

LYAPUNOV STABILITY OF THE GENERALIZED STOCHASTIC PANTOGRAPH EQUATION

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ABSTRACT. The purpose of the paper is to study stability properties of the generalized stochastic pantograph equation, the main feature of which is the presence of unbounded delay functions. This makes the stability analysis rather different from the classical one. Our approach consists in linking different kinds of stochastic Lyapunov stability to specially chosen functional spaces. To prove stability, we check that the solutions of the equation belong to a suitable space of stochastic processes, instead of searching for an appropriate Lyapunov functional. This gives us possibilities to study moment stability, stability with probability 1 and many other stability properties in an efficient way. We show by examples how this approach works in practice, putting emphasis on delay-independent stability conditions for the generalized stochastic pantograph equation. The framework can be applied to any stochastic functional differential equation with finite dimensional initial conditions.

1. INTRODUCTION

In this paper we study Lyapunov stability of the stochastic pantograph equation (see e.g. [4], [21], [23]):

$$(1) \quad dx(t) = \left[Ax(t) + \sum_{j=0}^{m_0} A_{j0}x(g_{j0}t) \right] dt + \sum_{l=1}^{m-1} \sum_{j=0}^{m_l} A_{jl}x(g_{jl}t) d\mathcal{B}_l(t),$$

where $0 < g_{jl} < 1$, and its generalizations (see the equation (5) in Section 4). A very good and comprehensive description of the role of the classical pantograph equation and its stochastic counterpart, including historical comments, can be found in the paper [21]. Let us only mention that generalizations of the pantograph equations have also attracted attention of many mathematicians, see e.g. [5], [7], [8], [11], [15], [17], [20], [22] and the references therein.

Stability analysis of the equations (1) and (5) has a special feature: the delay is unbounded, so that many methods, including those based on Lyapunov-Krasovskii functionals, are inapplicable. One uses therefore various special techniques, which can e.g. be found in the papers [1], [2] (the stochastic case) and [7] (the deterministic case). These techniques help to produce verifiable stability criteria, mostly in the case of the classic pantograph equation (1).

Our approach goes back to the framework developed in the monographs [16] (for linear differential equations in Banach spaces) and [3] (for linear deterministic functional differential equations), where Lyapunov stability is replaced by *input-to-state stability*, i.e. the property of the equation where its solutions belong to certain linear topological spaces and continuously depend (in the corresponding topology) on the initial data. In the stochastic case this approach is outlined in [12]. On the other hand, the equations (1) and (5) possess a very specific property: their initial conditions are finite dimensional, i.e. identical to the ones for ordinary differential equations. This considerably simplifies the analysis of the input-to-state stability, as all linear finite dimensional operators are bounded, and we only need to prove that

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all solutions of the equation belong to a certain topological space. For brevity, we will call this property *M-stability* keeping in mind that this is, in fact, a particular case of the input-to-state stability for linear equations with finite dimensional spaces of initial data.

The idea of how to verify the property of input-to-state stability for linear deterministic functional differential equations goes back to the papers of N.V. Azbelev and his students (see [3] and the references therein) who call their technique *the W-method*. It is somewhat similar to Lyapunov's direct method. But instead of seeking a Lyapunov function(al) one aims to find a suitable reference equation which possesses the prescribed asymptotic property and which then is used to regularize the original equation. Like Lyapunov's method, the *W-method* also provides necessary and sufficient stability conditions. The *W-method* proven to be rather efficient for many classes of delay equations, especially those where searching for Lyapunov functionals seems to be difficult. Equations with infinite delays can serve as a prominent example of such a class.

In [9], the method was for the first time applied to linear stochastic functional differential equations and developed further by the authors in the series of publications (see the review article [12]). The first efficient stability conditions for stochastic differential equations with unbounded delays, obtained by the *W-method*, were presented in the paper [11]. In the present paper we develop this approach further by concentrating on specific stochastic equations with unbounded delays and finite dimensional initial conditions: the pantograph equation (1) and its generalization (5). In the examples below (see Section 4) we only use the simplest reference equation $\dot{x} + \alpha x = g$, where $\alpha > 0$ is a parameter, variation of which ensures best possible stability conditions. More sophisticated reference equations (e.g. those including delays) can be found in other publications of the authors.

The paper is organized as follows.

In Section 2 we introduce some notation and define the general linear stochastic functional differential equation, which is used in Section 3 in the definitions of different kinds of stochastic stability. In this section we also offer a precise definition of *M-stability* for different spaces of stochastic processes. The central result of Section 3 describes relationship between stochastic Lyapunov stability and *M-stability*, where we relate specially defined spaces of stochastic processes to different kinds of stochastic Lyapunov stability.

Let us remark that the role of the definitions and results presented in Section 3 goes far beyond the applications to the stability analysis of the generalized stochastic pantograph equation. Having in mind these future applications, we chose to formulate and prove the results of Section 3 for the case of the general functional differential equation (2). In addition to the equation (5), the equation (2) covers integro-differential equations arising e.g. in electrical circuit analysis [13].

Section 4 contains applications to the stability analysis of the generalized stochastic pantograph equation (5), but we stress that most results are also new for the stochastic pantograph equation (1). This includes e.g. conditions of *p-stability* for $p \neq 2$, stability with probability 1 and stability conditions in the vector case.

Finally, Section 5 contains a short overview of the main results of the paper as well as some suggestions on further applications of the developed method.

2. NOTATION AND PRELIMINARIES

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a stochastic basis (see e.g. [14]), where Ω is a set of elementary probability events, \mathcal{F} is a σ -algebra of all events on Ω , $(\mathcal{F}_t)_{t \geq 0}$ is a right continuous family of σ -subalgebras of \mathcal{F} , P is a probability measure on \mathcal{F} ; all the above σ -algebras are assumed to be complete with respect to (w.r.t. in what follows)

the measure P , i.e. they contain all subsets of zero measure; the symbol E stands for the expectation related to the probability measure P .

In the sequel, we use an arbitrary yet fixed norm $|\cdot|$ in R^n , the real-valued index p satisfying the assumption $0 \leq p \leq \infty$ and a continuous positive function $\gamma(t)$ defined for all $t \geq 0$.

By $Z = (z_1, \dots, z_m)^T$ we denote an m -dimensional semimartingale (see e.g. [14]), while $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_m)^T$ stands for the standard Brownian motion (the Wiener process), the particular case of Z .

The general linear stochastic functional differential equation is defined as follows (see e.g. [12]):

$$(2) \quad dx(t) = (Vx)(t)dZ(t) \quad (t \geq 0),$$

and the initial condition reads in this case as

$$(3) \quad x(0) = x_0 \in R^n.$$

Here V is a k -linear Volterra operator (see below), which is defined in certain linear spaces of vector-valued stochastic processes.

By the k -linearity of the operator V we mean the property

$$V(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 Vx_1 + \alpha_2 Vx_2,$$

which holds for all \mathcal{F}_0 -measurable, bounded and scalar random values α_1, α_2 and all stochastic processes x_1, x_2 belonging to the domain of the operator V .

According to the paper [9] the following classes of linear stochastic equations can be rewritten in the form (2):

- a). *Systems of linear ordinary (i.e. non-delay) stochastic differential equations driven by an arbitrary semimartingale (in particular, systems of ordinary Itô equations);*
- b). *Systems of linear stochastic differential equations with discrete delays driven by a semimartingale (in particular, systems of Itô equations with discrete delays);*
- c). *Systems of linear stochastic differential equations with distributed delays driven by a semimartingale (in particular, systems of Itô equations with distributed delays);*
- d). *Systems of linear stochastic integro-differential equations driven by a semimartingale (in particular, systems of Itô integro-differential equations).*
- e). *Systems of linear stochastic functional difference equations driven by a semimartingale (in particular, systems of Itô functional difference equations).*

Of course, the stochastic pantograph equation (1) and its generalization (5) are also particular cases of the general functional differential equation (2).

3. LYAPUNOV STABILITY AND M -STABILITY

In this section we study different kinds of stochastic Lyapunov stability of the zero solution of the linear equation (2) with respect to the initial data (3). Let us start with the precise definitions.

Definition 3.1. *The zero solution of the equation (2) is called*

- (1) *weakly stable in probability if for any $\epsilon > 0, \delta > 0$ there is $\eta(\epsilon, \delta) > 0$ such that $P\{\omega \in \Omega : |x(t, x_0)| > \epsilon\} < \delta$ for all $|x_0| < \eta$ and $t \geq 0$;*
- (2) *asymptotically weakly stable in probability if it is weakly stable in probability and if, in addition, for any $\epsilon > 0$ and all $x_0 \in R^n$ one has $P\{\omega \in \Omega : |x(t, x_0)| > \epsilon\} \rightarrow 0$ as $t \rightarrow +\infty$;*
- (3) *stable in probability if for any $\epsilon, \delta > 0$ there is $\eta(\epsilon, \delta) > 0$ such that $P\{\omega \in \Omega : \sup_{t \geq 0} |x(t, x_0)| > \epsilon\} < \delta$ for all $|x_0| < \eta$;*

- (4) *asymptotically stable in probability if it is stable in probability and if, in addition, for any $\epsilon > 0$ and all $x_0 \in R^n$ one has $P\{\omega \in \Omega : |x(t, x_0)| > \epsilon\} \rightarrow 0$ as $t \rightarrow +\infty$;*
- (5) *p-stable if for any $\epsilon > 0$ there is $\eta(\epsilon) > 0$ such that $|x_0| < \eta$ implies $E|x(t, x_0)|^p \leq \epsilon$ for all $t \geq 0$;*
- (6) *asymptotically p-stable if it is p-stable and, in addition, $\lim_{t \rightarrow +\infty} E|x(t, x_0)|^p = 0$ for all $x_0 \in R^n$;*
- (7) *exponentially p-stable if there exist positive constants K, β such that the inequality $E|x(t, x_0)|^p \leq K|x_0|^p \exp\{-\beta t\}$ holds true for all $t \geq 0$ and all $x_0 \in R^n$;*
- (8) *stable with probability 1 if $\sup_{t \geq 0} |x(t, x_\nu)| \rightarrow 0$ with probability 1 whenever $|x_\nu| \rightarrow 0$ as $\nu \rightarrow +\infty$;*
- (9) *asymptotically stable with probability 1 if it is stable with probability 1 and if, in addition, $|x(t, x_0)| \rightarrow 0$ as $t \rightarrow +\infty$ for all $x_0 \in R^n$;*
- (10) *strongly stable with probability 1 if for any $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that $P\{\omega \in \Omega : \sup_{t \geq 0} |x(t, x_0)| \leq \epsilon\} = 1$ whenever $|x_0| < \eta$;*
- (11) *strongly asymptotically stable with probability 1 if it is strongly stable with probability 1 and if, in addition, for any $\epsilon > 0$ $x(t, x_0)$ tends to 0 with probability 1 as $t \rightarrow +\infty$ for all $x_0 \in R^n$.*

Remark 3.2. The initial condition x_0 can also be random. In this case the norm of x_0 should be adjusted accordingly.

For brevity, we will also write "the equation (2) is stable" in a certain sense instead of "the zero solution of the equation (2) is stable" in this sense.

In the sequel the following linear spaces of stochastic processes will be used:

- $L^n(Z)$ consists of all predictable $n \times m$ -matrix stochastic processes on $[0, +\infty)$, the rows of which are locally integrable w.r.t. the semimartingale Z (see e.g. [14]);
- D^n consists of all n -dimensional stochastic processes on $[0, +\infty)$, which can be represented as

$$x(t) = x(0) + \int_0^t H(s) dZ(s),$$

where $x(0) \in R^n$, $H \in L^n(Z)$.

The spaces below ("M-spaces") are crucial for studying the stochastic Lyapunov stabilities listed above.

$$M_0^\gamma = \{x : x \in D^n \text{ such that for any } \delta > 0 \text{ there is } K > 0, \text{ for which } \sup_{t \geq 0} P\{\omega : \omega \in \Omega, |\gamma(t)x(t)| > K\} < \delta\};$$

$$\hat{M}_0^\gamma = \{x : x \in D^n \text{ such that for any } \delta > 0 \text{ there is } K > 0, \text{ for which } P\{\omega : \omega \in \Omega, \sup_{t \geq 0} |\gamma(t)x(t)| > K\} < \delta\};$$

$$M_p^\gamma = \{x : x \in D^n, \sup_{t \geq 0} E|\gamma(t)x(t)|^p < \infty\} \quad (0 < p < \infty);$$

$$\hat{M}_p^\gamma = \{x : x \in D^n, E \sup_{t \geq 0} |\gamma(t)x(t)|^p < \infty\} \quad (0 < p < \infty);$$

$$M_\infty^\gamma = \hat{M}_\infty^\gamma = \{x : x \in D^n, \text{ess sup}_{(t, \omega) \in [0, +\infty[\times \Omega} |\gamma(t)x(t)| < \infty\};$$

For $\gamma(t) = 1$ ($t \geq 0$) we also put $M_p^1 = M_p$ and $\hat{M}_p^1 = \hat{M}_p$ ($0 \leq p \leq \infty$).

Let B be a linear subspace of the space $L^n(Z)$ equipped with some norm $\|\cdot\|_B$. For a given positive and continuous function $\gamma(t)$ ($t \in [0, \infty)$) we define $B^\gamma = \{f : f \in B, \gamma f \in B\}$. The latter space becomes a linear normed space if we put $\|f\|_{B^\gamma} := \|\gamma f\|_B$. By this, the linear spaces $M_p^\gamma, \hat{M}_p^\gamma$ become normed spaces if $1 \leq p \leq \infty$.

Remark 3.3. The above spaces can also be described as follows. Let $L_\infty(X)$ be the space consisting of all essentially bounded functions $g : [0, \infty) \rightarrow X$, while $\mathcal{L}_p(Y)$ be the space of measurable ($p = 0$), p -integrable ($0 < p < \infty$), essentially bounded ($p = \infty$) functions $h : \Omega \rightarrow Y$, where X and Y are arbitrary separable Banach spaces. Then it is easy to see that $M_p^\gamma = L_\infty(\mathcal{L}_p(R^n))$ and $\hat{M}_p^\gamma = \mathcal{L}_p(L_\infty(R^n))$ for all $0 \leq p \leq \infty$ and an arbitrary positive and continuous function $\gamma : [0, \infty) \rightarrow R$. This means that the above list of the M -spaces covers all possible combinations of Lebesgue spaces with respect to the variable $\omega \in \Omega$ and spaces of essentially bounded functions with respect to the variable $t \in [0, \infty)$. As we will see, this list covers also all types of stochastic Lyapunov stability described in the definition 3.1.

Below we use the following assumptions on a continuous positive function $\gamma(t)$, $t \in [0, \infty)$:

Property $\gamma 1$: the function γ satisfies the conditions $\gamma(t) \geq \sigma$ ($t \in [0, +\infty)$), $\sigma > 0$ and $\lim_{t \rightarrow +\infty} \gamma(t) = +\infty$.

Property $\gamma 2$: $\gamma(t) = \exp\{\beta t\}$ for some $\beta > 0$.

Definition 3.4. Let $x(\cdot, x_0)$ be the solution of the initial value problem (2-3) defined on $[0, \infty)$ and \mathcal{S} be a certain subspace of the space D^n . We say that the equation (2) is \mathcal{S} -stable if $x(\cdot, x_0) \in \mathcal{S}$ for any $x_0 \in R^n$.

Our first theorem describes relationships between the different kinds of the stochastic Lyapunov stability and the associated \mathcal{S} -stabilities for the equation (2).

Theorem 3.5. The following statements describe relationship between stochastic stability properties of the equation (2) and the M -spaces.

- (1) weak stability in probability is equivalent to the M_0 -stability;
- (2) weak asymptotic stability in probability is equivalent to the M_0^γ -stability for some γ satisfying Property $\gamma 1$;
- (3) stability in probability is equivalent to the \hat{M}_0 -stability;
- (4) if $0 < p < \infty$, then p -stability is equivalent to the M_p -stability;
- (5) if $0 < p < \infty$, then asymptotic p -stability is equivalent to the M_p^γ -stability for some γ satisfying Property $\gamma 1$;
- (6) if $0 < p < \infty$, then exponential p -stability is equivalent to the M_p^γ -stability for some γ satisfying Property $\gamma 2$;
- (7) stability with probability 1 is equivalent to the \hat{M}_0 -stability;
- (8) strong stability with probability 1 is equivalent to the M_∞ -stability;

Proof. We consider all cases separately.

(1) \Leftrightarrow . Choosing arbitrary positive numbers ϵ and δ and minding that $x(t, x_0) = X(t)x_0$ for all $x_0 \in R^n$ and all $t \geq 0$, we first find $K(\delta) > 0$ such that $\sup_{t \geq 0} P\{\omega \in \Omega : \|X(t)\| > K(\delta)\} < \delta$.

Letting $\eta = \frac{\epsilon}{K(\delta)}$ we observe that $|x_0| < \eta$ implies $\sup_{t \geq 0} P\{\omega \in \Omega : |x(t, x_0)| > \epsilon\} < \delta$ for all $x_0, |x_0| < \eta$, which is equivalent to weak stability in probability.

\Rightarrow). Assume that the zero solution of (2) is weakly stable in probability. Using the notation from the previous part of the proof, we assume, in addition, that there

exists $x_0 \in R^n$ such that $x(\cdot, x_0)$ does not belong to M_0 . Hence there is $\delta_0 > 0$ such that for any $\epsilon > 0$

$$P\{\omega \in \Omega : |x(t(\epsilon), x_0)| > \epsilon\} > \delta_0$$

for at least one $t(\epsilon) > 0$. In such a case, the solution $\hat{x}(t) \equiv \eta'x(t, x_0)/|x_0|$, where $0 < \eta' < \eta$ satisfies the estimate $P\{\omega \in \Omega : |x(t(\epsilon))| > \epsilon\} > \delta_0$, although $|x_0| < \eta$, which contradicts the assumption.

(2) \Leftarrow). The weak stability in probability follows from the observation that M_0^γ is topologically imbedded into M_0 . Let us now take arbitrary $x_0 \in R^n$, $\epsilon > 0$, $\delta > 0$. As $x(\cdot, x_0) \in M_0^\gamma$, there exists $K > 0$, for which

$$\sup_{t \geq 0} P\{\omega \in \Omega : |\gamma(t)x(t, x_0)| > K\} < \delta.$$

Now, choosing $T > 0$ satisfying $K\gamma^{-1}(t) < \epsilon$ for all $t \geq T$ we obtain

$$\sup_{t \geq T} P\{\omega \in \Omega : |x(t, x_0)| > \epsilon\} \leq \sup_{t \geq T} P\{\omega \in \Omega : |\gamma(t)x(t, x_0)| > K\} < \delta,$$

which implies weak asymptotic stability.

\Rightarrow). From the weak asymptotic stability in probability and the representation $x(t, x_0) = X(t)x_0$ for all $x_0 \in R^n$, we deduce that the function $\bar{\gamma}(t) = E(\|X(t)\| \wedge 1) \rightarrow 0$ as $t \rightarrow \infty$, as $E(|x - y| \wedge 1)$ is the metric in the space L_0 . Then the function $\gamma(t) = \frac{1}{\bar{\gamma}(t)}$ satisfies Property $\gamma 1$ with $\sigma = 1$. Now, we take an arbitrary $x_0 \in R^n$, $|x_0| \leq 1$ and define $K = 2C\delta^{-1}$ where $C = \sup_{t \geq 0} \bar{\gamma}(t)$. Let also choose

$T > 0$ satisfying the estimate $P\{\omega \in \Omega : \|X(t)\| > 1\} < \frac{\delta}{2}$ for all $t \geq T$. Then we obtain

$$\begin{aligned} P\{\omega \in \Omega : |x(t, x_0)| > K\} &\leq P\{\omega \in \Omega : \|X(t)\| > K\} \\ &\leq P\{\omega \in \Omega : \|X(t)\| > 1\} + P\{\omega \in \Omega : \|X(t)\| \wedge 1 > K\} \\ &\leq \frac{\delta}{2} + \frac{\delta}{2C} E(\|X(t)\| \wedge 1) = \frac{\delta}{2} + \frac{\delta}{2C} \bar{\gamma}(t) < \delta \end{aligned}$$

for all $t \geq T$. This yields $\sup_{t \geq T} P\{\omega \in \Omega, |\gamma(t)x(t)| > K\} < \delta$, which is sufficient

to conclude that $x(\cdot, x_0) \in M_0^\gamma$, as $x(\cdot, x_0)$ is continuous with probability 1 and hence bounded in probability on the interval $[0, T]$.

(3) \Leftarrow). This proof is similar to the one for case (1), where we only need to replace $\sup_{t \geq 0} P\{\omega \in \Omega : \|X(t)\| > K(\delta)\}$ and $\sup_{t \geq 0} P\{\omega \in \Omega : |x(t, x_0)| > \epsilon\}$ by $P\{\omega \in \Omega : \sup_{t \geq 0} \|X(t)\| > K(\delta)\}$ and $P\{\omega \in \Omega : \sup_{t \geq 0} |x(t, x_0)| > \epsilon\}$, respectively.

\Rightarrow). Assume that there exists $x_0 \in R^n$ such that $x(\cdot, x_0)$ does not belong to \hat{M}_0 . Hence there is $\delta_0 > 0$ such that for any natural number $\nu > 0$

$$P\{\omega \in \Omega : \sup_{t \geq 0} |x(t, x_0)| > \nu\} > \delta_0.$$

Then the solutions $x_\nu(t) = x(t, x_0\nu^{-1})$ satisfy $P\{\omega \in \Omega : \sup_{t \geq 0} |x_\nu(t)| > 1\} > \delta_0$,

although the sequence $x_{\nu u}(0) = x_0\nu^{-1}$ tends to 0. This contradicts the property of stability in probability.

(4) \Leftarrow). Let e_k , $k = 1, \dots, n$ be the standard basis in R^n , the latter being equipped with the norm $|b| = \sum_{k=1}^n |b_k|$ (all norms are equivalent). Due to p -stability we have $\sup_{t \geq 0} E|x(t, e_k)|^p \leq K$ for some $K > 0$ and any $k = 1, \dots, n$. Using the associated matrix norm $\|[a_1, \dots, a_n]\| = \max_{1 \leq k \leq n} |a_k|$, where $a_k \in R^n$, yields $\sup_{t \geq 0} E\|X(t)\|^p \leq K$.

For any $\epsilon > 0$ we now define $\eta = (\epsilon/K)^{1/p}$ obtaining

$$\sup_{t \geq 0} E|x(t, x_0)|^p \leq \sup_{t \geq 0} E\|X(t)\|^p |x_0|^p \leq K^p \eta^p = \epsilon$$

for any $|x_0| < \eta$, which completes the proof of p -stability.

\Rightarrow). Assume, on the contrary, that for some $x_0 \in R^n$ the solution $x(\cdot, x_0)$ does not belong to the space M_p . Hence for any natural number ν there exists t_ν such that $E|x(t_\nu)|^p > \nu^p$. As $x_\nu(0) = x_0\nu^{-1}$ can be arbitrarily small in the norm, while $E|x_\nu(t_\nu)|^p > 1$, this contradicts the property of p -stability.

(5) \Leftarrow). As the space M_p^γ is imbedded in M_p , the p -stability is trivial. On the other hand, for any $x_0 \in R^n$ there exists a positive number K such that $\sup_{t \geq 0} E\gamma(t)|x(t, x_0)|^p \leq K$.

Therefore $\sup_{t \geq 0} E|x(t, x_0)|^p \leq \gamma^{-1}(t) \rightarrow 0$ as $t \rightarrow \infty$ by Property $\gamma 1$.

\Rightarrow). Using the same norm as in part (5), we obtain $E\|X(t)\|^p \rightarrow 0$ as $t \rightarrow \infty$, so that $\sup_{t \geq T} E\|X(t)\|^p < 1$ for some $T > 0$. Choosing $\bar{\gamma}(t) = E(\|X(t)\|^p \wedge 1)$ and putting $\gamma(t) = 1/\bar{\gamma}(t)$ (which satisfies Property $\gamma 1$ with $\sigma = 1$), we obtain

$$\sup_{t \geq T} E(\gamma(t)|x(t, x_0)|^p) \leq \sup_{t \geq T} E(\gamma(t)\|X(t)\|^p)|x_0|^p = |x_0|^p.$$

As $\sup_{0 \leq t \leq T} E(\gamma(t)|x(t, x_0)|^p)$ is bounded due to continuity of the solutions, we deduce that $\sup_{t \geq T} E(\gamma(t)|x(t, x_0)|^p) < \infty$, so that $x(\cdot, x_0) \in M_p^\gamma$ for any $x_0 \in R^n$.

(6) \Leftarrow). Assume that $\gamma(t) = \exp\{\beta t\}$. Using the norm from (5), we conclude that $\sup_{t \geq 0} \gamma(t)E|x(t, e_k)|^p \leq K$ for all standard basis vectors e_k . This yields $\sup_{t \geq 0} \gamma(t)E\|X(t)\|^p \leq K$ and thus

$$E|x(t, x_0)|^p \leq E\|X(t)\|^p|x_0|^p \leq K|x_0|^p \exp\{-\beta t\}.$$

\Rightarrow). If $E|x(t, x_0)|^p \leq K|x_0|^p \exp\{-\beta t\}$ for all $t > 0$ and $x_0 \in R^n$, then $\sup_{t \geq 0} \gamma(t)E|x(t, x_0)|^p \leq K|x_0|^p$ is finite, so that $x(\cdot, x_0) \in M_p^\gamma$ for any $x_0 \in R^n$.

(7) \Leftarrow). We use again the norm from (5). According to the assumptions, for any $\delta > 0$ there is $K = K(\delta) > 0$, for which $P\{\omega \in \Omega : \sup_{t \geq 0} |x(t, e_k)| > K\} < \frac{\delta}{n}$ for all standard basis vectors e_k . Let $\Omega_k(\delta)$ contain all $\omega \in \Omega$ where $\sup_{t \geq 0} |x(t, e_k)| \leq K$ and $\Omega(\delta) = \bigcap_{1 \leq k \leq n} \Omega_k(\delta)$. Evidently, $P\Omega(\delta) \geq 1 - \delta$. As $\|X(t)\| = |x(t, e_k)|$ for some e_k (dependent on t and ω), we have that $\sup_{t \geq 0} \|X(t)\| \leq K(\delta)$ for all $\omega \in \Omega(\delta)$. This means that $\sup_{t \geq 0} \|X(t)\|$ is almost surely finite.

Taking now an arbitrary sequence $x_\nu \in R^n$ going to 0 as $\nu \rightarrow \infty$, we conclude that

$$\sup_{t \geq 0} |x(t, x_\nu)| \leq \sup_{t \geq 0} \|X(t)\||x_\nu|$$

tends to 0 almost surely. This completes the proof of this part.

\Rightarrow). Given $x_0 \in R^n$, the stability with probability 1 implies that the sequence $\sup_{t \geq 0} |x(t, x_0 n^{-1})|$ converges to zero almost surely as $n \rightarrow \infty$. Thus, this sequence converges in probability, and therefore, for any $\delta > 0$ there exists a number $N = N(\delta)$ such that $P\{\omega \in \Omega : \sup_{t \geq 0} |x(t, x_0/n)| \geq 1\} < \delta$ for all $n \geq N$. Choosing $K(\delta) = N(\delta)$ we see that $P\{\omega \in \Omega : \sup_{t \geq 0} |x(t, x_0)| \geq K\} < \delta$, so that $x(\cdot, x_0) \in \hat{M}_0$.

(8) \Leftarrow). Utilizing the norm from part (5) and the definition of the space M_∞ , we can write that for some sufficiently large K , the estimates $\sup_{t \geq 0} |x(t, e_k)| \leq K$ hold for all standard basis vectors e_k , $k = 1, \dots, n$ and all $\omega \in \Omega_k$, where $P\Omega_k = 1$. Therefore, on the set $\hat{\Omega} = \bigcap_{1 \leq k \leq n} \Omega_k$ of full measure we have

$$\sup_{t \geq 0} \|X(t)\| \leq \sup_{t \geq 0} \max_{1 \leq k \leq n} |x(t, e_k)| = \max_{1 \leq k \leq n} \sup_{t \geq 0} |x(t, e_k)| \leq K.$$

Hence, given $\delta > 0$ we set $\eta = \delta/K$ and obtain

$$P\{\omega : \omega \in \Omega, \sup_{t \geq 0} |x(t, x_0)|\} \leq P\{\omega : \omega \in \Omega, \sup_{t \geq 0} \|X(t)\| |x_0|\} \leq \delta$$

as long as $x_0 \in R^n, |x_0| < \eta$. By this, we verified strong stability with probability 1.

\Rightarrow). Assume that there exists a solution $x(\cdot, x_0)$ that does not belong to the space M_∞ . This means that there exists $\delta_0 > 0$ such that for any natural number ν $P\{\omega : \omega \in \Omega, \sup_{t \geq 0} |x(t, x_0)| \leq \nu\} \leq 1 - \epsilon_0$. Defining $x_\nu(t) = x(t, x_0 \nu^{-1})$ we see

that $P\{\omega : \omega \in \Omega, \sup_{t \geq 0} |x_{\nu\nu}(t)| \leq 1\} \leq 1 - \epsilon_0$, although $x_\nu(0) = x_0 \nu^{-1}$ goes to 0 as $\nu \rightarrow \infty$. This contradicts the property of strong stability with probability 1. \square

Remark 3.6. Note that the M -stabilities listed in Theorem 3.5 imply the respective stability properties not only w.r.t. the initial values $x_0 \in R^n$, but also w.r.t. the random initial values, provided that the increments of the semimartingale Z are independent of the initial σ -algebra \mathcal{F}_0 . This is e.g. the case if the family $(\mathcal{F}_t)_{t \geq 0}$ is generated by the Brownian motion. This comment applies to all results proven below.

Lemma 3.7. *Let $p, q \in [0, \infty]$, and γ, κ be continuous functions defined on $[0, \infty)$ and satisfying Property $\gamma 1$. Then the following inclusions are topological embeddings:*

- (1) $M_p^\gamma \subset M_q^\gamma$ and $\hat{M}_p^\gamma \subset \hat{M}_q^\gamma$ if $p \geq q$;
- (2) $M_p^\gamma \subset M_p^\kappa$ and $\hat{M}_p^\gamma \subset \hat{M}_p^\kappa$ if $\gamma(t) \geq \kappa(t)$ for $t \geq t_0$ ($t_0 \geq 0$ is some real number);
- (3) $\hat{M}_p^\gamma \subset M_p^\gamma$.

Proof. The first statement simply follows from the standard relationship between Lebesgue spaces generated by finite measure sets. The second and the third statements are a direct consequence of the definitions of the spaces M_p^γ and \hat{M}_p^γ and the inequality $\sup_{t \geq 0} E|y(t)| \leq E \sup_{t \geq 0} |y(t)|$, which holds for any measurable stochastic process $y(t)$. \square

Theorem 3.8. *The following is valid for the stochastic functional differential equation (2):*

- (1) if $0 < q < p < \infty$, then p -stability (resp. asymptotic, exponential p -stability) implies q -stability (resp. asymptotic, exponential q -stability);
- (2) if $0 < p < \infty$, then p -stability (resp. asymptotic p -stability) implies weak stability in probability (resp. weak asymptotic stability in probability);
- (3) stability in probability (resp. asymptotic stability in probability) implies weak stability with probability 1 (resp. weak asymptotic stability with probability 1);
- (4) stability in probability is equivalent to stability with probability 1.

Proof. The first and the second part follow directly from the first statement of Lemma 3.7 combined with statements 4-6 and 1-2, 4-5 of Theorem 3.5, respectively. The third part follows from the second statement of Lemma 3.7 combined with statements 1-4 of Theorem 3.5. To prove the last part first we observe that statements 3 and 8 of of Theorem 3.5 contains the same space M_0^γ , so that stability in probability and stability with probability 1 are equivalent for the equation (2). \square

In the next theorem, we describe more relations between different kinds of the stochastic Lyapunov stability. Some of these results are used in the examples below.

Theorem 3.9. *Let $p \in [0, \infty]$. Then*

- (1) \hat{M}_p -stability implies stability with probability 1.
- (2) \hat{M}_p^γ -stability with γ satisfying Property $\gamma 1$ implies asymptotic stability with probability 1.
- (3) \hat{M}_∞^γ -stability with γ satisfying Property $\gamma 1$ implies strong asymptotic stability with probability 1.

Proof. (1). This follows from statement (7) of Theorem 3.5 and the inclusion $\hat{M}_p \subset \hat{M}_0$.

(2). Due to the first statement of the theorem and the inclusion $\hat{M}_p^\gamma \subset \hat{M}_0^\gamma$ we obtain the property of stability with probability 1. To prove asymptotic stability with probability 1, we proceed as in the proof of statement (7) of Theorem 3.5 adding $\gamma(t)$ to $x(t, x_0)$ and $X(t)$, which gives almost everywhere boundedness of the random function $\xi = \xi(\omega) = \sup_{t \geq 0} \gamma(t) \|X(t)\|$. Therefore

$$\gamma(t)|x(t, x_0)| \leq \sup_{t \geq 0} \gamma(t) \|X(t)\| |x_0| = \xi |x_0|,$$

so that

$$|x(t, x_0)| \leq \frac{\xi |x_0|}{\gamma(t)} \rightarrow 0 \text{ almost everywhere as } t \rightarrow \infty.$$

(3). As $\hat{M}_\infty^\gamma \subset \hat{M}_\infty$, we readily obtain strong asymptotic stability with probability 1 from the last statement of Theorem 3.5. Adding $\gamma(t)$ to $x(t, x_0)$ and $X(t)$ and following the proof of this statement yield the estimate $\sup_{t \geq 0} \gamma(t) \|X(t)\| \leq K$ for some (non random) constant K . Hence

$$\gamma(t)|x(t, x_0)| \leq \sup_{t \geq 0} \gamma(t) \|X(t)\| |x_0| \leq K |x_0|,$$

and $|x(t, x_0)| \leq \frac{K|x_0|}{\gamma(t)} \rightarrow 0$ with probability 1 as $t \rightarrow \infty$. \square

4. DELAY-INDEPENDENT STABILITY CONDITIONS FOR THE GENERALIZED PANTOGRAPH EQUATION DRIVEN BY THE BROWNIAN MOTION

Delay-independent stability is also called absolute stability (see e.g. [19]). This property is important if the delay functions are unknown or difficult to determine.

It is well-known (see e.g. [23]) that using Itô's formula usually results in a delay-dependent stability condition for stochastic pantograph equations. Below we present an alternative method which is based on the theory developed in Section 3.

We start with the analysis of p -stability, where we use the integral form of the Marcinkiewicz-Zygmund inequality

$$(4) \quad E \left| \int_0^t \zeta(s) d\mathcal{B}(s) \right|^{2p} \leq \rho_p^{2p} E \left(\int_0^t |\zeta(s)|^2 ds \right)^p,$$

which holds true for any predictable stochastic process $\zeta(s)$ ($0 \leq s \leq t$), any $t > 0$ and the scalar standard Brownian motion $\mathcal{B}(s)$. The constant ρ_p depends on the number p ($0 < p < \infty$), only. In 1988 D.L. Burkholder proved (see for example [6], [18]) that the constant $\rho_p = 2p - 1$ in the Marcinkiewicz-Zygmund inequality (4) is best possible for all for $p \geq 1$.

Consider the following equation

$$(5) \quad dx(t) = \left[Ax(t) + \sum_{j=0}^{m_0} A_{j0} x(h_{j0}(t)) \right] dt + \sum_{l=1}^{m-1} \sum_{j=0}^{m_l} A_{jl} x(h_{jl}(t)) d\mathcal{B}_l(t) \quad (t \geq 0).$$

For all $0 \leq l \leq m-1$, $0 \leq j \leq m_l$, the $n \times n$ -matrices A, A_{jl} contain constant, real entries, the standard Wiener processes $B_l(t)$ are independent, and the functions h_{jl} are Borel measurable functions satisfying $0 \leq h_{jl}(t) \leq t$ on $[0, \infty)$.

In the next theorem we put $A_l = \sum_{j=0}^{m_l} \|A_{jl}\|$ for all $l = 0, \dots, m-1$, where $\|\cdot\|$ is the matrix norm induced by the chosen vector norm $|\cdot|$.

Theorem 4.1. *Let p satisfy the inequality $1 \leq p < \infty$. Assume that there exists a positive number α such that*

$$(6) \quad \|A + \alpha I_n\| + A_0 + \rho_p \sqrt{\alpha/2} \sum_{l=1}^{m-1} A_l < \alpha,$$

where I_n is the $n \times n$ identity matrix. Then the equation (5) is $2p$ -stable.

Proof. We apply the fourth statement from Theorem 3.5. To show that the solutions $x(t)$ of the equation (5) belong to the space M_{2p} for all $x(0) \in R^n$, we replace this equation by an equivalent integral equation

$$(7) \quad \begin{aligned} x(t) = & e^{-\alpha t} x(0) + \int_0^t e^{-\alpha(t-s)} \left[(A + \alpha I_n)x(s) + \sum_{j=0}^{m_0} A_{j0}x(h_{j0}(s)) \right] ds \\ & + \sum_{l=1}^{m-1} \sum_{j=0}^{m_l} \int_0^t e^{-\alpha(t-s)} A_{jl}x(h_{jl}(s)) dB_l(s) \quad (t \geq 0). \end{aligned}$$

Using the Marcinkiewicz-Zygmund inequality (4) we obtain

$$\begin{aligned} & \left(\sup_{t \geq 0} E|x(t)|^{2p} \right)^{1/2p} \leq |x(0)| + \left(\sup_{t \geq 0} E \left| \int_0^t e^{-\alpha(t-s)} (A + \alpha I_n)x(s) ds \right|^{2p} \right)^{1/2p} \\ & + \sum_{j=0}^{m_0} \left(\sup_{t \geq 0} E \left| \int_0^t e^{-\alpha(t-s)} A_{j0}x(h_{j0}(s)) ds \right|^{2p} \right)^{1/2p} \\ & + \sum_{l=1}^{m-1} \sum_{j=0}^{m_l} \left(\sup_{t \geq 0} E \left| \int_0^t e^{-\alpha(t-s)} A_{jl}x(h_{jl}(s)) dB_l(s) \right|^{2p} \right)^{1/2p} \\ & \leq |x(0)| + \alpha^{-1} \|A + \alpha I_n\| \left(\sup_{t \geq 0} E|x(t)|^{2p} \right)^{1/2p} \\ & + \alpha^{-1} \sum_{j=0}^{m_0} \|A_{j0}\| \left(\sup_{t \geq 0} E|x(h_{j0}(t))|^{2p} \right)^{1/2p} \\ & + \rho_p \sum_{l=1}^{m-1} \sum_{j=0}^{m_l} \left(\sup_{t \geq 0} E \left(\int_0^t e^{-2\alpha(t-s)} \|A_{jl}\|^2 |x(h_{jl}(s))|^2 ds \right)^p \right)^{1/2p} \\ & \leq |x(0)| + K \left(\sup_{t \geq 0} E|x(t)|^{2p} \right)^{1/2p}, \end{aligned}$$

where $K = \alpha^{-1} \|A + \alpha I_n\| + \alpha^{-1} A_0 + \rho_p (2\alpha)^{-1/2} \sum_{l=1}^{m-1} A_l$. In the estimates above we used the assumptions $0 \leq h_{jl}(t) \leq t$ for all $t \geq 0$, $0 \leq j \leq m_l$, $0 \leq l \leq m-1$. Due to (6), the constant K is less than 1, so that

$$\left(\sup_{t \geq 0} E|x(t)|^{2p} \right)^{1/2p} \leq (1 - K)^{-1} |x(0)|.$$

Therefore $x \in M_{2p}$ for all $x(0) \in R^n$. \square

Consider the generalized scalar pantograph equation

$$(8) \quad dx = [ax(t) + bx(h(t))]dt + [cx(t) + dx(g(t))]dB(t) \quad (t \geq 0),$$

where a, b, c, d are real constants, $h(t), g(t)$ ($t \geq 0$) are Borel measurable functions satisfying $0 \leq h(t) \leq t$, $0 \leq g(t) \leq t$ ($t \geq 0$), and $\mathcal{B}(t)$ is the standard scalar Brownian motion.

Corollary 4.2. *Let ξ_0 be the (only) positive root of the quadratic polynomial $P(\xi) = \xi^2 - K\xi - |b|$, where $K = \rho_p \frac{|c|+|d|}{\sqrt{2}}$. If $a < -\xi_0^2$, then the equation (8) is $2p$ -stable.*

Proof. Inequality (6) reads in this case as follows:

$$|a + \alpha| + |b| + K\sqrt{\alpha} < \alpha.$$

Assuming that $a < -\xi_0^2$ and putting $\alpha = \xi_0^2 > -a > 0$ we observe that

$$\alpha - |a + \alpha| - |b| - K\sqrt{\alpha} = \xi_0^2 - |b| - K\xi_0 > 0,$$

as $\xi > \sqrt{-a} > \xi_0$. □

Let us compare Corollary 4.2 with the following well-known result (see [4] or [21]): the equation (8) with the linear delays $h(t) = g(t) = kt$ ($0 < k < 1$) is 2 -stable if $2a + 2|b + cd| + c^2 + d^2 < 0$.

In the case when $b = 0$, this result and Corollary 4.2 give the same condition: $a < -\frac{(|c|+|d|)^2}{2}$.

If $c = 0$, then the result from [4] gives, in fact, a better stability condition in the case of linear delays: $a < -\frac{d^2}{2} - |b|$ vs. $a < -\frac{d^2}{4} - |b| - \frac{d}{\sqrt{2}}\sqrt{-\frac{d^2}{8} + |b|}$, but our result covers more general delays and the case of p -stability for $p \neq 2$.

Now we briefly describe explicit stability conditions for one particular case of the vector equation (5).

Corollary 4.3. *Let the matrix A have only real eigenvalues λ_i ($1 \leq i \leq n$) and A_{jl} be scalar matrices:*

$$A_{jl} = \text{diag}[a_{jl}, \dots, a_{jl}] \quad (0 \leq l \leq m-1; 1 \leq j \leq m_l).$$

Then the equation (5) is $2p$ -stable if $\lambda_i < -\xi_0^2$ ($1 \leq i \leq n$), where ξ_0 is the only positive root of the quadratic polynomial

$$P_i(\xi) = \xi^2 - K\xi - a_0$$

and

$$K = \frac{\rho_p}{\sqrt{2}} \sqrt{\sum_{l=1}^{m-1} \sum_{j=0}^{m_l} \|a_{jl}\|}.$$

Proof. Let us choose $\alpha = \xi_0^2$. To verify the condition (6) of Theorem 4.1 we first observe that for any positive $\epsilon > 0$ there exists a basis in R^n , in which the Euclidean matrix norm $\|\cdot\|$ satisfies $\|A + \alpha I_n\| < |\lambda + \alpha| + \epsilon$, where λ is the largest eigenvalue of the matrix A . At the same time, the Euclidean norms of the scalar matrices A_{jl} ($0 \leq l \leq m-1; 1 \leq j \leq m_l$) are independent of the choice of the basis. Denoting

$$A_l = \sum_{j=0}^{m_l} \|a_{jl}\|$$

for all $l = 0, \dots, m-1$ we obtain, exactly in the same manner as in the proof of Corollary 4.2, that

$$\|A + \alpha I_n\| + A_0 + \rho_p \sqrt{\alpha/2} \sum_{l=1}^{m-1} A_l < |\lambda + \alpha| + \epsilon + A_0 + K\sqrt{\alpha} < \alpha$$

for sufficiently small ϵ . □

In the next theorem, where we study stability with probability 1, we use the Doob inequality

$$E \sup_{t \geq 0} \left| \int_0^t \zeta(s) d\mathcal{B}(s) \right|^2 \leq 4E \left(\sup_{t \geq 0} \int_0^t |\zeta(s)|^2 ds \right)$$

where \mathcal{B} is the scalar standard Brownian motion, ζ is an arbitrary predictable stochastic process on $[0, \infty)$.

Theorem 4.4. *Assume that there exists a positive number α such that*

$$(9) \quad \|A + \alpha I_n\| + A_0 + \sqrt{2\alpha} \sum_{l=1}^{m-1} A_l < \alpha,$$

where I_n is the $n \times n$ identity matrix. Then the equation (5) is stable with probability 1.

Proof. We want to apply the first statement from Theorem 3.9 by checking that the solutions $x(t)$ of the equation (5) belong to the space \hat{M}_2 for all $x(0) \in R^n$. This is done exactly in the same way as in the proof of Theorem 4.1 provided that the norm $\left(\sup_{t \geq 0} E|x(t)|^{2p} \right)^{1/2p}$ is replaced by the norm $\left(E \sup_{t \geq 0} |x(t)|^{2p} \right)^{1/2p}$ and the Doob inequality is used instead of the Marcinkiewicz-Zygmund inequality. \square

5. CONCLUSIONS AND OUTLOOK

In the paper we described and justified a new framework for stability analysis of stochastic functional differential equations in the case when initial data are finite dimensional. In particular, this framework covers the generalized stochastic pantograph equation. The main feature of our analysis consists in replacing Lyapunov stabilities with input-to-state stabilities by choosing appropriate spaces of stochastic processes. It is shown that this approach is applicable to all known kinds of stochastic Lyapunov stability.

We demonstrated the efficiency of this idea by applying it to the generalized stochastic pantograph equation, where the emphasis was put on delay-independent stability conditions. In particular, we studied $2p$ -stability ($p \geq 1$) and stability with probability 1.

The future development of the suggested framework may include the following topics:

- (1) Stability analysis of the stochastic pantograph equations may be extended to the case of non-constant coefficients. In this case, Theorem 3.5 and other results of Section 3 could still be applied.
- (2) More attention should be paid to the vector case. In particular, coefficient-based stability conditions may be derived from Theorems 4.1 and 4.4 to get more general stability tests than those offered in Section 4.
- (3) Asymptotic p -stability and asymptotic stability with probability 1, are only described in terms of M -stability, but not in the form of specific stability conditions, so that further analysis of these asymptotic properties should be continued.
- (4) The stochastic pantograph equation driven by an arbitrary semimartingale can also be studied using the techniques developed in Section 3.
- (5) The main results of Section 3 stretch far beyond the generalized pantograph equation. As it is mentioned in Section 2, wide classes of stochastic functional differential equations can be represented as the equation (2). The

technique developed in this paper can be e.g. directly applied to integro-differential equations, where

$$(Vx)(t) = \left(\int_{(0,t)} d_s \mathcal{R}_1(t,s)x(s), \dots, \int_{(0,t)} d_s \mathcal{R}_m(t,s)x(s) \right),$$

\mathcal{R}_i are vector functions defined on $\{(t,s) : t \in [0, \infty), 0 < s \leq t\}$ for $i = 1, \dots, m$. But this analysis is beyond the scope of the present paper.

- (6) Stochastic hereditary equations with more general initial conditions $x(s) = \varphi(s)$, $s \leq 0$ can easily be transformed into the equation (2) as well (see e.g. [12]). However, a possible generalization of the main results of Section 3 will not be straightforward, as in this case, one will, in addition, need to prove boundedness of the infinite dimensional linear operators assigning φ to the associated solution $x(t)$.

6. CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

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