SMOOTH DYNAMICS BECOMES HYBRID IN THE LIMIT

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ABSTRACT. We show the appearance of an essentially nonlocal dynamics describing the limit behavior of trajectories of a class of dynamical systems defined by classical autonomous ODEs with smooth right-hand sides containing a small parameter and becoming discontinuous in the formal limit. The limit dynamics is shown to be described by an explicitly constructed Nerode-Kohn *hybrid* dynamical system consisting of a continuous plant (ODE) and a finite state machine which are interacting and producing hybrid dynamics with possible memory effects. We remark however that, from the "statistical" point of view, the limit behavior of an ensemble of trajectories can still be described by an ODE, with possibly time-dependent and discontinuous right-hand side depending on the chosen ensemble.

1. Introduction

We consider the following system of ODEs with respect to the unknown $x \in \mathbb{R}^n$

$$\dot{x}_i = f_i(z, x_i), \qquad i = 1, \dots n, \tag{1.1}$$

where $f_i \colon [0,1]^n \times \mathbb{R} \to \mathbb{R}$ is the given smooth function depending on the feedback vector $z \in \mathbb{R}^n$ of the form

$$z_i := H_{q,\theta_i}(x_i),$$

$H_{q,\theta_i}(x_i)$ being the Hill function depending on parameters $q > 0$ and $\theta_i \in \mathbb{R}$ and defined by

$$H_{q,\theta_i}(x_i) := \frac{x_i^{1/q}}{x_i^{1/q} + \theta_i^{1/q}},$$

and $i = 1, \dots n$. Here $q > 0$ is a small parameter (responsible for the steepness of the function H_{q,θ_i}), so that the trajectories of (1.1) depend on q and one is interested in their behavior as $q \to 0^+$. Formally, plugging $q := 0$ into (1.1) (i.e. passing to a limit as $q \to 0^+$ in the right-hand side) one gets a system of ODEs with discontinuous right-hand side, with discontinuity occurring at the hyperplanes $\{x_i = \theta_i\}$. It is easy to observe that far away from these hyperplanes the limit dynamics of (1.1) as $q \to 0^+$ is determined by this formal limit, while the real problem is to determine the dynamics near those hyperplanes.

One of the important examples of such a system is given by a general model of a gene regulatory network [17], where n is the number of genes in a given population, x_i represents the concentration of the protein produced by the gene i, z_i are regulatory functions describing interactions within the network through activation of the respective genes, $f_i(z, x_i) := F_i(z) - G_i(z)x_i$, and $F_i, G_i \colon \mathbb{R}^n \to \mathbb{R}^+$, $i = 1, \dots n$, are given multilinear functions representing the production rates and

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the relative degradation rates of the genes. The case of polynomial F_i and G_i was studied in [19].

We prove in Theorem 3.10 that the limit behavior of trajectories turns out to be governed by a Nerode-Kohn *hybrid* dynamical system [15] consisting of an ODE and a finite state machine producing hybrid dynamics, and we provide an explicit construction of both the limit ODE and the finite state machine. Note that the possibility of getting a *switched* system as the limit of smooth ones has been already shown in [7]. However, here we discover an essentially different phenomenon, namely, the limit dynamics will be shown to be qualitatively different from both the original smooth ones and the switched one: in fact, it typically has memory effects inherent to hybrid systems (see Example 3.12). The detailed description of the limit hybrid dynamical system requires several technical notions; however, we provide a simple two-dimensional Example 3.9 showing all the possible features of hybrid dynamics and explaining the necessary notions.

Note that in the particular context of gene regulatory networks it is generally accepted, since the publication of the seminal paper [10], that steep sigmoidal nonlinearities can be adequately modeled by the Heaviside step functions (see e.g. the review paper [12]), thus leading to a system of ODE's with discontinuous right-hand side. However, our analysis confirms the insight well-known in systems biology, that the description of the dynamics just by such a system of ODEs may be insufficient; in fact, it has been already noticed in [17] that the original Glass-Kauffman paradigm, only based on simple switchings, does not always represent the dynamics, due to the existence of sliding modes. Further, in genetics memory effects are well-known and are typically modeled by introducing artificial delays. Our result shows that both sliding modes and memory effects naturally appear as a consequence of the same hybrid dynamics that is a limit of standard smooth genetic network models.

We show however in Proposition 5.1 that when one looks at the limit behavior of trajectories of (1.1) as $q \to 0^+$ from the "statistical" point of view, i.e. considers the limit behavior of an ensemble of trajectories with starting points defined by some measure on the phase space $\mathbb{R}^n$, rather than that of each single trajectory, then qualitatively one obtains quite a different result. Namely, the limit dynamics of an ensemble of trajectories (viewed as a flow of measures represented by a Young measure) can still be described by a usual system of ODEs with possibly time-dependent and discontinuous right-hand side depending on the chosen ensemble. Precisely, it means that the ensemble as a whole behaves *as if* each its trajectory were a solution to an ODE. This is however typical for many continuous dynamical systems and not a peculiarity of the chosen class of the latter. In fact, by superposition principle for the continuity PDE (theorem 12 from [1]), to possess such a property, it would be enough for a dynamical system to produce a flow of measures satisfying the continuity equation in the weak sense. The latter property is guaranteed automatically, for instance, for flows of measures which are absolutely continuous curves in the space of measures endowed by some Kantorovich-Wasserstein metric W_p, with $p > 1$ (this is somewhat more than just being continuous in the narrow topology of measures) [2, §8]. For more discussion of the "macroscopic" (Eulerian) representation of curves of measures as flows satisfying continuity equation and their "microscopic" (Lagrangian) representation by ODEs, see [18].

To get the limit behavior of trajectories of (1.1) as $q \to 0^+$ near hyperplanes $\{x_i = \theta_i\}$, we make a change of variables passing from the part of the unknowns x to the unknowns z and getting in this way a classical singularly perturbed system of ODEs. This substitution has been first suggested for the particular case of gene regulatory network models (i.e. when f_i are linear in x_i) and, moreover, with f_i linear in each z_i, in [17], where the classical Tikhonov theory of singularly perturbed systems was applied to study the limit behavior of the trajectories of the transformed system in z variables. This theory however works only in the case when the invariant measures of the formal limit of the transformed system as $q \to 0^+$ are given by stationary points of the latter (i.e. are Dirac measures concentrated over stationary points). In general however the formal limit of the transformed system as $q \to 0^+$ may possess more complicated invariant measures even when the right-hand sides of (1.1) are multilinear (see [14], as well as the Examples 3.5 and 3.6), hence the Tikhonov theory is not applicable, and therefore we use the more general theory developed by Artstein and Vigodner in [4] (it is also worth mentioning that this theory found recently a lot of applications, in particular, in control problems [5] and numerical analysis [6]).

We assume further on that the system (1.1) has some bounded invariant open set $\Omega \subset \mathbb{R}^n$ for every $q > 0$. In particular, this holds when for every $x \in \partial\Omega$ and every $z \in [0,1]$ one has $f(z,x) := (f_1(z,x_1),\dots f_n(z,x_n))$ is directed inside of x (i.e. has strictly positive component along the inner normal at x to $\partial\Omega$, if the latter is smooth). In the application to gene regulatory networks this is satisfied with

$$\Omega := \{x \in \mathbb{R}^n \colon 0 < x_i < \max_{z\in[0,1]} F_i(z)/ \min_{z\in[0,1]} G_i(z), i = 1,\dots,n\}$$

(under the naturally admitted assumption that $\min_{z\in[0,1]} G_i(z) > 0$, $F_i(z) \geq 0$, $F_i \not\equiv 0$ for some $z \in [0,1]$, $i = 1,\dots,n$).

2. Notation and preliminaries

We will use the concept of a *hybrid dynamical system* that suits our purposes as the pair of objects, the "discrete" one, and the "continuous" ("smooth") one. A discrete component of a hybrid dynamical system is a *finite state machine* represented by a directed graph $\mathcal{F} = (\Sigma, E)$, with the finite set of vertices Σ interpreted as a set of admissible states containing finitely many states $\sigma \in \Sigma$ and the finite set of edges $E \subset \Sigma \times \Sigma$, each edge $e = (\sigma, \sigma^+) \in E$ representing an admissible transition between two states σ and σ^+. A continuous (smooth) component is represented by a collection of smooth (C^1) dynamical systems governed by an ODE $\dot{x}_\sigma = F(x_\sigma, \sigma)$, $\sigma \in \Sigma$, $t \geq 0$, defined in an open subset $\mathcal{D} \subset \mathbb{R}^n$. Interactions between the discrete and the continuous component are described through a family of *guards* $G_e \subset \mathcal{D}$, $e \in E$, which in this paper are assumed to be disjoint. A transition from state σ to state σ^+ occurs if and only if $e = (\sigma,\sigma^+) \in E$ and $x_\sigma(t') \in G_e$ for some $t' \geq 0$. In this case the next piece of the solution $x_{\sigma^+}(t)$ starts at $x_\sigma(t')$, which ensures continuity of the entire solution in $\mathcal{D}$. A *hybrid trajectory* is a pair $(x_{\sigma(\cdot)}(\cdot), \sigma(\cdot))$, where σ depends on $x(\cdot)$. Thus, given a point $x_0 \in \mathcal{D}$, we observe that, in principle, the projection of two different hybrid trajectories may assume equal values even at equal instants of time (see Example 3.12 below). In this case we speak of a "memory effect", since the continuous component of the hybrid trajectory "remembers" where it comes from.

For any set $D \subset \mathbb{R}^n$ we let $\bar{D}$ be the closure of D, $D^c := \mathbb{R}^n \setminus D$, $\operatorname{dist}(x, D) := \inf\{|x-y| : y \in D\}$ whenever $x \in \mathbb{R}^n$, $|\cdot|$ standing for the usual Euclidean norm.

Throughout the paper we will use the classical notation from measure theory. In particular, for a Borel measure μ over a metric space X and a Borel map $f\colon X \to Y$ the notation $f_{\#}\mu$ stands for the measure over the metric space Y defined by $(f_{\#}\mu)(B) := \mu(f^{-1}(B))$ for every Borel $B \subset Y$. Let also $e_t\colon x(\cdot) \in C([0,T];\mathbb{R}^n) \mapsto x(t) \in \mathbb{R}^n$, where $C([0,T];\mathbb{R}^n)$ stands for the usual class of continuous R^n-valued functions over $[0,T]$. The customary notation $C_0^1(0,T)$ and $C_0^\infty(\mathbb{R}^n)$ stands for the classes of continuously differentiable functions with compact support in $(0,T)$ and of infinitely many times continuously differentiable functions with compact support in $\mathbb{R}^n$.

3. Limit dynamics

Clearly, far away from the hyperplanes $\{x_i = \theta_i\}$ which will further be called *singular* (as well as their intersections), the system (1.1) is just a usual system of ordinary differential equations with smooth right-hand side. Note that $H_{q,\theta_i}(x_i) \to b_i \in \{0,1\}$ as $q \to 0^+$ for each $x_i \neq \theta_i$. Thus, one has that $(\cup_{i=1}^n \{x_i = \theta_i\})^c := \sqcup_{j=1}^{2^n} \Omega_j$, where in each open set $\Omega_j \subset \mathbb{R}^n$ (if $n = 2$, it is a quadrant, if $n = 3$, it is an octant etc.)

$$\{H_{q,\theta_i}(x_i)\}_{i=1}^n \to z^j \in \{0,1\}^n$$

as $q \to 0^+$ for each $x = \{x_i\} \in \Omega_j$.

We analyze here the local behavior of solutions near the "singular" hyperplanes as $q \to 0$. Introduce the following notation: let

$$H_{0,\theta}(x) := \begin{cases} 0, & x < \theta, \\ 1/2, & x = \theta, \\ 1, & x_i > \theta. \end{cases}$$

For a set $S \subset \{1, \ldots, n\}$ and a vector $b \in \{0,1\}^{n-\#S}$, indexed by $R := \{1, \ldots, n\} \setminus S$ (i.e. $b = \{b_r\}_{r \in R}$), denote

- by $Z(S, b)$ the $\#S$-dimensional face of the n-dimensional cube $Z^n := [0,1]^n$ determined by the relationship

  $$Z(S, b) := \{z \in Z^n \,:\, z_r = b_r \text{ for all } r \in R\}$$

  and by $\mathrm{int} Z(S, b)$ its relative interior, namely,

  $$\mathrm{int} Z(S, b) := \{z \in Z^n \,:\, z_r = b_r, z_s \in (0,1) \text{ for all } s \in S, r \in \{1, \ldots, n\} \setminus S\},$$

  with the convention $\mathrm{int} Z(\emptyset, b) := Z(\emptyset, b)$;
- by $X(S, b)$ the $(n - \#S)$-dimensional affine subvariety determined by the relationship

  $$X(S, b) := \{x \in \mathbb{R}^n \,:\, x_s = \theta_s, H_{0,\theta_r}(x_r) = b_r \text{ for all } s \in S \text{ and } r \in R\}.$$

  Note that the sets $X(S, b)$ with different pairs (S, b) are mutually disjoint.

Mind that the case $S = \emptyset$ is not excluded, namely, if so, $Z(S, b)$ become 0-dimensional faces (i.e. vertices) of the cube Z^n. Clearly, the total number of $\#S$-dimensional faces of Z^n is $C_n^{\#S} 2^{n-\#S}$ (where $C_n^{\#S}$ stands for the usual binomial coefficient).

Consider the system of equations

$$z_j' = \frac{z_j(1 - z_j)}{\theta_j} f_j(z, \theta_j), \qquad j = 1, \ldots, n. \tag{3.1}$$

Denote by $z^\tau \colon \mathbb{R}^n \to \mathbb{R}^n$ its solution flow. Observe that each face $Z(S, b)$ is invariant with respect to z^τ. We also consider this system restricted to each face $Z(S, b)$, namely, the same system of equations with the additional requirement $z(t) \in Z(S, b)$. Denoting $z_S := \{z_s\}_{s \in S}$ we get for the latter system of equations the representation

$$z_s' = \frac{z_s(1 - z_s)}{\theta_s} f_s((z_S, b), \theta_s), \qquad s \in S, \tag{3.2}$$

with respect to the unknown z_S, where the vector $z := (z_S, b)$ stands for the vector with components z_s for all $s \in S$ and $z_r := b_r$ for all $r \in R$. The solution flow to (3.2) will be denoted $z_{S,b}^\tau \colon \mathbb{R}^n \to \mathbb{R}^n$.

We make the following fundamental assumption on (3.2).

Assumption 3.1. *For every $S \subset \{1, \ldots, n\}$ and a vector $b \in \{0,1\}^{n-\#S}$ there is*

(A) *a finite number of disjoint sets* $G_i(S,b) \subset Z(S,b)$ *invariant with respect to the solution flow* $z^\tau_{S,b}$, *relatively open in* $Z(S,b)$, *such that for their relative closures* $\bar{G}_i(S,b)$ *one has*

$$Z(S,b) = \bigcup_i \bar{G}_i(S,b);$$

(B) *for each* $G_i(S,b)$ *there is a unique minimal compact* ω*-limit set* $K_i(S,b) \subset G_i(S,b)$ *for the trajectories of* (3.2) *(i.e. such that*

$$\mathrm{dist}\,(z^\tau_{S,b}(x), K_i(S,b)) \to 0$$

as $\tau \to +\infty$ *for all* $x \in G_i(S,b)$*), the minimality being understood in the usual sense, that is, there is no proper subset* $K \subset K_i(S,b)$ *such that* $\mathrm{dist}\,(z^\tau_{S,b}(x), K) \to 0$ *as* $\tau \to +\infty$ *for all* $x \in G$ *for some* $G \subset G_i(S,b)$ *relatively open in* $Z(S,b)$ *satisfying* $K \subset G$. *Moreover, for each* $G_i(S,b)$ *there is a unique positively invariant probability measure* $\nu_{i,S,b}$ *for the solution flow* $z^\tau_{S,b}$ *concentrated over* $K_i(S,b)$, *namely, such that*

$$(z^\tau_{S,b})_\# \nu_{i,S,b} = \nu_{i,S,b} \quad \textit{for all } t > 0;$$

(C) *for every* $K_i(S,b)$ *letting* $Z(\tilde{S},\tilde{b})$ *be a face of* Z^n *of minimal dimension such that* $K_i(S,b) \subset Z(\tilde{S},\tilde{b})$ *one has that* $K_i(S,b)$ *belongs to the interior of* $Z(\tilde{S},\tilde{b})$, *i.e.* $K_i(S,b) \subset \mathrm{int} Z(\tilde{S},\tilde{b})$*;*

(D) *for every* $K_i(S,b)$ *there is a* $G_j(S',b')$ *such that* $\#S' = \#S + 1$ *and* $Z(S,b)$ *is a face of* $Z(S',b')$, *unless* $K_i(S,b) \subset Z(\tilde{S},\tilde{b})$ *for some face* $Z(\tilde{S},\tilde{b})$ *of dimension* $\#\tilde{S} \le \#S$.

The above Assumption 3.1, though technically looking, is quite natural and is satisfied in most examples (see e.g. Examples 3.5, 3.6, 3.9, 3.12 as well as all the examples from [17, 14, 19]). On the contrary, the situations when it does not hold are quite degenerate, as can be seen even in the following simple two-dimensional example ($n = 2$) with $\theta_1 = \theta_2 = 1$ and f_1, f_2 such that

$$(z_1 - 1/2)z_1(1-z_1)f_1(z_1,z_2,\theta_1) + (z_2 - 1/2)z_2(1-z_2)f_2(z_1,z_2,\theta_2) = 0 \quad \text{and} \tag{3.3}$$

$$f_1(z_1,z_2,\theta_1)^2 + f_2(z_1,z_2,\theta_2)^2 \neq 0 \quad \text{unless } z_1 = z_2 = 1/2. \tag{3.4}$$

Transforming the system (3.1) from cartesian coordinates (z_1, z_2) in the phase space into polar coordinates (ρ, θ) with center in $(1/2, 1/2)$ in the same space, i.e. $z_1 = 1/2 + \rho\cos\theta$, $z_2 = 1/2 + \rho\sin\theta$, we will have $r' = (z_1'(z_1 - 1/2) + z_2'(z_2 - 1/2))/r = 0$, and thus among the trajectories of (3.1) infinitely (even uncountably) many belong to circles centered at $(1/2, 1/2)$ with radii strictly less that $1/2$ (so as to fit into $Z^2 = [0,1]^2$). Further, condition (3.4) guarantees that these trajectories in fact cover all these circles. Thus, there are no ω-limit sets for the system (3.1) in the whole Z^2, but there are infinitely (even uncountably) many invariant sets for the flow. Note however, that it is easy to make a slight change of the functions f_1 and f_2 even in C^1 topology so as to destroy the "degeneracy" condition (3.3) producing a system satisfying Assumption 3.1 and having finitely many ω-limit sets. For general dynamical systems governed by smooth ODEs the property of having a finite number of attractors generically (up to a small change in the right-hand side of the respective ODE) is known as the *Palis conjecture* [16], which is proven for some particular classes of dynamical systems. We think it also worth mentioning here that the situation can be even more complicated because there are quite exotic examples of systems possessing a just a single attractor but multiple invariant probability measure (see the example of Furstenberg [9] where the respective degeneracy can still be destroyed by a slight change of the dynamical system). Note however that although the above example does not satisfy Assumption 3.1 and thus formally is not covered by Theorem 3.10 below, the respective dynamics can be still obtained by the same method, at least when the "degeneracy" condition (3.3) is satisfied only inside some compact set inside the open square $(0,1)^2$ (in fact, in this particular case the dynamics is only influenced by the behavior of the systems (4.7) on the boundaries of Z^2).

Several remarks regarding Assumption 3.1 are worth being mentioned.

Remark 3.2. Clearly under Assumption 3.1 one has that $Z(S,b) \setminus \cup_i G_i(S,b)$ is nowhere dense in $Z(S,b)$.

Remark 3.3. We mention, for the sake of clarity, and as a matter of example, that for every $b \in \{0,1\}^n$ each zero-dimensional face $Z(\emptyset, b)$ (i.e. the vertex of the cube Z^n) is covered by a unique region as in Assumption 3.1(A), namely, $G_1(\emptyset, b) := Z(\emptyset, b)$, and, of course, in this case also $K_1(\emptyset, b) = Z(\emptyset, b)$.

Remark 3.4. For every $K_i(S,b)$ and the face $Z(S',b')$ adjacent to $Z(S,b)$ with $\#S' = \#S+1$ (i.e. with $S' := S \cup \{s'\}$ and b' differing from b by a single component $b_{s'}$) there is at most one $G_j(S',b')$ satisfying $K_i(S,b) \subset G_j(S',b')$. In fact, the sets $G'_j(S,b) := G_j(S',b') \cap Z(S,b)$ are disjoint, relatively open in $Z(S,b)$ and invariant with respect to the solution flow $z^\tau_{S,b}$, and

$$\begin{aligned}\bigcup_j \bar{G}'_j(S,b) &= \bigcup_j \bar{G}_j(S',b') \cap Z(S,b) \\ &= Z(S,b) \cap \bigcup_j \bar{G}_j(S',b') = Z(S,b) \cap Z(S',b') = Z(S,b).\end{aligned}$$

Now, $K_i(S,b) \cap G'_j(S,b) \neq \emptyset$ for at most one value of $j \in \mathbb{N}$, since otherwise this would contradict the minimality of $K_i(S,b)$ (Assumption 3.1(B)). Thus $K_i(S,b) \cap G_j(S',b') \neq \emptyset$ for at most one value of $j \in \mathbb{N}$.

The sets $G_j(S,b)$ in the Assumption 3.1 are, as usual, called *attraction basins*, and $K_j(S,b)$ are called minimal ω-limit sets relative to $G_j(S,b)$. An example of an invariant measure $\nu_{i,S,b}$ for the solution flow $z^\tau_{S,b}$ is a Dirac measure concentrated over a stationary point (in this case $K_i(S,b)$ is a singleton supporting this measure). However it is important to understand that there may be more complicated invariant measures as the following examples show.

Example 3.5. Assume in (1.1) $n=2$, $\theta_1 = \theta_2 = 1$ and f_i, $i=1,2$, be given by

$$f_1(z_1,z_2,x_1) := \lambda\left(z_1 - \frac{1}{2}\right) + z_2 - \frac{1}{2} - \left(z_1 - \frac{1}{2}\right)\left(\left(z_1 - \frac{1}{2}\right)^2 + \left(z_2 - \frac{1}{2}\right)^2\right) x_1,$$

$$f_2(z_1,z_2,x_2) := \lambda\left(z_2 - \frac{1}{2}\right) - z_1 + \frac{1}{2} - \left(z_2 - \frac{1}{2}\right)\left(\left(z_1 - \frac{1}{2}\right)^2 + \left(z_2 - \frac{1}{2}\right)^2\right) x_2.$$

Then the system of differential equations (3.1) becomes

(3.5)
$$\begin{aligned} z_1' &= z_1(1-z_1)\left(\lambda\left(z_1 - \frac{1}{2}\right) + z_2 - \frac{1}{2} - \left(z_1 - \frac{1}{2}\right)\left(\left(z_1 - \frac{1}{2}\right)^2 + \left(z_2 - \frac{1}{2}\right)^2\right)\right), \\ z_2' &= z_2(1-z_2)\left(\lambda\left(z_2 - \frac{1}{2}\right) - z_1 + \frac{1}{2} - \left(z_2 - \frac{1}{2}\right)\left(\left(z_1 - \frac{1}{2}\right)^2 + \left(z_2 - \frac{1}{2}\right)^2\right)\right). \end{aligned}$$

When the parameter λ passes through the Hopf bifurcation point $\lambda = 0$ and becomes positive, the above system (3.5) produces an asymptotically stable limit cycle inside the square $Z = [0,1]^2$. In the x-domain one then gets asymptotically stable spiral trajectories approaching $x_1 = x_2 = 1$ in the limit.

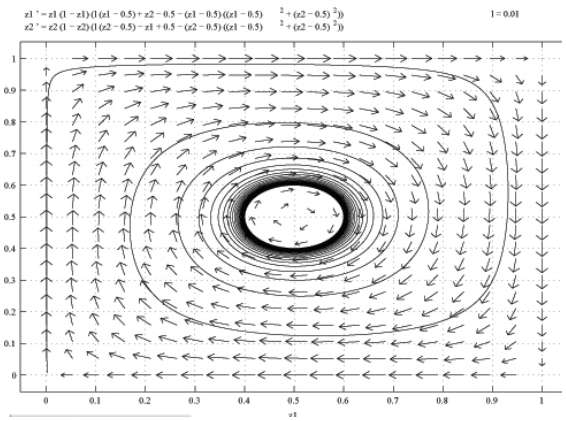


FIGURE 1. A cyclic attractor in the Example 3.5 with $\lambda = 0.1$.

Example 3.6. Assume in (1.1) $n = 3$, $\theta_1 = \theta_2 = \theta_3 = 0.25$ and f_i be given by

$$
\begin{aligned}
f_1(z_1, z_2, z_3, x_1) &:= 10\,(z_2 - z_1) + 75\theta_1 - 75x_1,\\
f_2(z_1, z_2, z_3, x_2) &:= \left(z_1 - \frac{1}{2}\right)(28 - 60z_3) - z_2 + \frac{1}{2} + 75\theta_2 - 75x_2,\\
f_3(z_1, z_2, z_3, x_3) &:= 60\left(z_1 - \frac{1}{2}\right)\left(z_2 - \frac{1}{2}\right) - \frac{8}{3}z_3 + 75\theta_3 - 75x_3.
\end{aligned}
$$

In the z-domain one obtains a nonlinear system admitting the Lorenz attractor [14]. Note that all these functions are multilinear.

We define now the finite state machine $\mathcal{F}$. We declare

- the set of admissible states $\Sigma(\mathcal{F})$ of $\mathcal{F}$ to be the set of all $K_i(S, b)$ (with all admissible S and b). If $K_i(S, b)$ belongs simultaneously to different faces of Z^n, and $Z(\tilde{S}, \tilde{b})$ is the face of minimal dimension containing $K_i(S, b)$, that is,
$$K_i(S, b) \subset \bigcap_{\{l\colon\, K_i(S,b) \subset Z(S_l, b_l)\}} Z(S_l, b_l) = Z(\tilde{S}, \tilde{b}),$$
then we will consider $K_i(S, b)$ as belonging to $Z(\tilde{S}, \tilde{b})$, i.e. as a minimal ω-limit set relative to some (relatively open) attraction basin $G \subset Z(\tilde{S}, \tilde{b})$;
- the transition between two states $K_i(S, b)$ and $K_j(\tilde{S}, \tilde{b})$ to be admissible whenever
$$
\begin{aligned}
&K_i(S, b) \subset G_j(S', b'),\\
&K_j(\tilde{S}, \tilde{b}) = K_l(S', b') \subset G_j(S', b')
\end{aligned}
$$
for some $G_j(S', b') \subset Z(S', b')$, the face $Z(S', b')$ of dimension $\#S' = \#S + 1$ being adjacent to $Z(S, b)$ (i.e. with $S' := S \cup \{s'\}$ and b' differing from b by a single component $b_{s'}$), and some $l \in \mathbb{N}$, i.e. $K_j(\tilde{S}, \tilde{b})$ is a minimal ω-limit set relative to the attraction basin $G_j(S', b')$ in $Z(S', b')$.

Finally, we define the following hybrid dynamical system $H(\mathcal{F})$ making use of the finite state machine $\mathcal{F}$. In each admissible state $K_i(S, b)$ of $\mathcal{F}$ we consider the state $x(\cdot)$ of the hybrid dynamical system to be governed by the system of equations

$$
\begin{aligned}
&\dot{x}_r = \int_{Z(S,b)} f_r((z_S, b), x_r)\, d\nu_{i,S,b}(z_S), \qquad r \in R,\\
&x_s = \theta_s, \qquad\qquad s \in S
\end{aligned}
\tag{3.6}
$$

(we think of the invariant measure $\nu_{i,S,b}$ as of a control to this system). The flow of the respective system of ODEs will be denoted by $x^t_{\nu_{i,S,b}}$. In the case $S = \emptyset$, hence $R = \{1, \dots, n\}$, this system reduces to

$$\dot{x}_j = f_j(b, x_j), \qquad j = 1, \dots, n, \tag{3.7}$$

because $\nu_{i,S,b} = \delta_b$, $b \in \{0,1\}^n$.

Define inductively the exceptional set E of the initial data. For every $X(S,b)$ let

- $\tilde{E}_0(S,b)$ stand for the set of $x_0 \in X(S,b)$ such that $x^t_{\nu_{i,S,b}}(x_0) \in X(\hat{S},\hat{b})$ for some $t > 0$, $i \in \mathbb{N}$ (i.e. for some $\nu_{i,S,b}$) and $(\hat{S},\hat{b})$ where $\#\hat{S} > \#S + 1$,
- $\tilde{E}_j(S,b)$, where $j = 1, \dots, \#S$, stand for the set of $x_0 \in X(\tilde{S},\tilde{b})$ such that $x^t_{\nu_{i,\tilde{S},\tilde{b}}}(x_0) \in \tilde{E}_{j-1}(S,b)$ for some $t > 0$, $\overline{X}(\tilde{S},\tilde{b}) \supset X(S,b)$, $i \in \mathbb{N}$ (i.e. for some $\nu_{i,\tilde{S},\tilde{b}}$),
- $E(S,b) := \bigcup_{j=0}^{\#S} \tilde{E}_j(S,b)$.

Finally, let

$$E := \bigcup_{X(S,b)} E(S,b).$$

It is easy to observe that all $\tilde{E}_j(S,b)$ are finite unions of pieces of manifolds of dimension at most $n-1$, so that E has zero Lebesgue measure.

We define now the switching rule for the finite state machine $\mathcal{F}$. Suppose that at time t_0 the finite state machine $\mathcal{F}$ is in the state $K_i(S,b)$, and $x_0 \in X(S,b) \setminus E$. Then

- either the trajectory $x(\cdot)$ of (3.6) with $x(t_0) = x_0$ remains in $X(S,b)$ for all $t \geq t_0$, in which case we will assume that $\mathcal{F}$ remains in state $K_i(S,b)$ forever (i.e. for all $t \geq t_0$),
- or there is some $t' > t_0$ such that this trajectory exits for the first time $X(S,b)$ hitting $X(S',b')$ with $S' := S \cup \{s'\}$ and b' differing from b by a single component $b_{s'}$ (i.e. $\#S' = \#S + 1$ and $Z(S',b')$ is adjacent to $Z(S,b)$). Note that $x(t') \notin X(\hat{S},\hat{b})$ for any $\hat{S}$ with $\#\hat{S} > \#S'$, because $x_0 \notin \tilde{E}_0(S,b) \subset E$. Then finding the unique (by Remark 3.4) attraction basin $G_j(S',b') \subset Z(S',b')$, such that $K_i(S,b) \subset G_j(S',b')$, and the unique minimal ω-limit set $K_j(S',b') \subset G_j(S',b')$ relative to $G_j(S',b')$, we say that $\mathcal{F}$ switches to the state $K_j(S',b')$ at time t'. Note that, as usual, if $K_j(S',b')$ belongs to different faces inside $Z(S',b')$, then we will consider it as belonging to the face $Z(\tilde{S},\tilde{b})$ of minimal dimension in which it is contained, i.e. as a minimal ω-limit set $K_l(\tilde{S},\tilde{b})$ relative to some (relatively open) attraction basin $G \subset Z(\tilde{S},\tilde{b})$.

In this way, the set of guards of $H(\mathcal{F})$ is $\{X(S,b)\colon S \neq \emptyset\}$.

The following assertion is valid.

Proposition 3.7. *Under Assumption 3.1 the above definition of the hybrid dynamical system $H(\mathcal{F})$ correctly defines for all $x_0 \in \bigcup_{b\in\{0,1\}^n} X(\emptyset,b) \setminus E$ (hence for a.e. $x_0 \in \mathbb{R}^n$) the unique hybrid trajectory $(x(\cdot),\sigma(\cdot))$ with*

$$x\colon [0,T^*) \to \mathbb{R}^n, \qquad \sigma\colon [0,T^*) \to \Sigma(\mathcal{F})$$

for some $T^* \in (0,+\infty]$ and $x(0) = x_0$, $\sigma(0) := Z(\emptyset,b)$ (i.e. is the vertex of the cube Z^n), where $b \in \{0,1\}^n$ is such that $x_0 \in X(\emptyset,b)$, while $x(t)$ satisfies (3.6) *whenever $\Sigma(t) = K_i(S,b)$.*

Proof. The hybrid trajectory is defined inductively according to the switching rule of $\mathcal{F}$. It suffices to observe that when $\Sigma(t) = K_i(S,b)$ for $t \in [t_0,t')$, $x(t_0) \in X(S,b) \setminus E$ and at time $t' > t_0$ the finite state machine switches to $K_j(\tilde{S},\tilde{b})$, which means $x(t) \in X(S,b)$ for all $t \in [t_0,t')$, $x(t') \in X(S',b')$ with $S' := S \cup \{s'\}$ and

b' differing from b by a single component $b_{s'}$, and one has $x(t') \notin E$. In fact, otherwise $x(t') \in \tilde{E}_j(S,b)$ for some $X(S,b)$ and $j \in \{0, \#S-1\}$, which would imply $x_0 = x(t_0) \in \tilde{E}_{j+1}(S,b)$ contradicting the assumption $x_0 \notin E$. □

Remark 3.8. One can have either $T^* = +\infty$ or $T^* < +\infty$. In the latter case one has the *bouncing ball* effect, which means that the limit trajectories of (1.1) (as $q \to 0^+$) may travel through the discontinuity hyperplanes infinitely many times, yet they reach the final point within a finite time interval. A simple example of such a behavior can be found in [8], which in our notation reads as follows: in (1.1) we assume $n = 2$, $\theta_1 = \theta_2 = 1$ and f_i, $i = 1, 2$, be given by

$$f_1(z_1, z_2, x_1) := 2 - 2z_2 - z_1 z_2 - x_1,$$
$$f_2(z_1, z_2, x_2) := 2z_1 - x_2.$$

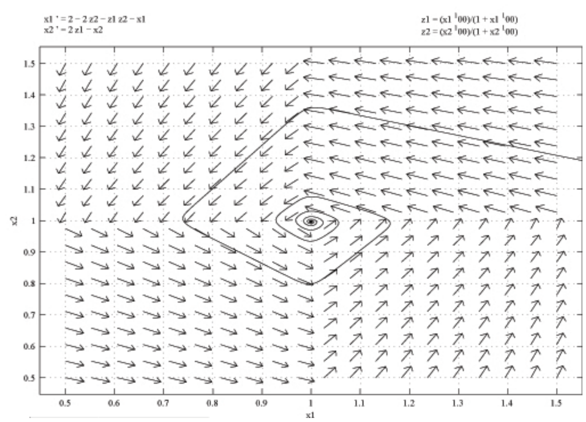


FIGURE 2. A bouncing ball behavior from the Remark 3.8 with $q = 0.01$.

Before stating the result of convergence of trajectories of (1.1) as $q \to 0^+$ to those of the hybrid dynamical system $H(\mathcal{F})$, we give the following simple example showing the dynamics of the hybrid dynamical system $H(\mathcal{F})$ and explaining all the introduced notions.

Example 3.9. Let $n = 2$, $\theta_1 = \theta_2 = 1$, and the functions f_i, $i = 1, 2$ be defined as

$$f_1(z_1, z_2, x_1) := (1 - z_1 - z_2 + 2z_1 z_2) - 0.4x_1,$$
$$f_2(z_1, z_2, x_2) := z_1 - 0.4x_2.$$

The system (3.1) looks then as follows:

$$z_1' = z_1(1 - z_1)((1 - z_1 - z_2 + 2z_1 z_2) - 0.4),$$
$$z_2' = z_2(1 - z_2)(z_1 - 0.4).$$

The finite state machine $\mathcal{F}$ has therefore 5 admissible states shown on Figure 3.9(a). Each admissible state is a singleton, and the respective invariant probability measures are just Dirac measures. Note that $K_3(\{1\}, 0) = \{(0.6, 0)\}$ is an attractive stationary point of (3.2) relative to the 1-dimensional face $Z(S,b) = [0,1] \times \{0\}$ of $Z^2 = [0,1]^2$ (here $S = \{1\}$, $R = \{2\}$, $b = b_2 = 0$) and $K_5(\emptyset, (1,1)) = \{1,1\}$ is the global attractive stationary point of (3.2) relative to the whole Z^2 and also to its faces $\{1\} \times [0,1]$ and $[0,1] \times \{1\}$ (and, of course, trivially, to the zero-dimensional face $(1,1)$ also). Note that according to our rule, we assume view this state as belonging to the face $Z(S,b)$ of minimal dimension (in this case zero, i.e. $S = \{1,2\}$, $R = \emptyset$, $b = (1,1)$). This state is responsible for the possible exit of a limit (as $q \to 0^+$) trajectory of (1.1) (i.e. a trajectory of the hybrid dynamical system $H(\mathcal{F})$) from the affine hyperplane of codimension 2 (in this case, just a point (θ_1, θ_2)) into the space (i.e. codimension zero), in this case the phase plane $\mathbb{R}^2$. The other admissible states are the just the remaining vertices of Z^2. The arrows on Figure 3.9(a) show schematically the dynamics of (3.2) in the respective faces of Z^2.

On Figure 3.9(a) one can observe three qualitatively different trajectories of the hybrid dynamical system $H(\mathcal{F})$. Name, the trajectory I starting at some point of the quadrant $\{(x_1, x_2)\colon x_1 < \theta_1, x_2 > \theta_2\}$ for some time evolves according to the system of ODEs

$$\begin{aligned} \dot{x}_1 &= -0.4x_1, \\ \dot{x}_2 &= -0.4x_2 \end{aligned}$$

(this corresponds to the state $K_1(\emptyset, (0,1))$ of $\mathcal{F}$). The trajectories of the latter system of equations are rays tending to the attractive point $P_{01} = (0,0)$. After hitting at some finite instant the line $\{x_2 = \theta_2\}$ (one of the guards of the system), the finite state machine switches to the state $K_2(\emptyset, (0,0))$, the trajectory passing to the quadrant $\{(x_1, x_2)\colon x_1 < \theta_1, x_2 > \theta_2\}$ where it obeys the law

$$\begin{aligned} \dot{x}_1 &= 1 - 0.4x_1, \\ \dot{x}_2 &= -0.4x_2 \end{aligned}$$

(the trajectories of this system converge to the attractive point $P_{00} = (5/2, 0)$). After some finite time it hits the line $\{x_1 = \theta_1\}$ (another guard), $\mathcal{F}$ switches to the state $K_3(\{1\}, 0)$ and the trajectory enters in the sliding mode along this line now obeying the law

$$\dot{x}_2 = 0.6 - 0.4x_2.$$

Again at a finite instant of time it arrives at the point (θ_1, θ_2) (singular manifold of codimension 2), $\mathcal{F}$ switches to the state and the trajectory exits into the quadrant $\{(x_1, x_2)\colon x_1 > \theta_1, x_2 > \theta_2\}$ where it will follow the system of ODEs

$$\begin{aligned} \dot{x}_1 &= 1 - 0.4x_1, \\ \dot{x}_2 &= 1 - 0.4x_2, \end{aligned}$$

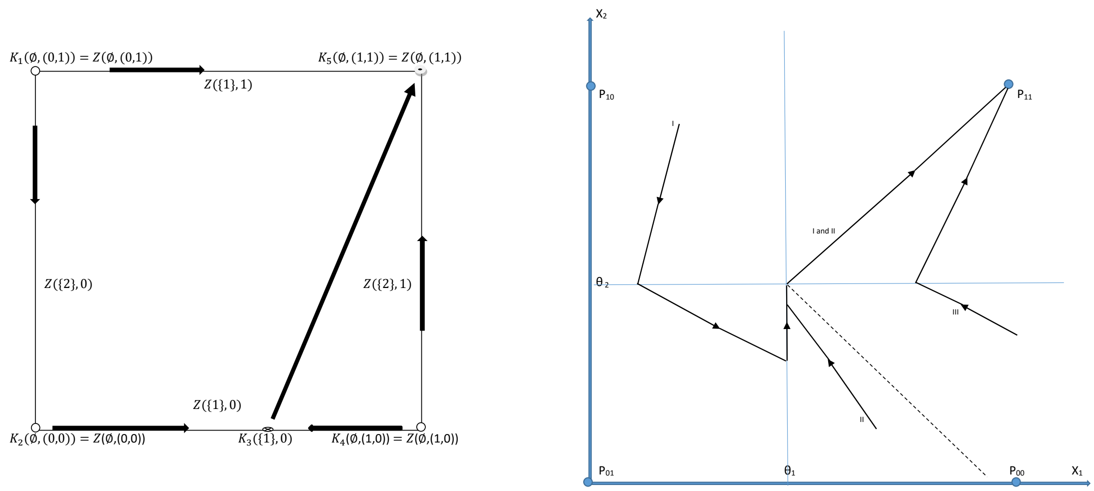
approaching, as $t \to +\infty$, the point $P_{11} = (5/2, 5/2)$. The trajectory II starts from the quadrant $\{(x_1, x_2)\colon x_1 > \theta_1, x_2 < \theta_2\}$, where it follows the law

$$\begin{aligned} \dot{x}_1 &= -0.4x_1, \\ \dot{x}_2 &= 1 - 0.4x_2, \end{aligned} \tag{3.8}$$

the finite state machine $\mathcal{F}$ being in the state $K_4(\emptyset, (1,0))$ (the trajectories of this system converge to the attractive point $P_{10} = (0, 5/2)$), then hits the line $\{x_1 = \theta_1\}$ having after that the same behavior as the trajectory I. Note that the trajectories I and II intersect, which would be impossible for the classical dynamics of the smooth ODEs (for $q > 0$). Finally, the trajectory III also starts from the quadrant $\{(x_1, x_2)\colon x_1 > \theta_1, x_2 < \theta_2\}$, so is similar to II, but it hits first the line $\{x_2 = \theta_2\}$, $\mathcal{F}$ switching from the state $K_4(\emptyset, (1,0))$ directly to the $K_5(\emptyset, (1,1))$, and the trajectory passes to the quadrant $\{(x_1, x_2)\colon x_1 > \theta_1, x_2 > \theta_2\}$ where it haves the same behavior as I and II. The dashed line on Figure 3.9(b) denotes the border separating the zone of trajectories of type III from that of trajectories of type I, and is in fact a piece of trajectory of (3.8) hitting the point (θ_1, θ_2). According to our definition, it belongs to the exceptional set E.

We now are able to state the following theorem which is the main result of this paper.

Theorem 3.10. *Let $x^q(\cdot)$ be solutions to* (1.1) *satisfying $x^q(0) = x_0^q$, while $x_0^q \to x_0 \in \bigcup_{b\in\{0,1\}^n} X(\emptyset, b) \setminus E$ as $q \to 0^+$. Then one has that $x^q(\cdot) \to x(\cdot)$ uniformly over every finite time interval $[0,T]$ with $T < T^*$, as $q \to 0^+$, where $x(\cdot)$ is the trajectory defined by the hybrid dynamical system $H(\mathcal{F})$ with $x(0) = x_0$.*



(a) (b)

FIGURE 3. Example 3.9. (a) Admissible states in $Z^2 = [0,1]^2$. (b) Three possible trajectories of the hybrid dynamics.

Remark 3.11. The choice of the Hill function for H_{q,θ_i} to describe transition from smooth to discontinuous systems is customary in applications (see e.g. [17] and references therein). Our results remain valid if we replace the Hill function with any similar function satisfying the properties described in [17] which however should be common for all discontinuities (otherwise one would not be able to apply the singular perturbation theory).

We emphasize that the limit dynamics described by Theorem 3.10 as a hybrid one, is qualitatively different from that of both smooth and switched systems (see e.g. [13]). In fact, it presents memory effects which are absent in the latter cases as the following example shows.

Example 3.12. Let in (1.1) $n = 2$ with $\theta_1 = 1$, $f_i(z, x_i) := F_i(z_1) - \gamma_i x_i$, $\gamma_i > 0$, $i = 1, 2$. Assume that $F_1(0) < \gamma_1 < F_1(1)$, so that

$$\varphi(z_1) := F_1(z_1) - \gamma_1$$

satisfies $\varphi(0) > 0$, $\varphi(1) < 0$. Assume further that φ changes sign exactly three times over $(0, 1)$, the respective roots being denoted P^1, P^2 and P^3 (with $0 < P^1 < P^2 < P^3 < 1$). Then the limit trajectories starting outside of the line $\{x_1 = \theta_1\}$ hit this line in finite time. The trajectories coming from the "left" half-plane $\{x_1 < \theta_1\}$ after hitting this line obey

$$\dot{x}_2 = F_2(P_1) - \gamma_2 x_2, \quad x_1 = \theta_1,$$

while those coming from the "right" half-plane $\{x_1 > \theta_1\}$ after hitting this line obey

$$\dot{x}_2 = F_2(P_3) - \gamma_2 x_2, \quad x_1 = \theta_1.$$

In particular, if we choose $F_2(P_1) \neq F_2(P_3)$, then the two limit trajectories coming from the half-planes $\{x_1 < \theta_1\}$ and $\{x_1 > \theta_1\}$ respectively, and hitting the line $\{x_1 = \theta_1\}$ between $F_2(P_1)/\gamma_2$ and $F_2(P_3)/\gamma_2$ will proceed after that in opposite directions (and one may easily choose initial data for both trajectories so that the hitting point and the hitting instance of time be the same; or, alternatively, so that after hitting they will meet each other in finite time). In other words, here we have two different sliding modes over the same set coexisting for some finite interval of time.

Clearly, this limit dynamics cannot be described by an ODE, namely, there is no Borel function $g\colon \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ such that each limit trajectory $x(\cdot)$ with initial data in the admissible region satisfies $\dot{x}(t) = g(t, x(t))$ for a.e. $t \in \mathbb{R}^+$. In fact, such a function g, if existed, would have to be defined for an interval of time and for every x in some line segment in the plane $\{x_1 = \theta_1\}$ between $F_2(P_1)/\gamma_2$ and $F_2(P_3)/\gamma_2$ as a vector looking simultaneously upwards and downwards, which is impossible.

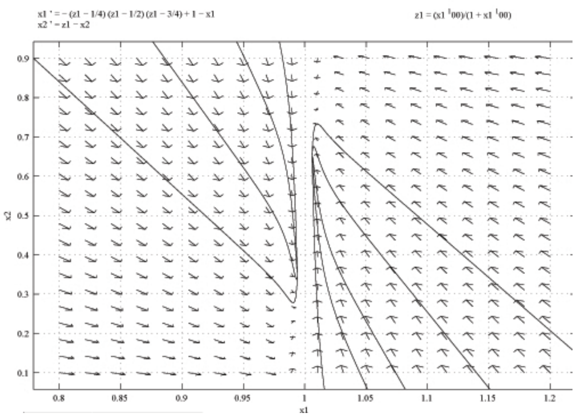


FIGURE 4. "Memory effects" in Example 3.12 with $q = 0.01$ and $P_1 = 1/4$, $P_2 = 1/2$, $P_3 = 3/4$.

We stress that, unlike multiple sliding modes constructed in [11] with the help of different nonlinear functions, our sliding trajectories are obtained by a single perturbation based on the Hill function.

4. Proof of Theorem 3.10

To prove Theorem 3.10, we need several auxiliary statements.

Lemma 4.1. *Let $(x(\cdot), \sigma(\cdot))$ be the trajectory of the hybrid dynamical system $H(\mathcal{F})$ over the time interval $[t_0, t_1]$, satisfying*

$$x(t) \in X(S, b), \qquad \sigma(t) = K_i(S, b) \tag{4.1}$$

for all $t \in [t_0, t_1)$, where $K_i(S, b) \subset \operatorname{int} Z(S, b)$. If $x^q(\cdot)$ are solutions to (1.1) satisfying $x^q(t_0) \to x(t_0)$ and

$$\operatorname{dist}(z_s^q(t_0), K_i(S, b)) \to 0 \tag{4.2}$$

as $q \to 0^+$, where

$$z_s^q(t) := H_{q,\theta_s}(x_s^q(t)), \qquad s \in S,$$

then $x^q(\cdot) \to x(\cdot)$ uniformly over $[t_0, t_1]$ as $q \to 0^+$.

Remark 4.2. Condition (4.1) means in particular that $x_s(t) = \theta_s$ for all $s \in S$ and for all $t \in [t_0, t_1]$. Therefore, the requirement $x^q(t_0) \to x(t_0)$ implies $x_s^q(t_0) \to \theta_s$ as $q \to 0^+$ for all $s \in S$.

Remark 4.3. Clearly, condition (4.2) is nonvoid only if $S \neq \emptyset$.

Proof. For every $s \in S$ we plug into (1.1) the expression

$$x_s := H_{q,\theta_s}^{-1}(z_s), \tag{4.3}$$

letting z_s to be a new unknown, and getting therefore

$$q\dot{z}_s = \frac{z_s(1-z_s)}{H^{-1}_{q,\theta_s}(z_s)} f_s\left((z_S, z_R(x_R)), H^{-1}_{q,\theta_s}(z_s)\right) \tag{4.4}$$

for all $s \in S$, where

$$z_S := \{z_s\}_{s\in S},$$
$$x_R := \{x_r\}_{r\in R},$$
$$z_R(x_R) := \{z_r(x_r)\}_{r\in R},$$

and $z_r(x_r) = H_{q,\theta_r}(x_r)$ for all $r \in R$. For $r \in R$ we have

$$\dot{x}_r = f_r\left((z_S, z_R(x_R)), \left(\{H^{-1}_{q,\theta_s}(z_\sigma)\}_{\sigma\in S}, x_R\right)\right), \tag{4.5}$$

The system of equations (4.4), (4.5) gives for every $q > 0$ the solution $(x^q_R(\cdot), z^q_S(\cdot)$ which uniquely determines the respective solution to (1.1) (by applying the substitution (4.3)).

Passing to the "rapid" time $\tau := t/q$ in (4.4), we get

$$z'_s = \frac{z_s(1-z_s)}{H^{-1}_{q,\theta_s}(z_s)} f_s\left((z_S, z_R(x_R)), H^{-1}_{q,\theta_s}(z_s)\right) \tag{4.6}$$

where now the unknowns are considered to be functions of τ, and z'_s stands for the derivative of z with respect to τ. Note that, as it is customary in the singular perturbation theory, we use the same letter z for both solutions of (4.4) (in the original time t) and of (4.6) (in the rapid time τ); the distinction is usually clear from the context (in particular, from different notation for derivatives and for the time variable).

It is worth observing at this point that (3.2) is just the formal limit as $q \to 0^+$ of the equation (4.6) minding that $x_R \in X(S, b)$. In the same vein, the formal limit as $q \to 0^+$ of the equation (4.5) is given by

$$\dot{x}_r = f_r\left((z_S, b_R), x_r\right), \tag{4.7}$$

where $b_R := \{b_r\}_{r\in R}$.

Let $\varepsilon_0 > 0$ be such that $(K_i(S, b))_{\varepsilon_0} \Subset G_i(S, b)$, where $(D)_\varepsilon$ stands for the ε-neighborhood of D. By (4.2), for every $\varepsilon \in (0, \varepsilon_0)$ there is a $q_0 = q_0(\varepsilon)$ such that

$$z^q_s(t_0) \in (K_i(S, b))_\varepsilon \Subset G_i(S, b)$$

for $q \in (0, q_0)$. Hence, by theorem I of [4] one has for $q \to 0^+$ the convergence $x^q_r(\cdot) \to x_r(\cdot)$ for all $r \in R$ uniformly over $[t_0, t_1]$, where $x(\cdot)$ is a solution to (3.6), and the narrow convergence of the Young measures $\delta_{z^q_S}$ over $[t_0, t_1] \times Z(S, b)$ corresponding to the functions z^q_S to the measure $\mathcal{L}^1 \llcorner (t_0, t_1) \otimes \nu_{i,S,b}$.

Here and below by $Car_b([a, b]; X)$ we denote the class of bounded Carathéodory functions $f : [a, b] \times X \to \mathbb{R}$ with X a separable metric space. Letting $x^q_s := H^{-1}_{q,\theta_s}(z^q_s)$ for all $s \in S$, we have that the Young measures $\delta_{x^q_S}$ over $[t_0, t_1] \times Z(S, b)$ corresponding to the functions x^q_S converge in the narrow sense to δ_{x_S}, where $x_s(\cdot) \equiv \theta_s$ for all $s \in S$. In fact, for every $f \in Car_b([t_0, t_1]; Z(S, b))$ one has

$$\begin{aligned}\int_{t_0}^{t_1} f(t, x^q_s(t))\, dt &= \int_{t_0}^{t_1} f(t, H^{-1}_{q,\theta_s}(z^q_s)(t))\, dt \\ &= \int_{[t_0,t_1]\times Z(S,b)} f(t, H^{-1}_{q,\theta_s}(\omega))\, d\delta_{z^q_s}(t, \omega) \\ &\to \int_{[t_0,t_1]\times Z(S,b)} f(t, \theta_s)\, dt\, d\nu_{i,S,b}(\omega)\end{aligned}$$

as $q \to 0^+$, because $H^{-1}_{q,\theta_s}(\cdot) \to \theta_s$ uniformly over every compact set $K \subset \operatorname{int} Z(S,b)$, and $\operatorname{supp} \nu_{i,S,b} \subset \operatorname{int} Z(S,b)$ (unless, of course, $\dim Z(S,b) = 0$, which means $S = \emptyset$). Now, since $\nu_{i,S,b}$ is a probability measure, then the above relationship means

$$\int_{t_0}^{t_1} f(t, x_s^q(t))\, dt \to \int_{[t_0,t_1]} f(t, \theta_s)\, dt,$$

hence, up to a subsequence, $x_s^q(t) \to x_s(t) \equiv \theta_s$ for all $s \in S$ pointwise over the interval $[t_0, t_1]$. It remains to observe that $|\dot{x}_s^q|$ are uniformly bounded, so that in fact this convergence of $x_s(\cdot)$ is uniform over the interval $[t_0, t_1]$ by the Ascoli-Arzelà theorem for all $s \in S$. □

Lemma 4.4. *Let $(x(\cdot), \sigma(\cdot))$ be the trajectory of the hybrid dynamical system $H(\mathcal{F})$ over the time interval $[t', t_1]$, satisfying*

$$\sigma(t) = \begin{cases} K_i(S,b) \subset \operatorname{int} Z(S,b), & t \in [t_0, t'), \\ K_j(\tilde{S},\tilde{b}) \subset \operatorname{int} Z(\tilde{S},\tilde{b}), & t \in [t', t_1), \end{cases}$$

where $\#\tilde{S} \le \#S' = \#S + 1$, $K_i(S,b) \subset G_l(S',b')$, the face $Z(S',b')$ is adjacent to $Z(S,b)$ and $K_j(\tilde{S},\tilde{b}) = K_l(S',b') \subset G_l(S',b')$ is the minimal ω-limit set relative to $G_l(S',b')$, $Z(\tilde{S},\tilde{b})$ is an $\#\tilde{S}$-dimensional face of $Z(S',b')$ which is the minimal face containing $K_j(\tilde{S},\tilde{b})$ (see Figure 5), $t' \in (t_0, t_1)$ being the instant of switching of $\mathcal{F}$ from $K_i(S,b)$ to $K_j(\tilde{S},\tilde{b})$, so that $x(t) \in X(S,b)$, $t \in [t_0, t')$, $x(t') \in X(S',b')$.

If $x^q(\cdot)$ are solutions to (1.1) satisfying $x^q(t_0) \to x(t_0) \in X(S,b) \setminus E$ and (4.2) holds as $q \to 0^+$, then $x^q(\cdot) \to x(\cdot)$ uniformly over $[t_0, t_1]$ and

$$\begin{aligned} &\operatorname{dist}(z^q_{\tilde{S}}(t'), K_j(\tilde{S},\tilde{b})) \to 0, \\ &z^q_{R^+}(t') \to b_+ \end{aligned} \tag{4.8}$$

as $q \to 0^+$, where $R_+ := S' \setminus \tilde{S}$ and $z^q_m(t) := H_{q,\theta_m}(x^q_m(t))$, $z^q_{R^+} := \{z^q_r\}_{r \in R^+}$, $z^q_{\tilde{S}} := \{z^q_s\}_{s \in \tilde{S}}$, $b_+ := \{b_r\}_{r \in R^+}$.

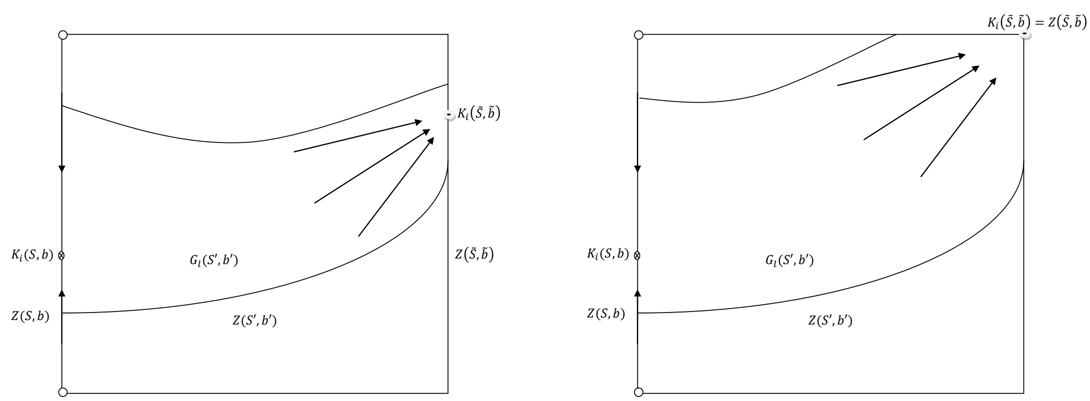


FIGURE 5. Possible switching treated in Lemma 4.4.

Remark 4.5. Under the conditions of Lemma 4.4 one clearly has $x(t) \in X(\tilde{S},\tilde{b})$ for all $t \in (t', t_1)$. In particular, if $\#\tilde{S} \le \#S$ and $S \ne \emptyset$, then the limit trajectory $x(\cdot)$ which was sliding along $X(S,b)$ before the instant t' at time t' hits $X(S',b')$ and then immediately leaves $X(S,b)$, so that sliding over $X(S,b)$ is finished at time t'. If $\tilde{S} = S = \emptyset$, then the limit trajectory $x(\cdot)$ at time t' hits $X(S',b')$ and immediately passes from $X(\emptyset,b)$ to $X(\emptyset,\tilde{b})$ without entering into sliding along $X(S',b')$.

Proof. By Lemma 4.1 one has $x^q(\cdot) \to x(\cdot)$ uniformly over $[t_0,t']$ as $q \to 0^+$, so that in particular $x^q(t') \to x(t') \in X(S',b') \setminus E$.

The proof of the limit behavior of trajectories over $[t',t_1]$ will be split in several steps.

Step 1. Denote $R_+ := S' \setminus \tilde{S}$, so that for $R' := \{1,\ldots,n\} \setminus S'$ and $\tilde{R} := \{1,\ldots,n\} \setminus \tilde{S}$ one has $\tilde{R} = R' \sqcup R_+$. Thus, $\tilde{b} := \{b_r\}_{r\in\tilde{R}}$ differs from $b' := \{b_r\}_{r\in R'}$ by $b_+ := \{b_r\}_{r\in R_+}$. It is worth noting that in the particular case $\tilde{S} = S'$ one has $R_+ = \emptyset$ and $\tilde{R} = R'$.

Let $\varepsilon_0 > 0$ be such that $(K_i(S,b))_{\varepsilon_0} \Subset G_l(S',b')$ relative to the face $Z(S',b')$. By (4.2) for every $\varepsilon \in (0,\varepsilon_0)$, there is a $q_0 = q_0(\varepsilon)$ such that

$$z_s^q(t_0) \in (K_i(S,b))_\varepsilon \Subset G_l(S',b')$$

(relative to the face $Z(S',b')$) when $q \in (0,q_0)$. Consider an arbitrary $\delta > 0$. By theorem III from [4]

$$\operatorname{dist}(z_{S'}^q(\cdot), K_j(\tilde{S},\tilde{b})) \to 0 \quad \text{as } q \to 0^+,$$

uniformly over $[t_0+\delta,t']$, so that in particular (4.8) is valid, and, moreover, there is a $q_1 = q_1(\varepsilon,\delta) \in (0,q_0)$ such that

$$z_{S'}^q(t') \in (K_j(\tilde{S},\tilde{b}))_\varepsilon \Subset G_l(S',b').$$

for $q \in (0,q_1)$. Taking an arbitrary $\bar{t} \in (t',t_1)$ and acting as in the proof of Lemma 4.1 with R', S', b' and t' instead of R, S, b and t_0, respectively, we get that as $q \to 0^+$ one has $x_r^q(\cdot) \to x_r(\cdot)$ for all $r \in R'$, uniformly over $[t',\bar{t}]$, where

$$\dot{x}_r = \int_{Z(S',b')} f_r((z_{S'},b'),x_r)\,d\nu_{l,S',b'}(z_{S'}), \qquad r \in R', \tag{4.9}$$

and the Young measures $\delta_{z_{S'}^q}$ over $[t',\bar{t}] \times Z(S',b')$ corresponding to the functions $z_{S'}^q$ converge to the measure $\mathcal{L}^1 \llcorner (t',\bar{t}) \otimes \nu_{l,S',b'}$ in the narrow sense. Note that although $G_l(S',b')$ is only relatively open in $Z(S',b')$, when using theorem I of [4] we may consider the right hand sides of the equations (4.4) and (4.5) to be extended in the Lipschitz continuous way to a neighborhood of $Z(S',b')$ in $\mathbb{R}^{\#S'}$ so that $K_l(S',b') = K_j(\tilde{S},\tilde{b})$ be a minimal ω-limit set relative to some open neighborhood G of this set in $\mathbb{R}^{\#S'}$ of the respective extended dynamical system (one can do it, say, by local reflections).

Observing that we may represent $\nu_{l,S',b'} = \nu_{j,\tilde{S},\tilde{b}} \otimes \delta_{b_+}$ (when $\tilde{S} \neq S'$, otherwise just $\nu_{l,S',b'} = \nu_{j,\tilde{S},\tilde{b}}$), we get that (4.9) can be rewritten as

$$\dot{x}_r = \int_{Z(\tilde{S},\tilde{b})} f_r((z_{\tilde{S}},\tilde{b}),x_r)\,d\nu_{j,\tilde{S},\tilde{b}}(z_{\tilde{S}}), \qquad r \in R'. \tag{4.10}$$

Further, in the same way as in the proof of Lemma 4.1, letting $x_s^q := H_{q,\theta_s}^{-1}(z_s^q)$ for all $s \in S'$, we have that the Young measures $\delta_{x_{\tilde{S}}^q}$ over $[t',\bar{t}] \times Z(\tilde{S},\tilde{b})$ corresponding to the functions $x_{\tilde{S}}^q$ converge in the narrow sense to $\delta_{x_{\tilde{S}}}$, where $x_s(\cdot) \equiv \theta_s$ for all

$s \in \tilde{S}$. Namely, for every $f \in Car_b([t', \bar{t}]; Z(\tilde{S}, \tilde{b}))$ and $s \in \tilde{S}$ one has

$$\begin{aligned}\int_{t'}^{\bar{t}} f(t, x_s^q(t))\, dt &= \int_{t'}^{\bar{t}} f(t, H_{q,\theta_s}^{-1}(z_s^q)(t))\, dt \\ &= \int_{[t',\bar{t}]\times Z(\tilde{S},\tilde{b})} f(t, H_{q,\theta_s}^{-1}(\omega))\, d\delta_{z_s^q}(t,\omega) \\ &\to \int_{[t',\bar{t}]\times Z(S,b)} f(t,\theta_s)\, dt\, d\nu_{i,S,b}(\omega) = \int_{t'}^{\bar{t}} f(t,\theta_s)\, dt\end{aligned}$$

as $q \to 0^+$, because $\operatorname{supp} \nu_{j,\tilde{S},\tilde{b}} \subset K_j(\tilde{S}, \tilde{b}) \subset \operatorname{int} Z(\tilde{S}, \tilde{b})$ and $H_{q,\theta_s}^{-1} \to \theta_s$ uniformly over every compact subset of $\operatorname{int} Z(S,b)$ (unless, of course, $\dim Z(S,b) = 0$, which means $S = \emptyset$). Therefore, $x_s^q(\cdot) \to \theta_s$, $s \in \tilde{S}$, uniformly over $[t', \bar{t}]$ as $q \to 0^+$.

Step 2. It remains now in the case $\tilde{S} \neq S'$ to study the limit dynamics of $x_{R_+}^q(\cdot)$ over $[t', t_1]$ as $q \to 0^+$ and show that the trajectories converge uniformly to the trajectory of the system of ODEs

$$\dot{x}_r = \int_{Z(\tilde{S},\tilde{b})} f_r((z_{\tilde{S}}, \tilde{b}), x_r)\, d\nu_{j,\tilde{S},\tilde{b}}(z_{\tilde{S}}), \qquad r \in R_+, \tag{4.11}$$

starting from the initial point $x_{R_+}(t')$. To this aim take a sequence $\{t_j\} \subset (t', t_1)$, $t_j \searrow t'$ as $j \to \infty$. Proceed now by induction. For $t = t_1$ consider a sequence Q_1 of q such that the sequence $\{x_{R_+}^q(\cdot)\}_{q\in Q_1}$ is convergent as $q \to 0^+$, $q \in Q_1$. Denoting x_{1,R_+} the limit of the latter, we have that $H_{\theta_r,0}(x_{1,r}) = \lim_q z_r^q(t_1) = b_r$ for every $r \in R_+$ as proven on Step 1. Acting as in the proof of Lemma 4.1 with R_+, $\tilde{S}$ and b_+ instead of R, S and b respectively over the interval $[t_1, \bar{t}]$ we get that as $q \to 0^+$, $q \in Q_1$, one has $x_r^q(\cdot) \to x_r(\cdot)$ for all $r \in R_+$, uniformly over $[t_1, \bar{t}]$, where $x_{R_+}(\cdot)$ is the solution to (4.11) satisfying the initial condition $x_{R_+}(t_1) = x_{1,R_+}$. Suppose now that for each $j = 1, \ldots, k-1$ one has chosen a sequence Q_j with $Q_j \subset Q_{j-1}$ for $j \neq 1$, such that $x_r^q(\cdot) \to x_r(\cdot)$ for all $r \in R'$, uniformly over $[t_j, \bar{t}]$, where $x_{R_+}(\cdot)$ is the solution to (4.11) satisfying the initial condition

$$x_{R_+}(t_j) = x_{j,R_+} := \lim_{q\to 0^+, q\in Q_j} x_{R_+}^q(t_j) = \lim_{q\to 0^+, q\in Q_k} x_{R_+}^q(t_j).$$

For $j = k+1$ choosing a subsequence Q_{k+1} of Q_k such that there exists a limit

$$x_{k+1,R_+} := \lim_{q\to 0^+, q\in Q_{k+1}} x_{R_+}^q(t_{k+1}),$$

we get again $H_{\theta_r,0}(x_{k+1,r}) = \lim_q z_r^q(t_{k+1}) = b_r$ for every $r \in R_+$ and acting once more as in the proof of Lemma 4.1 with R_+, $\tilde{S}$ and b_+ instead of R, S and b, respectively, over the interval $[t_{k+1}, \bar{t}]$ we get that as $q \to 0^+$, $q \in Q_{k+1}$, one has $x_r^q(\cdot) \to x_r(\cdot)$ for all $r \in R_+$, uniformly over $[t_{k+1}, \bar{t}]$, where $x_{R_+}(\cdot)$ is the solution to (4.11) satisfying the initial condition $x_{R_+}(t_{k+1}) = x_{k+1,R_+}$. Note that by uniqueness of the solution to this Cauchy problem one has $x_{R_+}(t_j) = x_{j,R_+}$ for all $j = 1, \ldots, k$.

In this way we have defined a trajectory $x_{R_+}(\cdot)$ of (4.11) over $(t', \bar{t}]$. Since $|x_{R_+}^q(t_j) - x_{R_+}^q(t')| \leq C|t_j - t'|$ for some $C > 0$ independent of q by uniform boundedness of $|\dot{x}_j^q|$, then passing in the above estimate to the limit as $q \to 0^+$, $q \in Q_j$, we get

$$|x_{R_+}(t_j) - x_{R_+}(t')| \leq C|t_j - t'|.$$

Thus $x_{R_+}(\cdot)$ extends by continuity to the whole interval $[t', \bar{t}]$ and starts from the initial point $x_{R_+}(t')$ as $t = t'$.

We show now that in fact for every sequence of $q \to 0^+$ one has that $x_r^q(t) \to x_r(t)$ for all $r \in R_+$ for all $t \in (t', \bar{t})$. Clearly, it is enough to show this for $t = t_j$ with an arbitrary $j \in \mathbb{N}$. Suppose the contrary, i.e. that for some $j \in \mathbb{N}$ there is a sequence Q'_j of positive numbers such that $x_{R_+}^q(t_j) \to x'_{j,R_+} \neq x_{R_+}(t_j)$ as $q \to 0^+$, $q \in Q'_j$, then proceeding as above by induction we define for all $k \in \mathbb{N}$, $k > j$, a subsequence $Q'_k \subset Q'_{k-1}$ such that as $q \to 0^+$, $q \in Q'_k$, one has

$$x^q_{R_+}(\cdot) \to x'_{R_+}(\cdot)$$

uniformly over $[t_k, \bar{t}]$, where $x'_{R_+}(\cdot)$ solves (4.11) satisfying the initial condition

$$x'_{R_+}(t_k) = x'_{k,R_+} := \lim_{q \to 0^+, q \in Q'_k} x^q_{R_+}(t_k).$$

Again by uniqueness of the solution to this Cauchy problem one has $x'_{R_+}(t_l) = x'_{l,R_+}$ for all $l = j, \ldots, k$. This defines a trajectory $x'_{R_+}(\cdot)$ of (4.11) over $(t', \bar{t}]$, and exactly as above one proves that

$$|x'_{R_+}(t_k) - x_{R_+}(t')| \leq C|t_k - t'|.$$

Thus $x'_{R_+}(\cdot)$ also may be extended by continuity to the whole interval $[t', \bar{t}]$ and starts from the same initial point $x_{R_+}(t')$ as $t = t'$. By uniqueness of the solution to the Cauchy problem for (4.11) one has thus that $x'_{R_+}(\cdot) = x_{R_+}(\cdot)$, which is not the case because by construction they do not coincide at $t = t_j$. This contradiction shows the claim.

It remains to observe that since $|\dot{x}^q_{R_+}|$ are uniformly bounded, then in fact the convergence of $x^q_{R_+}(\cdot)$ is uniform over the interval $[t', t_1]$ by the Ascoli-Arzelà theorem. □

We are now able to prove Theorem 3.10.

Proof of Theorem 3.10. Let Θ stand for the set of $T > 0$ such that $x^q(\cdot) \to x(\cdot)$ uniformly over $[0, T]$ as $q \to 0^+$, where $(x(\cdot), \sigma(\cdot))$ is the trajectory of $H(\mathcal{F})$ over the time interval $[0, T]$. We show first that, Θ is (relatively) open in $[0, T^*)$. In fact, if $T \in \Theta$, then the following can happen:

(i) $\sigma(t) = K_i(S, b)$ for some $K_i(S, b) \in \Sigma(F)$, $K_i(S, b) \in \mathrm{int}Z(S, b)$ and for all $t \in (T - \varepsilon, T + \varepsilon]$ with some $\varepsilon > 0$, that is, no switching of $\mathcal{F}$ occurs over this time interval. Then $x(t) \in X(S, b)$ over, say, $[T - \varepsilon/2, T + \varepsilon]$, hence, by Lemma 4.1 $x^q(\cdot) \to x(\cdot)$ uniformly over $[T - \varepsilon/2, T + \varepsilon]$ as $q \to 0^+$, and thus, in particular, $[T, T + \varepsilon] \subset \Theta$.

(ii) The situation is that of Lemma 4.4, that is, $t = T$, and the finite state machine $\mathcal{F}$ switches from the state $K_i(S, b) \subset \mathrm{int}Z(S, b)$ to the state $K_j(\tilde{S}, \tilde{b}) \subset \mathrm{int}Z(\tilde{S}, \tilde{b})$ and remains in the latter state for some time (say, for $t \in [T, T+\varepsilon]$ for some $\varepsilon > 0$), with $\#\tilde{S} \leq \#S' = \#S + 1$. We apply now Lemma 4.4 to show that $x^q(\cdot) \to x(\cdot)$ uniformly over $[T, T + \varepsilon]$ as $q \to 0^+$, which provides $[T, T + \varepsilon] \subset \Theta$.

Thus we have proven that Θ is relatively open in $[0, T^*)$.

On the other hand, Θ is closed. In fact, if $T_\nu \to T$ as $\nu \to \infty$ with $T_\nu \in \Theta$, then this clearly implies $x^q(t) \to x(t)$ as $q \to 0^+$ for all $t < T$ and thus, minding that $|\dot{x}^q_j| \leq C$, $j = 1, \ldots, n$, for some $C > 0$, we have for $t < T$ the estimate

$$\begin{aligned}
\lim_q |x_j(T) - x^q_j(T)| &\leq |x_j(T) - x_j(t)| + \lim_q |x^q_j(T) - x^q_j(t)| \\
&\quad + \lim_q |x_j(t) - x^q_j(t)| \\
&\leq |x_j(T) - x_j(t)| + C|T - t|.
\end{aligned}$$

Passing to the limit in $t \to T^-$, using the continuity of $x(\cdot)$, we have that $x^q(\cdot) \to x(\cdot)$ as $q \to 0^+$ pointwise and hence also uniformly (since $|\dot{x}^q_j| \leq C$) over $[0, T]$.

Since $\Theta \neq \emptyset$, we have $\Theta = [0, T^*)$ which concludes the proof. □

5. Limits of bunches of trajectories as flows of measures

As Example 3.12 shows, the dynamics of limit trajectories is essentially nonlocal and cannot be described by an ODE, namely, there is no Borel function $g\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ such that each limit trajectory $x(\cdot)$ with $x(0) \in \Omega$ satisfies $\dot{x}(t) = g(t, x(t))$ for a.e. $t \in \mathbb{R}^+$. Nevertheless, we show now, that if instead of considering the evolution under the limit dynamics of every single initial datum, one considers the evolution of an ensemble of initial data, then its behavior can be described by an ODE, i.e. the ensemble as a whole behaves *as if each single trajectory is a solution of an ODE* as above. Namely, if we consider an ensemble of initial data modeled by a finite positive Borel measure μ over Ω, then the evolution of this ensemble along the trajectories of the system of ODEs (1.1) at $q > 0$ is well-known to be given by a family of measures parameterized by time satisfying the continuity equation with the velocity field given by the right-hand side of (1.1). We show in a simple result below that also the evolution of this ensemble along the limit trajectories (as $q \to 0^+$) is given by a family of measures parameterized by time satisfying the continuity equation with some, possibly time-dependent, Borel velocity field g_t, and, moreover, μ-a.e. initial point $x \in \Omega$ is evolving along the trajectories of an ODE with the right-hand side given by the velocity field g_t. Note that as opposed to what has been done in the previous sections, here we model the bunch of trajectories supporting the flow of the initial measure as a Young measure representing this flow; the limit then is in the narrow sense of Young measures. This may be seen as a "statistical" point of view on the limit of trajectories.

For a $q > 0$ we denote by $x^{q,t}\colon \mathbb{R}^n \to \mathbb{R}^n$, the solution flow for (1.1), i.e. $x^{q,t}(x_0) := x(t)$, $t \geq 0$, where $x(\cdot)$ solves (1.1) with the initial condition $x(0) = x_0$.

Proposition 5.1. *Let μ be a finite positive Borel measure over Ω. Then there is a Borel function $g\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ such that, fixed an arbitrary $T > 0$, there is a finite positive Borel measure η over $C([0,T];\mathbb{R}^n)$ concentrated over absolutely continuous solutions of the ODE*

$$\dot{x}(t) = g(t, x(t)) \quad \text{for a.e. } t \in [0,T] \tag{5.1}$$

with $x(\cdot) \in \Omega$ such that

(i) *$x^{q,t}_{\#}\mu \otimes dt \rightharpoonup \mu_t \otimes dt$, in the narrow sense of Young measures over $[0,T] \times \Omega$ as $q \to 0^+$, where $\mu_t := e_{t\#}\eta$, are finite positive Borel measures over Ω of the same total mass (and in particular, $\mu_0 = \mu$);*

(ii) *the family of measures $\{\mu_t\}_{t\in[0,T]}$ coincides a.e. over $[0,T]$ with a narrow continuous curve of measures and satisfies the continuity equation*

$$\partial_t \mu_t + \operatorname{div} g_t \mu_t = 0 \tag{5.2}$$

in the weak sense (with $g_t(x) := g(t,x)$), i.e.

$$-\int_0^T \dot{\psi}(t)\,dt \int_{\mathbb{R}^n} \phi(x)\,d\mu_t(x) - \int_0^T \psi(t)\,dt \int_{\mathbb{R}^n} \nabla\phi(x)\cdot g_t(x)\,d\mu_t(x) = 0 \tag{5.3}$$

for every $\psi \in C_0^1(0,T)$, $\phi \in C_0^\infty(\mathbb{R}^n)$.

Remark 5.2. Proposition 5.1 may be interpreted as follows: disintegrating $\eta = \mu \otimes \eta_x$, where each η_x is a Borel probability measure concentrated over solutions of the Cauchy problem $\dot{x}(t) = g(t,x)$, $x(0) = x$, we have that the evolution of μ-a.e. $x \in \Omega$ under the limit dynamics (i.e. as $q \to 0^+$) is given by the measure $e_{t\#}\eta_x$, that is, this point is evolved along the trajectories of the above Cauchy problem.

Proof. Since

$$\frac{d}{dt} x^{q,t} = g^q(x^{q,t}),$$

where $g_i^q(x) := f\left(\{H_{q,\theta_j}\}_{j=1}^n, x_i\right)$, $i = 1, \ldots, n$, then, denoting for brevity by $\mu_t^q := x_\#^{q,t}\mu$, we have that μ_t^q are finite positive Borel measures over Ω satisfying the continuity equation

$$\partial_t \mu_t^q + \operatorname{div} g^q \mu_t^q = 0$$

in the weak sense, i.e.

$$-\int_0^T \dot{\psi}(t)\,dt \int_{\mathbb{R}^n} \phi(x)\,d\mu_t^q(x) - \int_0^T \psi(t)\,dt \int_{\mathbb{R}^n} \nabla\phi(x)\cdot g^q(x)\,d\mu_t^q(x) = 0$$

for every $\psi \in C_0^1(0,T)$, $\phi \in C_0^\infty(\mathbb{R}^n)$. Since Ω is compact, all μ_t^q have the same mass and g^q are uniformly bounded, then up to a subsequence of q (not relabeled), we have $\mu_t^q \otimes dt \rightharpoonup \mu_t \otimes dt$ and $g^q\mu_t^q \otimes dt \rightharpoonup v_t \otimes dt$ in the narrow sense of Young measures over $[0,T]\times\Omega$ as $q \to 0^+$, where μ_t are some positive Borel measures and v_t are some Borel vector measures (with values in $\mathbb{R}^n$), and

$$-\int_0^T \dot{\psi}(t)\,dt \int_{\mathbb{R}^n} \phi(x)\,d\mu_t(x) - \int_0^T \psi(t)\,dt \int_{\mathbb{R}^n} \nabla\phi(x)\cdot\,dv_t(x) = 0 \tag{5.4}$$

for every $\psi \in C_0^1(0,T)$, $\phi \in C_0^\infty(\mathbb{R}^n)$. But since for all $x_0 \in \Omega$ one has $g^q(x^{q,t}(x_0)) \in \Omega$, then in particular, for some $C > 0$ independent of q one has $|g^q(x_0)| \le C$ for μ_t^q-a.e. $x_0 \in \Omega$, $|\cdot|$ standing for the Euclidean norm in $\mathbb{R}^n$. Therefore $|v_t|\otimes dt \ll \mu_t \otimes dt$, i.e.

$$v_t \otimes dt = g_t \mu_t \otimes dt$$

for some Borel $g\colon [0,T]\times\Omega \to \mathbb{R}^n$, $g_t(x) := g(t,x)$, and thus (5.4) implies (5.3). Taking $T := m$ for $m \in \mathbb{N}$, by a diagonal procedure we may assume g to be extended over $\mathbb{R}^+\times\Omega$, and then extend it to $\mathbb{R}^+\times\mathbb{R}^n$ in an arbitrary way. Due to lemma 8.1.2 from [2], up to changing $\{\mu_t\}_{t\in[0,T]}$ over a set of zero Lebesgue measure of t, the latter family of measures is a narrow continuous in t. Finally, the existence of an η as claimed in the statement being proven such that $\mu_t := e_{t\#}\eta$ is due to the superposition principle for continuity equations (theorem 12 from [1]). □

Note that the above Proposition 5.1 does not say much about the limit dynamics; for instance, the dynamics of every finite ensemble of particles moving along a finite number of arbitrary absolutely continuous curves is as in this Proposition. Namely, it is easy to define a Borel velocity field g_t and, given a positive measure μ being a finite linear combination of Dirac masses in the initial points of those curves, to define also a measure η over $C([0,T];\mathbb{R}^n)$ concentrated over absolutely continuous solutions of the ODE (5.1) such that for $\mu_t := e_{t\#}\eta$, the continuity equation (5.2) (with $g_t(x) := g(t,x)$) being satisfied in the weak sense (moreover, the velocity field may be chosen time independent in this case). Moreover, for every curve of positive Borel measures $\{\mu_t\}_{t\in[0,T]}$ satisfying just the mild assumption of absolute continuity with respect to some Kantorovich-Wasserstein distance W_p, $p > 1$, one can claim the existence of a Borel vector field $g_t(x) := g(t,x)$ such that (5.2) holds in the weak sense and of a finite positive Borel measure η over $C([0,T];\mathbb{R}^n)$ concentrated over absolutely continuous solutions of the ODE (5.1) satisfying $e_{t\#}\eta = \mu_t$ for all $t \in [0,T]$ (see, e.g., theorem 8.2.1 from [2]). Note also that the function g is defined in fact just $\mu_t \otimes dt$-a.e. Therefore, the result of Proposition 5.1 is a very rough property of a quite large class of dynamical systems rather than being specific to the particular class considered.

6. Generalizations and open problems

(i) In (1.1) we always assumed that each variable x_i produces only one discontinuity hyperplane $x_i = \theta_i$. However, almost nothing will change if we let some variables give rise to several discontinuity hyperplanes $x_i = \theta_{ij}$, $j = 1, ..., n_j$. In this case, $z = (z_{ij})$ where $z_{ij} := H_{q,\theta_{ij}}(x_i)$, $i = 1, ..., n; j = j = 1, ..., n_j$. The discrete state machine is designed in a similar way as before, and the proofs of the main results remain valid up to the notational changes. In genetic models this case corresponds to a network with several activation/deactivation thresholds (see [17] for more details on how to deal with this case).

(ii) In the bouncing ball case, where trajectories approach a stationary point P called sometimes the *Zeno breaking point* [3] within a finite time interval $[0, T^*]$. Our result only justifies the limit behavior up to $t = T^*$. The trajectories of the smooth systems are also defined for $t > T^*$, but the limit behavior is not described by our result.

The problem of how to define hybrid dynamics beyond a Zeno breaking point is only partially studied. Some results for particular classes of hybrid systems can be found, e.g. in [3], where, however, the hybrid dynamics is studied without connections with approximating smooth dynamical systems. Numerical simulations around a Zeno breaking point show that inserting steep Hill functions into (1.1) may give a single asymptotically stable point, or may produce splitting of the breaking point in one stable and one unstable point. In either case, it would mean that the trajectories of the limit (as $q \to 0^+$) hybrid system never leave a Zeno breaking point after the bouncing ball regime is completed. Proving (or disproving) this conjecture seems to be a challenging open problem.

(iii) Another interesting (though probably more technical) open problem would be determining the limit behavior of the solutions of a more general system

$$\dot{x} = f(z, x),$$

where $x = (x_1, ..., x_n)$, $z = (z_1, ..., z_n)$, $z_i = H_{q,\theta_i}(x_i)$, $i = 1, \dots, n$, because the latter cannot be handled in a similar way as the system (1.1). The main obstacle is the definition of the corresponding discrete state machine, which would depend on continuous parameters x_i. This is readily seen from its description based on (3.2), which in our case is always independent of x_i.

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