Two-dimensional Amari neural field model with periodic microstructure: Rotationally symmetric bump solutions

Evgenii Burlakov[1,*], John Wyller[1], Arcady Ponosov[1]

Abstract

We investigate existence and stability of rotationally symmetric bump solutions to a homogenized two-dimensional Amari neural field model with periodic micro-variations built in the connectivity strength and by approximating the firing rate function with unit step function. The effect of these variations is parameterized by means of one single parameter, called the degree of heterogeneity. The bumps solutions are assumed to be independent of the micro-variable. We develop a framework for study existence of bumps as a function of the degree of heterogeneity as well as a stability method for the bumps. The former problem is based on the pinning function technique while the latter one uses spectral theory for Hilbert–Schmidt integral operators. We demonstrate numerically these procedures for the case when the connectivity kernel is modeled by means of a Mexican hat function. In this case the generic picture consists of one narrow and one broad bump. The radius of the narrow bumps increases with the heterogeneity. For the broad bumps the radius increases for small and moderate values of the activation threshold while it decreases for large values of this threshold. The stability analysis reveals that the narrow bumps remain unstable while the broad bumps are destabilized when the degree of heterogeneity exceeds a certain critical value.

Keywords: two-dimensional neural field models, neural field models with micro-structure, bump solutions, existence of solutions, stability
2010 MSC: 92B99, 35B27, 35B35

[*]Corresponding author
Email address: evgenii.burlakov@nmbu.no (Evgenii Burlakov)
[1]Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, 1432 Ås, Norway

1. Introduction.

Cortical networks are often investigated in the framework of firing rate neural field models. The most well-known and simplest model describing the coarse grained dynamics of such a network is the Amari model [1]

$$\partial_t u(t,x) = -u(t,x) + \int\limits_R \omega(x-x')f(u(t,x'))dx' \qquad t \geq 0, x \in R, \tag{1}$$

where the function $u(t,x)$ denotes the activity of a neural element at time t and position x. The connectivity function (spatial convolution kernel) $\omega(x)$ determines the coupling between the elements and the non-negative function $f(u)$ gives the firing rate of a neuron with activity u. Neurons at a position x and time t are said to be active if $f(u(t,x)) > 0$. Particular attention is usually given to the localized stationary, i.e. time-independent, solutions to (1) (so-called "bumps"), as they are expected to correspond to normal brain functioning. Existence and stability of these solutions have been investigated in numerous papers (see e.g. [1], [2], [3], [4]).

Most works on bumps are restricted to one spatial dimension, however. A more realistic modeling framework of the coarse grained activity in cortical tissue makes use of neural field models in two spatial dimensions. Yet, these models have been only occasionally studied in the literature. For example, rotationally symmetric bump solutions to the two-dimensional Amari model

$$\partial_t u(t,x) = -u(t,x) + \int\limits_{R^2} \omega(x-x')f(u(t,x'))dx' \qquad t \geq 0, x \in R^2, \tag{2}$$

were first examined in [5], [6]. Rigorous analysis of these solutions involving conditions for their existence and stability was given in [7] and [8] for the case when the connectivity function ω is expressed as a sum of modified Bessel functions.

The modeling framework (1) and its extensions are proposed to capture the features of the brain activity on the macroscopic level. However, they do not take into account the heterogeneity in the cortical structure. The first step in that direction has been taken by Coombes et al [9]. In that paper the heterogeneous nonlocal framework

$$\partial_t u_\varepsilon(t,x) = -u_\varepsilon(t,x) + \int\limits_R \omega_\varepsilon(x-x')f(u_\varepsilon(t,x'))dx', \\ t>0,\ x\in R, \tag{3}$$

in one spatial dimension was chosen as a starting point, where the connectivity kernel $\omega_\varepsilon(x) = \omega(x, x/\varepsilon)$ by assumption is periodic in the second variable. The powerful two-scale convergence method (see e.g. [10]) has been applied by Svanstedt et al [11] to the neural field models with spatial microstructure. It allows one to reduce (as $\varepsilon \to 0$) the integro-differential equation (3) with the heterogeneous connectivity kernel to

$$\partial_t u(t,x,y) = -u(t,x,y) + \int\limits_R \int\limits_{[0,1)} \omega(x-x', y-y')f(u(t,x',y'))dy'dx', \\ t>0,\ x\in R, \tag{4}$$

where y is the periodic fine-scale variable. This limit procedure is known as the homogenization procedure and the corresponding equation (4) is usually referred to as the *homogenized Amari equation*. Later on, this approach was applied in Svanstedt et al [12] and Malyutina et al [13] to the investigation of existence and stability of the single-bump and symmetric two-bump solutions, respectively, to the model (3).

This serves as a background and motivation for the present work. We consider the two-dimensional homogenized Amari model analogue of (4). We first develop a framework for studying the existence of the rotationally symmetric single-bump stationary solutions of this model. In the construction procedure we proceed in a way analogous to the method outlined in [12] and [13]: It is assumed that the firing rate function is approximated by means of the unit step function and that the solutions are independent of periodic microvariable. Next, we develop a stability method for the bumps based on the spectral properties of the Hilbert–Schmidt integral operators, also by following ideas of Svanstedt et al [12] and Malyutina et al [13]. The whole stability assessment then boils down to a study of maximal growth rate of the perturbations imposed on the bumps state, corresponding to the operator norm of the Hilbert–Schmidt operator. We demonstrate the bumps construction procedure and the stability assessment in detail when the connectivity kernel is modeled by means of Mexican hat function. The main challenge in this study was the complexity of the numerical simulations caused both by the problem of dimensionality and the fact that we were not able to use analytical expressions for the Hankel transform of the connectivity kernel (due to its heterogeneity) and, consequently, of its integrals, as it was done in Folias et al [14] and Owen et al [8].

This paper is organized in the following way. In Section 2 we develop the framework for construction of the rotationally symmetric single bumps solutions to the two-dimensional homogenized model with the unit step firing rate function and outline the stability method for such structures. In Section 3 we illustrate the theory developed with the concrete example of the Amari equation where the connectivity is modeled by the Mexican hat function. Concluding remarks and outlook are given in Section 4.

2. General theory.

2.1. Existence of single bumps.

The heterogeneous Amari neural field model

$$\partial_t u_\varepsilon(t,x) = -u_\varepsilon(t,x) + \int\limits_{R^2} \omega_\varepsilon(x-x') f(u_\varepsilon(t,x'))dx', \qquad t>0,\ x\in R^2, \tag{5}$$

in $2D$ serves as a starting point for our study. Here $u_\varepsilon(t,x)$ is the electrical activity at the time t and the point x of the neural field, f is the firing rate function, $\omega_\varepsilon(x) = \omega(x, x/\varepsilon)$ is the connectivity kernel which by assumption is continuous, vanishing at infinity with respect to the first argument and Y-periodic even function of the second argument $y = x/\varepsilon$ ($Y = [0,1)^2$). Proceeding in the way analogous to Svanstedt et al [12], we get the following homogenized equation

$$\partial_t u(t,x,y,\gamma) = -u(t,x,y,\gamma) + \int\limits_{R^2}\int\limits_{[0,1)^2} \omega(x-x', y-y', \gamma) f(u(t,x',y',\gamma))dy'dx', \qquad t>0,\ x\in R^2, \tag{6}$$

in the limit $\varepsilon \to 0$ where y is the fine-scale variable. The heterogeneity is parameterized by $\gamma \in \Gamma$. Here Γ is some admissible parameter set. Let us introduce polar coordinates (r,α) i.e. $x = (x_1, x_2) = (r\cos(\alpha), r\sin(\alpha))$. We are interested in existence and stability of solutions U of (6) that are radially symmetric, independent of the fine - scale variable y and time - independent. In polar coordinates this type of solution satisfies the following equation

$$U(r,\gamma) = \int_0^\infty \int_0^{2\pi} \int_{[0,1)^2} \omega(x-x',y',\gamma) f(U(r,\gamma)) dy' d\alpha' dr',$$

$$r \in [0,\infty),\ \gamma \in \Gamma,\ x' = (r'\cos(\alpha'), r'\sin(\alpha')).$$

In addition, we assume that the firing rate function is given by the unit step Heaviside function with the activation threshold h i.e. $f(u) = H(u-h)$. Moreover, we study stationary solutions U for which $U(r,\gamma) > h$ for $r < a$ and $U(r,\gamma) < h$ for $r > a$, where the bump radius a is determined by the equality $U(a,\gamma) = h$. These solutions are referred to as single bump solutions. The formal expression for these solutions is given by

$$U(r,\gamma) = \int_0^a \int_0^{2\pi} \langle\omega\rangle(x-x',\gamma) r' d\alpha' dr', \tag{7}$$

where $\langle\omega\rangle$ is the mean value

$$\langle\omega\rangle(x,\gamma) = \int_{[0,1)^2} \omega(x,y,\gamma) dy$$

is the mean value of the connectivity kernel over the period of the second variable y. We calculate the double integral in (7) using the two-dimensional Fourier transform of the radially symmetric function $\langle\omega\rangle(r,\gamma)$, expressed in polar coordinates,

$$\langle\omega\rangle(r,\gamma) = \int_0^\infty \langle\widetilde{\omega}\rangle(\rho,\gamma) \rho J_0(r\rho) d\rho,$$

where J_ν is the Bessel function of the first kind of order ν and $\langle\widetilde{\omega}\rangle$ denotes the Hankel transform of $\langle\omega\rangle$. See Bochner et al [15] for details. Following the procedure implemented in Folias et al [14], we finally get the formal expression

$$U(r,\gamma) = 2\pi a \int_0^a \langle\widetilde{\omega}\rangle(r',\gamma) J_0(rr') J_1(ar') dr' \tag{8}$$

for the bump solution. The bump radius a is determined by the threshold intersection condition

$$U(a,\gamma) = h \tag{9}$$

where

$$U(a,\gamma) = 2\pi a \int_0^a \langle\widetilde{\omega}\rangle(r',\gamma) J_0(ar') J_1(ar') dr' \tag{10}$$

The function $U(a,\gamma)$ given by the expression (10) is called the *pinning function* while the equation (9) is referred to as the *pinning equation*. Hence, for a given threshold value of h, the equation (10) defines a level curve in the a, γ - plane, showing the variation of the γ - dependent bumps radius a. For each γ, one inserts the corresponding bumps radius a into the expression (8) for the bump. In Section 3 we investigate this construction procedure when the connectivity function ω is expressed in terms of Mexican hat function.

2.2. Stability of single bumps.

We study stability of the stationary bump state (8) in the standard way, i.e. by perturbing the stationary solution

$$u(t,x,y,\gamma) = U(r,\gamma) + \Phi(t,x,y,\gamma),$$

where $\Phi(t,x,y,\gamma) = \varphi(x,y,\gamma)e^{\lambda t}$ (see e.g. [8], [13]). Expanding to first order in φ, we obtain

$$\varphi(x,y,\gamma) = \frac{a}{(\lambda+1)\big|\partial_r U(r,\gamma)|_{r=a}\big|} \int_0^{2\pi} \int_{[0,1)^2} \omega(|x-\overline{a}|, y-y', \gamma)\varphi(\overline{a}, y', \gamma) dy' d\theta,$$
$$\overline{a} = (a,\theta).$$

By inserting $r = a$ in the above expression and introducing

$$\mu = (\lambda+1)\big|\partial_r U(r,\gamma)|_{r=a}\big|,$$

we get the following operator equation

$$\mu\varphi = \mathbb{H}(a,\gamma)\varphi, \tag{11}$$

where

$$\varphi = \varphi((a,\alpha),y,\gamma),$$
$$\mathbb{H}(a,\gamma)\varphi((a,\alpha),y) =$$
$$a\int_0^{2\pi}\int_{[0,1)^2} \omega(\sqrt{2a^2-2a^2\cos(\alpha-\theta)}, y-y',\gamma)\varphi((a,\theta),y')dy'd\theta.$$

For each $a \in (0,\infty)$, $\gamma \in \Gamma$, the operator $\mathbb{H}(a,\gamma)$ is self-adjoint on the space $L_2([0,2\pi]\times[0,1)^2)$ with the norm

$$\|\psi\|_{L_2} = \sqrt{\langle \psi,\psi\rangle},$$
$$\langle \psi,\phi\rangle = \int_0^{2\pi}\int_{[0,1)^2} \psi((a,\alpha),y)\phi((a,\alpha),y)dyd\alpha.$$

Indeed, for each $a \in (0,\infty)$, $\gamma \in \Gamma$, and any $\phi,\psi \in L_2([0,2\pi]\times[0,1)^2)$, using the properties of the connectivity function together with an interchange of the integration order, we have

$$\langle \mathbb{H}(a,\gamma)\phi,\psi\rangle = \int_0^{2\pi}\int_{[0,1)^2} a\int_0^{2\pi}\int_{[0,1)^2} \omega(\sqrt{2a^2-2a^2\cos(\alpha-\alpha')}, y-y',\gamma)\times$$
$$\phi(\alpha',y')\psi(\alpha,y)dy'd\alpha'dyd\alpha =$$
$$= \int_0^{2\pi}\int_{[0,1)^2} a\int_0^{2\pi}\int_{[0,1)^2} \omega(\sqrt{2a^2-2a^2\cos(\alpha'-\alpha)}, y'-y,\gamma)\times$$
$$\psi(\alpha,y)\phi(\alpha',y')dyd\alpha dy'd\alpha' = \langle \phi,\mathbb{H}(a,\gamma)\psi\rangle.$$

In addition, for any $a \in (0,\infty)$, $\gamma \in \Gamma$, the operator $\mathbb{H}(a,\gamma)$ is compact as the integral operator having bounded continuous kernel. Thus, as it follows from Hilbert–Schmidt's theorem (see e.g. [16]), we have the following expressions for the eigenvalues μ_n and the corresponding growth/decay rates, respectively:

$$\mu_n = a\int_0^{2\pi}\int_{[0,1)^2} \omega(\sqrt{2a^2-2a^2\cos(\alpha-\theta)}, y-y',\gamma)dy'cos(2n\theta)d\theta,$$

$$\max_{\forall n}\{\mu_n\} = \|\mathbb{H}(a,\gamma)\|_{L_2},$$

$$\max_{\forall n}\{\lambda_n\} = \lambda_{max} = \frac{\|\mathbb{H}(a,\gamma)\|_{L_2}}{\left|\partial_r U(r,\gamma)|_{r=a}\right|} - 1. \tag{12}$$

The stability of the single bumps (8) - (10) can thus be assessed by means of the operator norm $\|\mathbb{H}(a,\gamma)\|_{L_2}$: When $\lambda_{\max} < 0 (> 0)$, then the bump is stable (unstable).

3. Example: Mexican hat connectivity function.

In this section we illustrate the theory developed in the previous section by letting the connectivity kernel be given as

$$\omega(x,y,\gamma) = \frac{1}{\sigma(y,\gamma)}\chi\Big(\frac{x}{\sigma(y,\gamma)}\Big).$$

with

$$\sigma(y,\gamma) = 1 + \gamma\cos(2\pi y_1)\cos(2\pi y_2),\ y = (y_1, y_2),\ \gamma \in \Gamma = [0,1).$$

and

$$\chi(x) = \frac{1}{2\pi}\Big(\frac{\exp(-|x|)}{2} - \frac{\exp(-|x|/2)}{4}\Big). \tag{13}$$

This connectivity kernel is referred to as the Mexican hat function. The bump radius a is then found by solving the pinning equation (10) numerically. In Fig. 1 the graph of the pinning function is shown for selected values of the heterogeneity parameter γ i.e. $\gamma = 0,\ 0.2,\ 0.5,\ 0.9$. The intersection between the fixed threshold value h and the graph of the pinning function yields the bumps radius. In the figure we have put $h = 0.1$. From this plot we infer the following result: The generic picture consists of one narrow and one broad bumps for each admissible activation threshold value, in a way analogous to single bumps in the 1D case. Moreover, we also observe that the bumps radius of both the narrow and the broad bump increases with the degree of heterogeneity for the selected value of the threshold value. We finally notice that for the translationally invariant case ($\gamma = 0$), our plot resembles the results obtained in Owen et al [8]. In order to study the variation of the bumps radius with the degree of heterogeneity in some detail, we conveniently make use of the level curve description (9) - (10). The result of this investigation is summarized in Fig. 2 and Fig. 3. Fig. 2

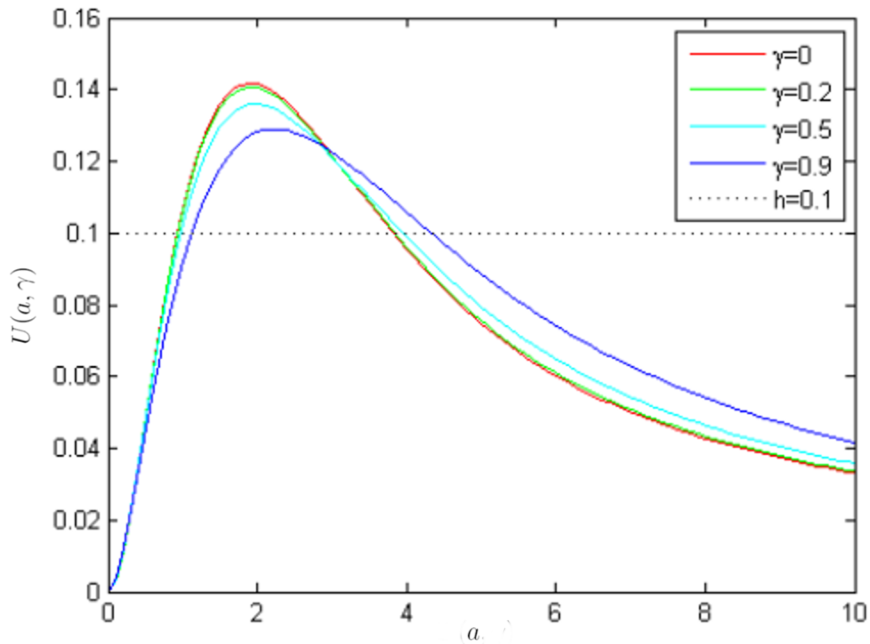

Figure 1: The graph of the pinning function (10) in the case of the Mexican hat connectivity function (13) for different values of the degree of heterogeneity γ. The activation threshold is kept constant and within the range of admissible values.

and Fig. 3 support the conclusion that bumps radius a of the narrow bump increases with the degree of heterogeneity γ. The bump radius for broad bump increases for small and moderate values of the activation threshold h, while it decreases with γ for larger values of h. Variation of the broad and the narrow bump shapes with the degree of heterogeneity parameter is shown in Fig. 4 and Fig. 5, respectively.

In order to investigate stability of the stationary solutions to (6) with the connectivity given by (13), we study the maximal growth rate (12) as function of the threshold value h for different values of the degree of heterogeneity. In order to do that, we need to estimate numerically the operator norm $\|\mathbb{H}(a,\gamma)\|_{L_2}$ in (12). The result of this investigation is summarized in Fig. 6. One readily observes that the narrow bumps remain unstable for all values of the degree of heterogeneity. For the broad bumps an increase in the degree of heterogeneity decreases the interval of activation threshold h for

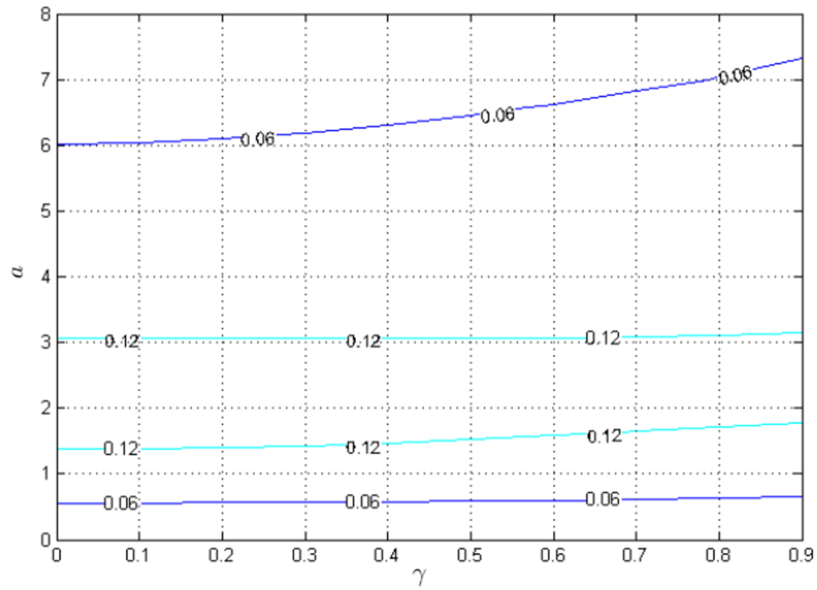

Figure 2: Level curves (9) - (10) in the case of the Mexican hat connectivity function (13) for different values of the activation threshold values. The curves are labeled with these values.

which the bumps are stable. When the degree of heterogeneity exceeds a certain threshold value, the bumps will be unstable for all values of h. The destabilization process is further detailed in Fig. 7. Notice that Fig. 6 (namely, the case $\gamma = 0$) reproduces qualitatively the same results as in Owen et al [8].

4. Conclusions and outlook

We have investigated the existence and stability of bump solutions in $2D$ of the homogenized Amari model. The starting point of this study is the homogenized Amari neural field equation. This model has previously been obtained as the limit of the parameterized heterogeneous neural field models by using the two-scale convergence technique.

The bumps solutions are assumed to be independent of the periodic microvariable and the firing rate function is modeled by the Heaviside function.

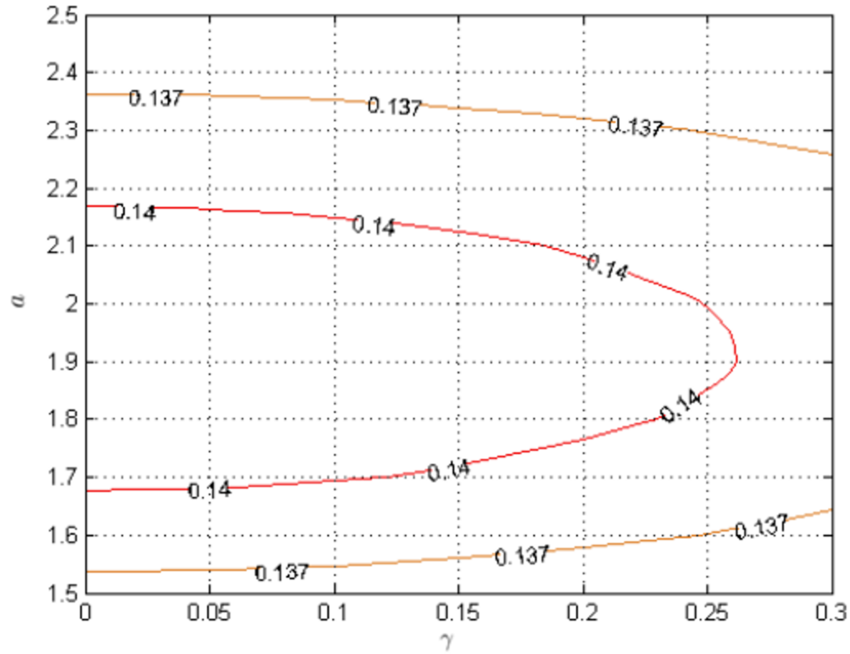

Figure 3: Magnification of the level curve description in Fig. 2 where the broad and the narrow bumps merge together. The curves are labeled with activation threshold values.

We use the pinning function technique to study the existence of the bumps while the stability method is based on spectral theory for Hilbert–Schmidt integral operators. The stability can be inferred from the maximal growth rate which in turn depends on the operator norm of the actual integral operator.

We apply these procedures to the case when the connectivity kernel is modeled by means of a Mexican hat function. The outcome of this analysis can be summarized as follows: The generic picture consists of one narrow and one broad bump for the set of admissible threshold values. The bumps radius of the narrow bump increases with the degree of heterogeneity γ. In the case of broad bumps the bumps radius increases for small and moderate values of the activation threshold h, while it decreases with γ for larger values of h. Numerical analysis in this example indicates that increase of the degree of heterogeneity acts to destabilize the broad bumps while the narrow bumps always remain unstable.

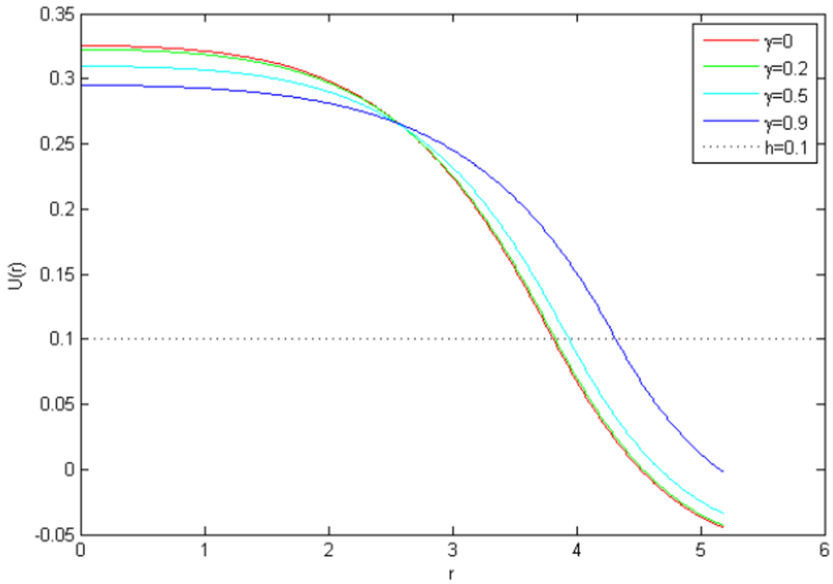

Figure 4: The variation of the broad bump shape with the heterogeneity parameter γ.

In future works we aim at proving existence and continuous dependence of the stationary bump solutions under transition from the Heaviside to Lipschitz continuous firing rate functions. The transition to piecewise-linear firing rate functions is of particular importance for the theory of neural fields possessing microstructure. The aforementioned continuous dependence results link the neural field homogenization theory developed in Svanstedt et al [11] for the case of convex firing rate functions to the numerical results obtained for the Heaviside firing rate in e.g. [12], [13], and also in the present study.

Acknowledgements. The authors wish to thank the Norwegian Metacenter for Computational Science (NOTUR), which provided computational resources for the present research.

References

[1] Amari S. Dynamics of Pattern Formation in Lateral-Inhibition Type Neural Fields. Biol Cybern 1977; 27: 77-87.

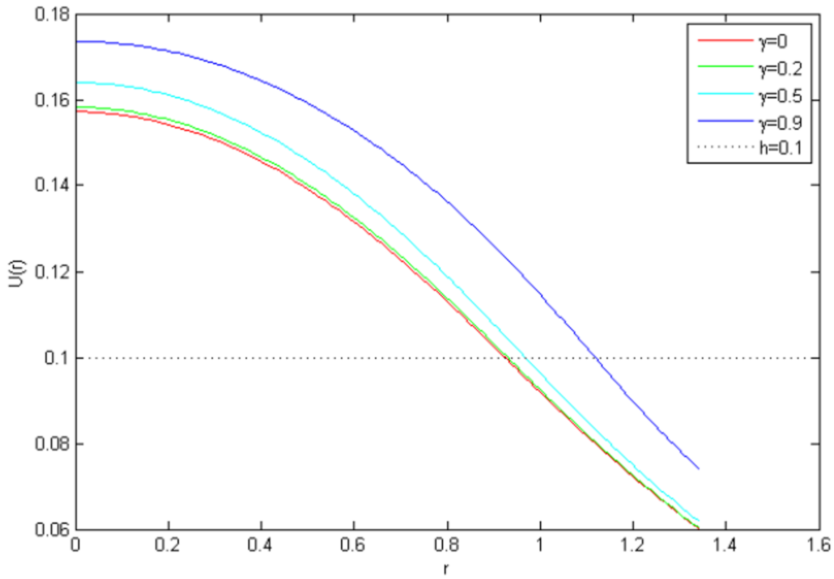

Figure 5: The variation of the narrow bump shape with the heterogeneity parameter γ.

[2] Pinto D, Ermentrout GB. Spatially structured activity in synaptically coupled neuronal networks: II. Lateral inhibition and standing pulses. SIAM J Appl Math 2001; 62: 226-43.

[3] Coombes S, Owen MR. Evans functions for integral neural field equations with Heaviside firing rate function. SIAM J Appl Dyn Syst 2004; 4: 574-600.

[4] Guo Y, Chow CC. Existence and stability of standing pulses in neural networks: II. Stability. SIAM J Appl Dyn Syst 2005; 4: 249-81.

[5] Taylor JG. Neural bubble dynamics in two dimensions: foundations. Biol Cybern 1999; 80: 393-409.

[6] Werner H, Richter T. Circular stationary solutions in two-dimensional neural fields. Biol Cybern 2001; 85: 211-7.

[7] Folias SE, Bressloff PC. Breathers in two-dimensional neural media. Phys Rev Lett 2005; 95: 208107.

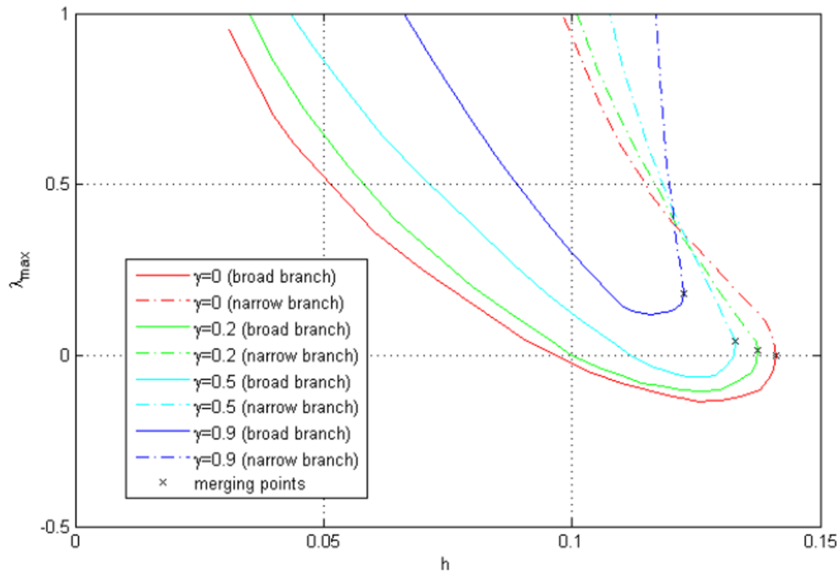

Figure 6: The maximal growth rate of the perturbation as a function of the activation threshold for different values of the degree of heterogeneity parameter γ.

[8] Owen MR, Laing CR, Coombes S. Bumps and rings in a two-dimensional neural field: splitting and rotational instabilities. New Journal of Physics 2007; 9: 378.

[9] Coombes S, Laing C, Schmidt H, Svanstedt N, Wyller J. Waves in random neural media. Discrete and Continuous Dynamical Systems, Series A 2011; 32: 2951-70.

[10] Visintin A. Towards a two-scale calculus. ESAIM Control Optim Calc Var 2006; 12: 371-97.

[11] N. Svanstedt N, Woukeng JL, Homogenization of a Wilson–Cowan model for neural fields. Nonlinear Analysis Real World Applications 2013; 14: 1705-15.

[12] Svanstedt N, Wyller J, Malyutina E. A one-population Amari model with periodic microstructure. Nonlinearity 2014; 27: 1394-417.

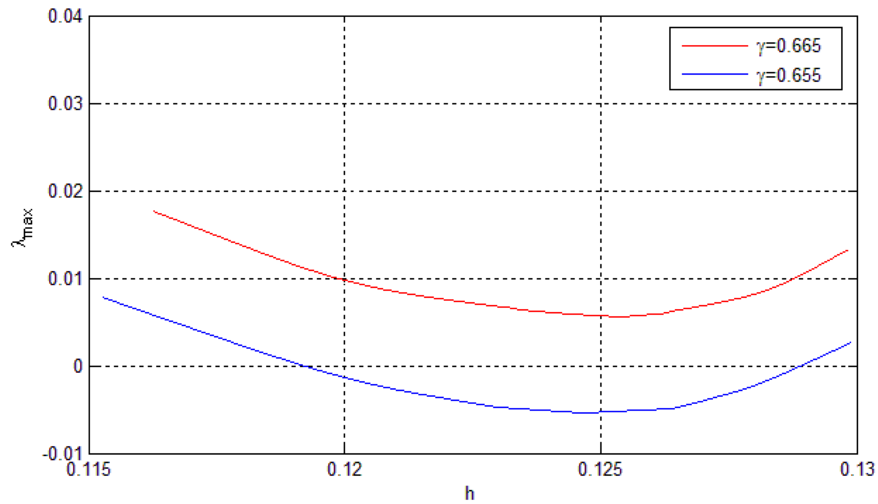

Figure 7: The destabilization regime of the broad bump solution for the case of Mexican hat connectivity function.

[13] Malyutina E, Wyller J, Ponosov A. Two bump solutions of a homogenized Amari model with periodic microstructure. Physica D 2014; 271: 19-31.

[14] Folias SE, Bressloff PC. Breathing pulses in an excitatory neural network. SIAM J Appl Dyn Syst 2004; 3: 378-407.

[15] Bochner S, Chandrasekharan K. Fourier Transforms. New Jersey: Princeton University Press; 1949.

[16] Kolmogorov AN, Fomin SV. Elements of the Theory of Functions and Functional Analysis. Vol 1. New York: Dover Publications Inc; 1961.