *If the function $u(t,a)$ describes the size of a population, then it must be positive, and this will be proven for the case of the boundary value problem (1), (6) provided that the assumptions on the sign of b and m in **(A)** are fulfilled. This also explains why these assumptions are a priori put on these entries. However, the positivity of u is less evident in the nonlinear case and may require additional conditions on the entries, see Theorem 4.*

To prove the existence and uniqueness of solutions of the linear problem (1), (6), we need a lemma for the deterministic version of this problem, where we by notational reasons replace the initial function χ and the birth function b in (6) with some functions v_1 and v_2, respectively.

Lemma 1. *Let $v_1(a)$, $v_2(t)$ and $m(t,a)$ ($t, a \in [0,\infty)$) be locally Lebesgue integrable functions. Then*

$$\bar{u}(t,a) = \begin{cases} v_1(a-t)\exp\{-\int_0^t m(s, a-t+s)ds\}, & t \leq a, \\ v_2(t-a)\exp\{-\int_0^a m(t-a+s, s)ds\}, & t > a, \end{cases} \tag{9}$$

is the unique solution of the following integral version of DMF (4)

$$\int_0^a (\bar{u}(t,s) - \bar{u}(0,s))ds + \int_0^t (\bar{u}(\sigma,a) - \bar{u}(\sigma,0))d\sigma = -\int_0^t \left(\int_0^a m(\sigma,s)\bar{u}(\sigma,s)ds\right) d\sigma, \tag{10}$$

satisfying the boundary conditions $\bar{u}(0,a) = v_1(a)$ and $\bar{u}(t,0) = v_2(t)$ for almost all $a \geq 0$ and $t \geq 0$, respectively.

Proof. The proof of the fact that (9) satisfies (10) and the corresponding boundary conditions can be found in [19, Lemma 1]. To prove uniqueness, let us assume that there are two solutions $\bar{u}_1$ and $\bar{u}_2$ of (10) satisfying the same boundary conditions. Then the function $w(t,a) = \bar{u}_1(t,a) - \bar{u}_2(t,a)$ obeys $w(0,a) = w(t,0) = 0$ $(t, a \geq 0)$ and

$$\int_0^a w(t,s)ds + \int_0^t w(\sigma,a)d\sigma = -\int_0^t \left(\int_0^a m(\sigma,s)w(\sigma,s)ds\right) d\sigma \quad (t,a \geq 0). \tag{11}$$

Let us pick an arbitrary C^2-function $\psi(t,a)$, $(t, a \in [0,\infty))$, supported by a compact subset of $\{t, a \in (0,\infty)\}$ and satisfying $0 \leq \psi(t,a) \leq 1$, and solve the equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial a} = m(t,a)w(t,a) + \psi(t,a) \ \ (t,a > 0). \tag{12}$$

If $0 \leq t \leq a$, then the substitution $\xi = a - t$ reduces (12) to a linear nonhomogeneous equation with the zero initial condition, so that

$$\begin{gathered} \phi(t,a) = \int_0^t \exp\{\int_\xi^t m(s, s+a-t)ds\}\psi(\xi, a-t+\xi)d\xi \\ = \int_{a-t}^a \exp\{\int_{\xi+t-a}^t m(s,s+a-t)ds\}\psi(\xi+t-a,\xi)d\xi \quad (t \leq a). \end{gathered}$$

Similarly, the substitution $\xi = a - t$ $(t > a)$ yields

$$\phi(t,a) = \int_{t-a}^t \exp\{\int_\xi^{\xi+a-t} m(s,s+a-t)ds\}\psi(\xi,\xi+a-t)d\xi \quad (t > a).$$

As ψ has a compact support, there is $R > 0$ for which $\psi(t,a) = 0$ for $t > 0, a > R$ and $a > 0, t > R$. Therefore,

$$\phi(t,a) = 0 \ \ (t > 0, a > t+R) \quad \text{and} \quad \phi(t,a) = 0 \ \ (a > 0, t > a+R). \tag{13}$$

Then integration by parts results in

$$\int_0^\infty da \left(\frac{\partial^2 \phi}{\partial a \partial t}(t,a)\int_0^a w(t,s)ds\right) = -\int_0^\infty w(t,a)\frac{\partial \phi}{\partial t}(t,a)da,$$

$$\int_0^\infty dt \left(\frac{\partial^2 \phi}{\partial t \partial a}(t,a)\int_0^t w(\sigma,a)d\sigma\right) = -\int_0^\infty w(t,a)\frac{\partial \phi}{\partial a}(t,a)dt$$

and

$$\begin{gathered} \int_0^\infty da \int_0^\infty dt \left(\frac{\partial^2 \phi}{\partial a \partial t}(t,a)\int_0^t\int_0^a m(\sigma,s)w(\sigma,s)d\sigma ds\right) \\ = \int_0^\infty\int_0^\infty w(t,a)m(t,a)\phi(t,a)dtda \end{gathered}$$

due to (13). Integrating the first two equalities from 0 to ∞ with respect to t and a, respectively, and adding the results yield

$$\begin{gathered} \int_0^\infty dt \int_0^\infty da \left(\frac{\partial^2 \phi}{\partial t \partial a}(t,a)\left(\int_0^a w(t,s)ds + \int_0^t w(\sigma,a)d\sigma\right)\right) = \\ -\int_0^\infty dt \int_0^\infty da \left(w(t,a)\left(\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial a}\right)\right) = -\int_0^\infty\int_0^\infty w(t,a)\left(\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial a}\right)dtda. \end{gathered}$$

Therefore,

$$0=\int_0^\infty dt\int_0^\infty da\left(\frac{\partial^2\phi}{\partial t\partial a}(t,a)\left(\int_0^a w(t,s)ds+\int_0^t w(\sigma,a)d\sigma+\int_0^t\int_0^a m(\sigma,s)w(\sigma,s)d\sigma ds\right)\right)$$
$$=\int_0^\infty\int_0^\infty w(t,a)\left(-\frac{\partial\phi}{\partial t}-\frac{\partial\phi}{\partial a}+m(t,a)\phi(t,a)\right)dtda=-\int_0^\infty\int_0^\infty w(t,a)\psi(t,a)dtda.$$

Thus, it is proven that

$$\int_0^\infty\int_0^\infty w(t,a)\psi(t,a)dtda=0 \tag{14}$$

for any C^2-function ψ having a compact support for $t,a>0$. To check that this necessarily implies $w(t,a)=0$ almost everywhere, let us assume on the contrary that there exists a Lebesgue measurable set G, $\text{mes}\,(G)=q>0$ (mes is the Lebesgue measure in the plane) such that $w(t,a)\ge h>0$ (the case $w(t,a)\le -h<0$ is similar). Without loss of generality, G can be assumed to be a subset of an open ball B. Pick $0<\varepsilon<\frac{qh}{4}$. Due to integrability of w on B, there exists $0<\delta<\frac{q}{4}$ such that $|\int\int_E w(t,a)dtda|<\varepsilon$ whenever $\text{mes}\,(E)<\delta$. Further, there exist finitely many open balls $B_i(r_i)\subset B$ in the plane, the union $\mathcal{O}$ of which satisfies the property $\text{mes}\,(\mathcal{O}\Delta G)<\delta$. Choose $0<\rho_i<r_i$ for which $\text{mes}\,(\mathcal{O}-\mathcal{C})<\delta$, where $\mathcal{C}$ is the union of the closed balls $\bar{B}_i(\rho_i)$, and construct a C^2-function ψ equalling 1 on the set $\mathcal{C}$ and 0 outside the set $\mathcal{O}$.

Then

$$\int_0^\infty\int_0^\infty w(t,a)\psi(t,a)dtda=\int\int_{\mathcal{O}}\psi(t,a)w(t,a)dtda$$
$$=\int\int_{\mathcal{C}-G}\psi(t,a)w(t,a)dtda+\int\int_{\mathcal{O}-\mathcal{C}}\psi(t,a)w(t,a)dtda+\int\int_{\mathcal{C}\bigcap G}\psi(t,a)w(t,a)dtda$$
$$\ge-\varepsilon-\varepsilon+h,\text{mes}\,(\mathcal{C}\cap G)>-2\varepsilon+h(q-2\delta)>\frac{hq}{4}>0,$$

as $\mathcal{C}-G\subset\mathcal{O}-G$ and hence $\text{mes}\,(\mathcal{C}-G)<\delta$; $\text{mes}\,(\mathcal{O}-\mathcal{C})<\delta$, $\psi(t,a)=1$ $((t,a)\in\mathcal{C})$, $w(t,a)\ge h$ $((t,a)\in G)$ and

$$\text{mes}\,(\mathcal{C}\cap G)\ge\text{mes}\,(G)-\text{mes}\,(G-\mathcal{O})-\text{mes}\,(\mathcal{O}-\mathcal{C})>q-2\delta.$$

This contradicts property (14) and completes, therefore, the proof of the uniqueness of solutions of Eq. (8). □

The first theorem of the paper justifies the existence and uniqueness of the solutions of the linear boundary value problem (1), (6), where we use the so-called 'stochastic exponential'

$$X(t)=\mathcal{E}\{\nu(t)\}:=\exp\{\int_0^t\gamma(s)dW(s)-\frac{1}{2}\int_0^t\gamma^2(s)ds\}, \tag{15}$$

which satisfies the equation $X(t)=1+\int_0^t\gamma(s)X(s)dW(s)$, $t\ge 0$.

Theorem 1. *If Assumptions **(A)** are fulfilled, then the boundary value problem (1), (6) has a unique nonnegative solution $u(t,a,\omega)$ $(t,a\geq 0,\omega\in\Omega)$ in the sense of Definition 1, and if $\chi(a)>0$ $(a\geq 0)$ and $b(t,\omega)>0$ $(t\geq 0)$ a.s., then also $u(t,a,\omega)>0$ $(t\geq 0)$ a.s. This solution is given explicitly by the formula*

$$u(t,a)=\begin{cases}\chi(a-t)\exp\{-\int_0^t m(s,a-t+s)ds\}\mathcal{E}\{\nu(t)\}, & t\leq a,\\ b(t-a)\exp\{-\int_0^a m(t-a+s,s)ds\}\mathcal{E}\{\nu(t)\}\mathcal{E}^{-1}\{\nu(t-a)\}, & t>a.\end{cases}\tag{16}$$

Proof. The existence part of this result was proven in [19, Th. 1]. In addition, it was proven there that the stochastic process $u(t,a)$ $(t,a\geq 0)$ satisfies (8) a.s. if and only if $\bar{u}(t,a)=u(t,a)\mathcal{E}^{-1}\{\nu(t)\}$ satisfies (10) a.s. Now we can apply Lemma 1 for almost all $\omega\in\Omega$, according to which the a.s. unique solution $\bar{u}(t,a)$ of (10) is given by (9), where $v_1(a)=\chi(a)$ and $v_2(t)=b(t)$. This gives the representation formula (16) and at the same time ensures uniqueness of solutions.

Finally, the explicit formulas (15) and (16) guarantee that this u is always nonnegative and even a.s. positive if χ is positive and b is a.s. positive. □

In the next theorem as well as in the next section we will use the differential-based notation for stochastic differential equations [21].

Theorem 2. *Let Assumptions **(A)** be fulfilled and $u(t,a)$ $(t,a\geq 0)$ be a solution of the boundary value problem (1), (2). Then for all $t\geq 0$ the stochastic processes $J(t)$ $(t\geq 0)$ and $A(t)$ introduced in (3) are well-defined, $\mathcal{F}_t$-adapted, continuous and satisfy on the interval $0\leq t\leq\tau$ the following system of SFDE:*

$$\begin{aligned}dJ(t)&=-\mathcal{D}_0\{t,X(\cdot)\}\}dt+b(t)dt-m_J(t)J(t)dt+\gamma(t)dW(t)\\ dA(t)&=\mathcal{D}_0\{t,X(\cdot)\}dt-m_A(t)A(t)dt+\gamma(t)dW(t)\\ dX(t)&=\gamma(t)X(t)dW(t),\end{aligned}\tag{17}$$

where

$$\mathcal{D}_0\{t,X(\cdot)\}=\chi(\tau-t)\exp\{-\int_0^t m_J(s)ds\}X(t),\tag{18}$$

and the initial conditions

$$J(0)=J_0,\quad A(0)=A_0,\quad X(0)=1,\tag{19}$$

where J_0,A_0 are real numbers defined as

$$J_0=\int_0^\tau\chi(s)ds,\quad A_0=\int_\tau^\infty\chi(s)ds.\tag{20}$$

On the interval (τ,∞) the stochastic processes (3) satisfy the system

$$dJ(t)=\beta_A(t)A(t)dt-m_J(t)J(t)dt-\mathcal{D}\{t,X(\cdot)\}b(t-\tau)dt+\gamma(t)J(t)dW(t), \tag{21}$$

$$dA(t) = -m_A(t)A(t)dt + D\{t,X(\cdot)\}b(t-\tau)dt + \gamma(t)A(t)dW(t), \tag{22}$$

$$dX(t) = \gamma(t)X(t)dW(t), \tag{23}$$

where

$$\mathcal{D}\{t,X(\cdot)\} = \exp\{-\int_{t-\tau}^{t} m_J(s)ds\}X(t)X^{-1}(t-\tau). \tag{24}$$

Proof. First of all, let us notice that the initial conditions (20) are satisfied due to the representation (3) of the variables J and A.

According to (16), the function $u(\cdot,a)$ is $\mathcal{F}_t$–adapted for all a. It is therefore straightforward to check that also $J(t)$ and $A(t)$ will be $\mathcal{F}_t$-adapted stochastic processes, provided that the integral for $A(t)$ converges in the proper sense (for this property, see (29)).

To derive equations for the variable $J(t)$, let us put $a=\tau$ in (8). Then

$$\begin{gathered}\int_0^\tau (u(t,s)-\chi(s))ds + \int_0^t (u(\sigma,\tau)-b(\sigma))d\sigma \\ = -\int_0^t \left(\int_0^\tau m(\sigma,s)u(\sigma,s)ds\right)d\sigma + \int_0^t \left(\int_0^\tau u(\sigma,s)ds\right)d\nu(\sigma).\end{gathered} \tag{25}$$

Then, using (3), we obtain

$$J(t)-J(0)+\int_0^t (u(\sigma,\tau)-b(\sigma))\,d\sigma = -\int_0^t m_J(\sigma)J(\sigma)d\sigma + \int_0^t J(\sigma)d\nu(\sigma). \tag{26}$$

because $0\le s\le\tau$, so that $m(\sigma,s)=m_J(\sigma)$.

Assume first that $0\le t\le\tau$. In this case, the term $u(t,\tau)$ must be calculated according to the first formula in the solution (16), i.e.

$$\begin{gathered}u(t,\tau) = \chi(\tau-t)\exp\{-\int_0^t m(s,\tau-t+s)ds\}\mathcal{E}\{\nu(t)\} = \\ \chi(\tau-t)\exp\{-\int_0^t m_J(s)ds\}\mathcal{E}\{\nu(t)\}\end{gathered} \tag{27}$$

due to the definition (7) of the mortality rate and the observation that $\tau-t+s\le\tau$ if $0\le s\le t$. Inserting (27) in (26) gives the first equation in (17).

To calculate $u(\sigma,\tau)$ in the case $t>\tau$ let us use the assumption $a\le\tau$, the second formula in (16) and again the formula (7), which yields

$$u(t,\tau) = b(t-\tau)\exp\{-\int_{t-\tau}^{t} m_J(s)ds\}\mathcal{E}\{\nu(t)\}\mathcal{E}^{-1}\{\nu(t-\tau)\}, \tag{28}$$

as $\int_0^\tau m(t-\tau+s,s)ds = \int_0^\tau m_J(t-\tau+s)ds = \int_{t-\tau}^t m_J(s)ds$ if $0 \le s \le \tau$. Plugging (28) in (26) justifies (21).

To derive the equation for the variable $A(t)$, we need to be sure about the convergence of the following improper integrals:

$$\lim_{a_1,a_2\to\infty} \mathbf{E}\left(\int_{a_1}^{a_2} u(t,a)da\right)^2 = 0, \tag{29}$$

$$\lim_{a\to\infty} \mathbf{E}\left(\int_0^t u(\sigma,a)d\sigma\right)^2 = 0, \tag{30}$$

$$\lim_{a\to\infty} \mathbf{E}\left(\int_0^t m_A(\sigma)d\sigma \int_a^\infty u(\sigma,s)ds\right)^2 = 0, \tag{31}$$

and

$$\lim_{a\to\infty} \mathbf{E}\left(\int_0^t d\nu(\sigma) \int_a^\infty u(\sigma,s)ds\right)^2 = 0. \tag{32}$$

The property (29) ensures, in particular, that the integral (3) converges in the mean-square sense, so that $A(t)$ in (3) is a well-defined $\mathcal{F}_t$-adapted stochastic process.

Properties (29)-(32) are checked in [19, p. 6-7]. By this reason, the proof, which utilizes some standard techniques for estimating stochastic integrals, is omitted here.

To derive the equation for the variable $A(t)$, let us now subtract (25) from (8) giving

$$\begin{aligned} &\textstyle\int_\tau^a (u(t,s)-\chi(s))ds + \int_0^t (u(\sigma,a)-u(\sigma,\tau))d\sigma \\ &= -\textstyle\int_0^t \left(\int_\tau^a m_A(\sigma)u(\sigma,s)ds\right)d\sigma + \int_0^t \left(\int_\tau^a u(\sigma,s)ds\right)d\nu(\sigma) \end{aligned}$$

Letting $a\to\infty$ and using (29)-(32) we get

$$A(t) - A(0) - \int_0^t u(\sigma,\tau)d\sigma = -\int_0^t m_A(\sigma)A(\sigma)d\sigma + \int_0^t A(\sigma)d\nu(\sigma). \tag{33}$$

If $0 \le t \le \tau$, then combining (33) with the formula (27) for the function $u(\sigma,\tau)$ yields the second equation in (17).

If $t > \tau$, then according to (7) and (16),

$$u(t,\tau) = b(t-\tau)\mathcal{E}\{\nu(t)\}\mathcal{E}^{-1}\{\nu(t-\tau)\}\exp\{-\int_0^\tau m_J(t-\tau+s)ds\} \quad (t>\tau).$$

Then from (33) and Assumptions **(A)** we derive (22).

The equations for the auxiliary variable $X(t) = \mathcal{E}\{\nu(t)\}$ follow immediately from the formula (15) for the stochastic exponential. □

Remark 4. *The system of functional differential equations (21)-(23) should be supplied by the prehistory conditions on the interval* $[0,\tau]$*:*

$$J(t)=\varphi_J(t),\quad A(t)=\varphi_A(t),\quad X(t)=\varphi_X(t)\quad (t\in[0,\tau]). \tag{34}$$

Based on the property of path continuity of the stochastic processes J*,* A *and* X*, we will in the main theorem of the next section define the right-hand sides in these conditions to be the solutions of the initial value problem (17)-(19). This explains, in particular, why the functions* $\varphi_J(t)$*,* $\varphi_A(t)$ *and* $\varphi_X(t)$ *must be random.*

3. The full boundary value problem.

In this section, we consider the boundary value problem (1), (2), where the entries m, β and γ depend on the aggregated variables (3), which makes this problem both nonlocal and nonlinear. The set of assumptions put on the coefficients is described below (Assumption set **(B)**).

We start with the following auxiliary definition [19]:

Definition 2. *We say that a real-valued (deterministic) function* $\alpha(t,x,y)$*,* $t\geq 0, x,y\in(-\infty,\infty)$ *belongs class* L *if it is Borel measurable (as a function of three variables), continuous with respect to* t *for any* x *and* y*, satisfies the uniform Lipschitz condition with respect to* x *and* y*:*

$$|\alpha(t,x_1,y_1)-\alpha(t,x_2,y_2)|\leq L(|x_1-y_1|+|x_2-y_2|)$$

for all $t\geq 0, x_1,x_2,y_1,y_2\in(-\infty,\infty)$ *and equals* 0 *outside the set* $t,x,y>0$*.*

The following assumptions replace Assumption set **(A)** in this section.

Assumption set (B):

- *The extinction rate* $m(t,a)\geq 0$ *is defined as*

$$m(t,a)=\begin{cases} m_J(t):=\mu_J(t,J(t),A(t)), & 0\leq a\leq\tau,\\ m_A(t):=\mu_A(t,J(t),A(t)), & a>\tau,\end{cases} \tag{35}$$

  where μ_J *and* μ_A *are class* L *functions.*
- *The birth rate* $\beta(t,a)\geq 0$ *is defined as*

$$\beta(t,a)=\begin{cases} 0, & 0\leq a\leq\tau,\\ \beta_A(t,J(t),A(t)), & a>\tau,\end{cases} \tag{36}$$

  where β_A *is a class* L *function.*

- *The stochastic noise is defined as*

$$\nu(t)=\int_0^t \gamma(s,J(s),A(s))dW(s), \tag{37}$$

where γ is a class L function and W is, as before, the standard scalar Wiener process (the Brownian motion) defined on the filtered probability space (5).

- *The initial age distribution function χ is deterministic, continuous and satisfies $\int_0^\infty \sup\limits_{s\geq a}\chi(s)da<\infty$.*

Remark 5.

- *By default, the birth rate of the juvenile population (i.e. β_J) is equal to 0; due to this assumption and the formula for $A(t)$ in (3), we get the following representation of the second boundary condition in (2):*

$$b(t)=u(t,0)=\int_0^\infty \beta(t,a)u(t,a)da=\beta_A(t,J(t),A(t))A(t). \tag{38}$$

- *The functions u, J and A should, in principle, be non-negative in the population model. However, by technical reasons we sometimes let them be negative, too.*
- *Assumption set **(B)** requires an adjustment of the definition of solutions of the boundary value problem (1)-(2), which is done in Definition 3.*

Definition 3. *A solution of the nonlocal boundary value problem (1), (2) satisfying Assumption set **(B)** is a $\mathcal{B}\otimes\mathcal{B}\otimes\mathcal{F}$-measurable function $u(t,a)=u(t,a,\omega)$ $(t,a\geq 0$, $\omega\in\Omega)$, for which the stochastic process $u(\cdot,a)$ is $\mathcal{F}_t$-adapted and continuous for all $a\geq 0$, for which the second integral in (3) is a.s. finite and which satisfies (2) and the integral equation (8) for $t,a\in[0,\infty)$ a.s.*

Before we proceed, let us compare the simplified and the full boundary value problem (1)-(2). The existence and uniqueness of solutions of the simplified problem was proven in Section 2 under Assumptions **(A)**. Then it was shown that the aggregated age variables (3) are well-defined and satisfy systems (17) and (21)-(23). In the case of the full problem, i.e. if Assumptions **(B)** are fulfilled, we have to proceed in a quite different way. We start with a 'guess' on what stochastic functional differential systems can describe the dynamics of the aggregated age variables, treat these systems separately and then use their properties to prove existence and uniqueness of solutions of the boundary value problem (1)-(2) under Assumptions **(B)**. In particular, this will imply that, indeed, our guess was correct. The necessity of using such a trick also explains why we below temporarily denote the solutions of these systems by $\tilde{J}(t), \tilde{A}(t), \tilde{X}(t)$, respectively. This is done to avoid the notational crash until the equalities $\tilde{J}(t) = J(t), \tilde{A}(t) = A(t), \tilde{X}(t) = X(t)$ are proven in Step 3 of the proof of Theorem 4.

For $0 \leq t \leq \tau$ the system is defined as

$$\begin{aligned} d\tilde{J}(t) = -\mathcal{D}_0\{t, \tilde{J}(\cdot), \tilde{A}(\cdot, \tilde{X}(\cdot))\}dt + \beta_A(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t)dt \\ -\mu_J(t, \tilde{J}(t), \tilde{A}(t))\tilde{J}(t)dt + \gamma(t, \tilde{A}(t), \tilde{J}(t))\tilde{J}(t)dW(t) \\ d\tilde{A}(t) = \mathcal{D}_0\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\}dt - \mu_A(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t)dt \\ +\gamma(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t)dW(t) \\ d\tilde{X}(t) = \gamma(t, \tilde{J}(t), \tilde{A}(t))\tilde{X}(t)dW(t), \end{aligned} \tag{39}$$

where

$$\mathcal{D}_0\{t, \tilde{J}(\cdot, \tilde{A}(\cdot)), \tilde{X}(\cdot)\} = \chi(\tau - t)\exp\{-\int_0^t \mu_J(s, \tilde{J}(s), \tilde{A}(s))ds\}\tilde{X}(t). \tag{40}$$

The initial conditions for the system (39) are, as in the previous section, given by

$$\tilde{J}(0) = J_0, \quad \tilde{A}(0) = A_0, \quad \tilde{X}(0) = 1, \tag{41}$$

where J_0, A_0 are the real numbers defined in (20).

For $t > \tau$ we put

$$\begin{aligned} d\tilde{J}(t) = \beta_A(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t)dt - \mu_J(t, \tilde{J}(t), \tilde{A}(t))\tilde{J}(t)dt \\ -\mathcal{D}\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\}\beta_A(t-\tau, \tilde{J}(t-\tau), \tilde{A}(t-\tau))\tilde{A}(t-\tau)dt \\ +\gamma(t, \tilde{A}(t), \tilde{J}(t))\tilde{J}(t)dW(t) \quad (t > \tau), \end{aligned} \tag{42}$$

$$\begin{aligned} d\tilde{A}(t) = -\mu_A(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t)dt \\ +D\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\}\beta_A(t-\tau, \tilde{J}(t-\tau), \tilde{A}(t-\tau))\tilde{A}(t-\tau)dt \\ +\gamma(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t)dW(t) \quad (t > \tau), \end{aligned} \tag{43}$$

$$d\tilde{X}(t) = \gamma(t, \tilde{J}(t), \tilde{A}(t))\tilde{X}(t)dW(t), \tag{44}$$

where

$$\mathcal{D}\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\} = \exp\{-\int_{t-\tau}^t \mu_J(s, \tilde{J}(s), \tilde{A}(s))ds\}\tilde{X}(t)\tilde{X}^{-1}(t-\tau). \tag{45}$$

is the integral operator standing for the distributed delay in the above system.

The prehistory conditions for the system (42)-(44) are defined as

$$\tilde{J}(t) = \varphi_J(t), \quad \tilde{A}(t) = \varphi_A(t) \quad \tilde{X}(t) = \varphi_X(t) \quad (t \in [0, \tau]), \tag{46}$$

where the right-hand sides are $\mathcal{F}_\tau$-measurable for all $0 \le t \le \tau$ and continuous stochastic processes.

The theorem below ensures the existence and uniqueness property of the formally defined systems of SFDE (39) and (42)-(44).

Theorem 3. *Let μ_J, μ_A, β_A, γ be class L functions. Then for any J_0, A_0 the initial value problem (39), (41) has a unique $\mathcal{F}_t$–adapted continuous solution on the interval $[0,\tau]$, while the system (42)-(44) has a unique $\mathcal{F}_t$–adapted continuous solution on the interval $0 \le t < \infty$ satisfying the prehistory conditions (46), where $\varphi_J(t)$, $\varphi_A(t)$ and $\varphi_X(t)$ $(0 \le t \le \tau)$ are arbitrary $\mathcal{F}_\tau$-measurable and continuous stochastic processes.*

Proof. We split the proof into two steps starting with the system (39) equipped with the initial conditions (41). This is a particular case of a stochastic hereditary equation with special initial conditions at time $t = 0$, i.e. without the prehistory condition for $t < 0$. Such equations are sometimes called the Doléans-Protter equations, see e.g. [15]. Let $\|\cdot\|$ be the sup-norm on the space of all continuous functions defined on $[0,\tau]$. It is straightforward to check that for some $\mathcal{F}_t$-adapted and càdlàg stochastic process $L(t)$, the right-hand side F_0 in (39) satisfies the Lipschitz condition $|(F_0x_1)(t) - (F_0x_2)(t)| \le L(t)\|x_1 - x_2\|$ almost surely for all $t \in [0,\tau]$, where $|\cdot|$ is the usual Euclidean norm. Using now the main result of the paper [16], based on the generalized contraction principle for stochastic functional differential equations, ensures existence and uniqueness of solutions of the initial value problem (39)-(41) defined on the interval $[0,\tau]$. This solution is continuous and $\mathcal{F}_t$-adapted $(0 \le t \le \tau)$.

The existence and uniqueness result for the second initial value problem is justified in step 2 of the proof. Following [18] we show how to obtain the Doléans-Protter equation from the system (42)-(44). By this, the existence and uniqueness property can be again established by referring to [16].

To simplify the notation, let us put $\varphi(t) = (\varphi_J(t), \varphi_A(t), \varphi_X(t))$ $(t \in [0,\tau])$ and $x(t) = (\tilde{J}(t), \tilde{A}(t), \tilde{X}(t))$ $(t \ge 0)$ and accumulate the right-hand sides of (42)-(44) into two Volterra operators F_1 and F_2 yielding

$$dx(t) = (F_1x)(t)dt + (F_2x)(t)dW(t), \quad t > \tau. \tag{47}$$

The prehistory conditions (46) become

$$x(t) = \varphi(t), \quad 0 \le t \le \tau. \tag{48}$$

Define

$$\varphi^-(t) = \begin{cases} \varphi(t), & 0 \le t < \tau, \\ 0, & t \ge \tau, \end{cases}$$

and introduce the nonlinear operators $\hat{F}_1$ and $\hat{F}_2$, both acting on $\mathcal{F}_t$-adapted continuous stochastic processes $x(t)$ $(\tau \le t < \infty)$, by setting $\hat{F}_i(x) = F_i(x^+ + \varphi^-)$ $(i = 1, 2)$, where

$$x^+(t) = \begin{cases} 0, & 0 \le t < \tau, \\ x(t), & t \ge \tau. \end{cases}$$

By construction, the solution $x(t)$ of the Doléan-Protter equation

$$dx(t) = (\hat{F}_1 x)(t)dt + (\hat{F}_2 x)(t)dW(t), \quad t \ge \tau, \tag{49}$$

satisfying the initial condition (at $t = \tau$)

$$x(\tau) = \varphi(\tau), \tag{50}$$

defines the solution of the initial value problem by the formula $x(t) = x^+(t) + \varphi^-(t)$, and vice versa: given the stochastic process $x(t)$ satisfying (47) and (48) on $[0, \infty)$, its restriction $x(t)$ to the interval $[\tau, \infty)$ is a solution of (49) satisfying the initial condition (50). Thus, we can replace the property of existence and uniqueness for (47)-(48) by that of (49)-(50).

Using again the Lipschitz condition $|(\hat{F}x_1)(t) - (\hat{F}x_2)(t)| \le L(t)\|x_1 - x_2\|$ on $\tau \le t < \infty$, where $\|\cdot\|$ is the sup-norm and the main result of the paper [16] yields existence and uniqueness of the solutions of the initial value problem (49)-(50). This completes the proof of the theorem. □

Remark 6. *Assumptions on the stochastic processes φ_J, φ_A and φ_X in the prehistory conditions (46) are, in particular, fulfilled if they coincide with the solutions of the initial value problem (39)-(41), which is defined on the interval $[0, \tau]$. In this case, we may call the triple $(\tilde{J}(t), \tilde{A}(t), \tilde{X}(t)$ $(t \ge 0)$ from Theorem 3* ***the solution of the initial value problem (39)-(46).*** *This terminology will be used in the next theorem.*

We are now ready to prove the main result of this paper.

Theorem 4. *Suppose that the entries m, χ, β, γ, ν satisfy Assumptions* ***(B)****. Then the nonlocal boundary value problem (1), (2) has a unique solution $u(t, a)$ $(0 \le t, a < \infty)$ in the sense of Definition 3.*

Moreover, $u(t,a) > 0$ *a.s.* $(0 \le t, a < \infty)$ *if and only if*

$$\beta_A(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t) > 0 \ a.\ s., \quad (0 \le t < \infty),$$

where the stochastic processes $\tilde{J}(t)$ *and* $\tilde{A}(t)$ *(together with* $\tilde{X}(t)$*) are the solutions of the initial value problem (39)-(46) and* J_0, A_0 *are given by (20).*

Proof. The proof of the theorem is divided in 5 steps. The existence of solutions is proven in Steps 1-4, while their uniqueness is justified in Step 5.

Step 1. Applying Theorem 3 we let the triple $\left(\tilde{J}(t),\, \tilde{A}(t),\, \tilde{X}(t)\right)$ be a unique solution of the initial value problem (39)-(46), where J_0 and A_0 are given by (20).

Let us define the function $u(t,a)$ $(t, a \ge 0)$ by

$$u(t,a) = \begin{cases} \chi(a-t)\exp\{-\int_0^t \tilde{m}(s, a-t+s)ds\}\mathcal{E}\{\tilde{\nu}(t)\}, & t \le a, \\ \tilde{b}(t-a)\exp\{-\int_0^a \tilde{m}(t-a+s, s)ds\}\mathcal{E}\{\tilde{\nu}(t)\}\mathcal{E}^{-1}\{\tilde{\nu}(t-a)\}, & t > a, \end{cases} \tag{51}$$

for χ satisfying the last assumption in **(B)** and $\tilde{m}$, $\tilde{b}$ and $\tilde{\nu}$ given by

$$\tilde{m}(t,a) = \begin{cases} \tilde{m}_J := \mu_J(t, \tilde{J}(t), \tilde{A}(t)), & 0 \le a \le \tau, \\ \tilde{m}_A := \mu_A(t, \tilde{J}(t), \tilde{A}(t)), & a > \tau, \end{cases} \tag{52}$$

$$\tilde{b}(t) = \beta_A(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t), \tag{53}$$

$$\tilde{\nu}(t) = \int_0^t \gamma(s, \tilde{J}(s), \tilde{A}(s))dW(s), \tag{54}$$

where the class L functions μ_J, μ_A, β_A and γ are taken from Assumptions **(B)**. Evidently, these formulas correspond to those in **(B)** and (38) if J and A are replaced by $\tilde{J}$ and $\tilde{A}$, respectively. As the solutions $\tilde{J}(t)$, $\tilde{A}(t)$ are $\mathcal{F}_t$-adapted and continuous, it is straightforward to see that the functions $\tilde{m}$, $\tilde{b}$ and $\tilde{\nu}$ satisfy the Assumptions **(A)**. Thus, we can apply Theorem 1 ensuring the function $u(t,a)$ to satisfy the integral version (8) of the SMF equation (1) as well as the boundary conditions (6). This solution is, in addition, unique in the class of stochastic processes described in Definition 1.

Step 2. Using the function u from Step 1 we define the aggregated variables $J(t)$ and $A(t)$, as it is done in (3). By Theorem 2, they satisfy the following systems of SFDE

$$\begin{gathered} dJ(t) = -\mathcal{D}_0\{t, \tilde{J}(\cdot), \tilde{A}(\cdot, \tilde{X}(\cdot))\}dt + \beta_A(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t)dt \\ -\mu_J(t, \tilde{J}(t), \tilde{A}(t))J(t)dt + \gamma(t, \tilde{A}(t), \tilde{J}(t))J(t)dW(t) \quad (0 \le t \le \tau) \\ dA(t) = \mathcal{D}_0\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\}dt - \mu_A(t, \tilde{J}(t), \tilde{A}(t))A(t)dt \\ +\gamma(t, \tilde{J}(t), \tilde{A}(t))A(t)dW(t) \quad (0 \le t \le \tau), \end{gathered} \tag{55}$$

where the integral operator $\mathcal{D}_0\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\}$ is defined in (40), and

$$\begin{aligned} dJ(t) &= \beta_A(t, \tilde{J}(t), \tilde{A}(t))\tilde{A}(t)dt - \mu_J(t, \tilde{J}(t), \tilde{A}(t))J(t)dt \\ &-\mathcal{D}\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\}\beta_A(t-\tau, \tilde{J}(t-\tau), \tilde{A}(t-\tau))\tilde{A}(t-\tau)dt \\ &+\gamma(t, \tilde{A}(t), \tilde{J}(t))J(t)dW(t) \quad (t > \tau), \end{aligned} \tag{56}$$

$$\begin{aligned} dA(t) &= -\mu_A(t, \tilde{J}(t), \tilde{A}(t))A(t)dt \\ &+\mathcal{D}\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\}\beta_A(t-\tau, \tilde{J}(t-\tau), \tilde{A}(t-\tau))\tilde{A}(t-\tau)dt \\ &+\gamma(t, \tilde{J}(t), \tilde{A}(t))A(t)dW(t) \quad (t > \tau), \end{aligned} \tag{57}$$

where the integral operator $\mathcal{D}$ is defined in (45).

Finally,

$$\begin{aligned} J(0) &= \textstyle\int_0^\tau u(0,s)ds = \int_0^\tau \chi(s)ds = \tilde{J}(0), \\ A(0) &= \textstyle\int_\tau^\infty u(0,s)ds = \int_\tau^\infty \chi(s)ds = \tilde{A}(0). \end{aligned} \tag{58}$$

Step 3. Now, we can prove that $\tilde{J}(t) = J(t)$ and $\tilde{A}(t) = A(t)$, $t \geq 0$. To see this, we observe that due to (58), these processes satisfy the same initial conditions, while (39), (42), (43) and (55)-(57) imply that they satisfy the same system of linear stochastic functional differential equations given by

$$\begin{aligned} dx_1(t) &= \Big(-c_1(t)x_2(t) + \tilde{b}(t) - \tilde{m}_J(t)x_1(t)\Big)\,dt + \tilde{\gamma}(t)x_1(t)dW(t) \\ dx_2(t) &= \left(c_1(t) - \tilde{m}_A(t)x_2(t)\right)dt + \tilde{\gamma}(t)x_2(t)dW(t) \quad (0 \leq t \leq \tau) \end{aligned}$$

and

$$\begin{aligned} dx_1(t) &= \Big(-c_2(t)x_2(t) + \tilde{b}(t) - \tilde{m}_J(t)x_1(t)\Big)\,dt + \tilde{\gamma}(t)x_1(t)dW(t) \\ dx_2(t) &= \left(c_2(t) - \tilde{m}_A(t)x_2(t)\right)dt + \tilde{\gamma}(t)x_2(t)dW(t) \quad (t > \tau), \end{aligned}$$

where

$$\begin{gathered} c_1(t) = \mathcal{D}_0\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\}, \\ c_2(t) = \mathcal{D}\{t, \tilde{J}(\cdot), \tilde{A}(\cdot), \tilde{X}(\cdot)\}, \beta_A(t-\tau, \tilde{J}(t-\tau), \tilde{A}(t-\tau)\tilde{A}(t-\tau), \\ \tilde{\gamma}(t) = \gamma(t, \tilde{A}(t), \tilde{J}(t)), \quad \tilde{X}(t) = \mathcal{E}\{\int_0^t \gamma(s, \tilde{J}(s), \tilde{A}(s))dW(s)\}, \end{gathered}$$

and $\tilde{m}_J(t)$, $\tilde{m}_A(t)$, $\tilde{b}(t)$ are defined in (52)-(53).

The uniqueness property of the solutions of the above systems for the variables x_1 and x_2 on the intervals $[0, \tau]$ and (τ, ∞), respectively, yields $x_1(t) = J(t) = \tilde{J}(t)$, $x_2(t) = A(t) = \tilde{A}(t)$ $(t \geq 0)$.

Step 4. Using the results obtained in step 3, we conclude that $m = \tilde{m}$, $b = \tilde{b}$ and $\nu = \tilde{\nu}$, see (52)-(54) and $A(t)$ calculated by (3) is well-defined. Hence the representations of $u(t,a)$, given by (16) and (51) coincide and this u solves the SMF equation (1). The first condition in (2) is satisfied by definition, while the second one is fulfilled, because

$$\int_0^\infty \beta(t,a)u(t,a)da = \beta_A(t, J(t), A(t)))A(t) = \beta_A(t, \tilde{J}(t), \tilde{A}(t)))\tilde{A}(t) = u(t,0)$$

due to the formulas (38) and (51), (53), respectively.

Finally, the positivity of $u(t,a)$ follows immediately from the formulas (51), (53) and the evident inequality $\mathcal{E}\{\tilde{\nu}(t)\} > 0$ for all $t \geq 0$.

Step 5. In the last step we prove the uniqueness of u. Assume, on the contrary, that there exist two different solutions $u_1(t,a)$ and $u_2(t,a)$ of the full boundary value problem (1)-(2). Then by the previous steps of the proof, the aggregated variables $J_i(t) = \int_0^\tau u_i(t,a)da$ and $A_i(t) = \int_\tau^\infty u_i(t,a)da$ $(i = 1,2)$ are well-defined and satisfy the initial value problem (39)-(46) with the same initial conditions $J_1(0) = J_2(0)$, $A_1(0) = A_2(0)$ due to the formulas (20). In particular, they are $\mathcal{F}_t$-adapted continuous stochastic processes on $[0,\infty)$. Using the uniqueness property for the solutions of this initial value problem proven in Theorem 3, we conclude that $J_1(t) = J_2(t)$ and $A_1(t) = A_2(t)$ a.s. $(t \geq 0)$. Let us define m, b and ν by the formulas (35), (38) and (37), respectively. Clearly, these stochastic processes satisfy Assumption **(A)**. Then u_1 and u_2 become the solutions of the same boundary value problem (1), (6), so that by Theorem 1 $u_1(t,a) = u_2(t,a)$ a.s. for all $t, a \geq 0$. □

4. Conclusions and outlook

The McKendrick-Von Foerster equation for an age-structured population with a stochastically perturbed extinction rate function and the non-local boundary conditions depending on the aggregated age variables was studied in this paper. The fundamental existence and uniqueness results for the simplified (linear) and the full (nonlinear) boundary value problems were proven. This may, in particular, serve as the final justification of several finite dimensional stochastic models for the aggregated age variables describing the size of juveniles and adults in the population, which were earlier derived in the paper [19].

However, the present study only covered continuous stochastic noises of the type 'weighted white noise'. More general cases, in particular, those including discontinuous stochastic noises (see [19]), should be investigated further.

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