Stationary solutions of continuous and discontinuous neural field equations

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Abstract

We study existence and continuous dependence of the solutions to the Hammerstein operator equation under the transition from continuous nonlinearities in the Hammerstein operator to the Heaviside nonlinearity in a vicinity of the solution, corresponding to the discontinuous nonlinearity case. We apply these results to corresponding problems arising in the neural activity modeling.

Keywords: Discontinuous Hammerstein equations, solvability, continuous dependence
47H30, 46T99, 47H11, 92B99

1. Introduction

We consider a special case of nonlinear operator equation with the Hammerstein operator, the nonlinear part of is either represented by the Heaviside unit step function, or by a bounded continuous function. We are studying existence and continuous dependence of the solutions to the Hammerstein operator equation under the transition from continuous nonlinearities in the Hammerstein operator to the Heaviside nonlinearity. To do this, we choose an appropriate topology, where the Hammerstein operator with the Heaviside nonlinearity becomes continuous in a vicinity of the solution, corresponding to the case of the discontinuous Hammerstein operator nonlinearity. Then we use methods of functional analysis and topological degree theory to establish

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the results needed. This study is strongly motivated by applications of some problems arising in the neural activity modeling. Below we give a detailed descriptions of these problems.

It is well-known (see e.g. [11], [9]) that electrical activity in the neocortex is naturally studied in the framework of cortical networks. However, since the number of neurons and synapses in even a small piece of cortex is immense, a suitable modeling approach is to take a continuum limit of the neural networks and, thus, consider so-called neural field models of the brain cortex (rigorous justification of this limit procedure can be found in e.g. [4]). The simplest model describing the macro-level neural field dynamics is the Amari model [1]

$$\partial_t u(t,x) = -u(t,x) + \int\limits_{\Xi} \omega(x-y) f(u(t,y))dy, \quad t \geq 0,\, x \in \Xi \subseteq R^m. \qquad (1)$$

Here $u(t,x)$ denotes the activity of a neural element u at time t and position x. The connectivity function ω determines the coupling strength between the elements and the non-negative function $f(u)$ gives the firing rate of a neuron with activity u. Neurons at a position x and time t are said to be active if $f(u(t,x)) > 0$. Typically f is a smooth function that has sigmoidal shape. Solvability of (1) in the case of a smooth firing rate function was proved in [23], [3]. Particular attention in the neural field theory is usually given to the localized stationary, i.e., time-independent, solutions to (1) (so-called "bump solutions", or simply "bumps"), as they correspond to normal brain functioning (see e.g [26]). Faugeras et al [8] proved existence and uniqueness of the stationary solution to (1) as well as obtained conditions for this solution to be absolutely stable, for the case of a bounded Ξ.

A common simplification of (1) consists of replacing a smooth firing rate function by the Heaviside function. This replacement simplifies numerical investigations of the model as well as allows to obtain closed form expressions for some important types of solutions (see e.g. [1], [22] [17]). Existence of the solution to (1) in the case of Heaviside firing rate function was proved by Potthast et al [23]. Stability of the stationary solutions to (1) is usually assessed by the Evans function approach (see e.g. [6], [22]). The analysis of existence and stability of localized stationary solutions for a special class of the firing rate functions, the functions that are "squeezed" between two unit step functions, was carried out in [13], [20], [15]. This analysis served as a connection between stability\instability properties of the solutions to the models with the "squeezing" Heaviside firing rate functions and the solution to the model with the "squeezed" smooth firing rate function. However, no rigorous mathematical justification of the passage from a smooth to discontinuous firing rate functions in the framework of neural field models was given until the work by Oleynik et al [21], where continuous dependence of the 1-bump stationary solution to (1) under the transition from a smooth firing rate function to the Heaviside function was proved in the 1-D case.

On the other hand, more advanced neural field models have not been studied in this respect. One example is the homogenized Amari model describing the neural field dynamics on both macro- and micro- levels

$$\begin{gathered}\partial_t u(t,x,x_{\mathrm{f}}) = -u(t,x,x_{\mathrm{f}}) + \int\limits_{\Xi}\int\limits_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f(u(t,y,y_{\mathrm{f}}))dy_{\mathrm{f}}dy,\\ t\geq 0,\ x\in\Xi,\ x_{\mathrm{f}}\in\mathcal{Y}\subset R^k,\end{gathered} \tag{2}$$

which was introduced in the pioneering work by Coombes et al [7]. Here x_{f} is the fine-scale spatial variable and $\mathcal{Y}$ is an elementary domain of periodicity in R^k. As it was shown in [24], the solution to (2) is a weak two-scale limit of solutions to the following family of heterogeneous neural field models

$$\begin{gathered}\partial_t u(t,x) = -u(t,x) + \int\limits_{\Xi} \omega^{\varepsilon}(x-y) f(u(t,y))dy,\\ \omega^{\varepsilon}(x) = \omega(x, x/\varepsilon),\ 0<\varepsilon\ll 1,\\ t\geq 0,\ x\in\Xi,\end{gathered} \tag{3}$$

as $\varepsilon\to 0$, where ε corresponds to the medium heterogeneity.

The starting point for the investigation of the solutions to (2) was assuming these solutions to be independent of the fine-scale variable, i.e. solutions to the equation

$$\begin{gathered}\partial_t u(t,x) = -u(t,x) + \int\limits_{\Xi}\int\limits_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f(u(t,y))dy_{\mathrm{f}}dy,\\ t>0,\ x\in\Xi\subseteq R^m,\ x_{\mathrm{f}}\in\mathcal{Y}.\end{gathered} \tag{4}$$

This assumption was also supported by numerical evidence of non-existence of the fine-scale-dependent solutions to (2) given in [19].

Existence and stability of the single bump and double bump stationary solutions to (4) in 1-D were investigated in [25] and [18], respectively, for the case of the Heaviside firing rate function. Existence and stability of the radially symmetric single bump stationary solutions to (4) in 2-D when f is represented by the Heaviside unit step function were investigated in [5].

In the present research we extend the results of [21] to the homogenized Amari model and, in addition to the single bump solutions in 1-D, consider symmetric double bump solutions in 1-D and radially symmetric bump solutions in 2-D. We formulate the following two main theorems: the theorem on continuous dependence of the stationary solutions to (4) under the transition from continuous firing rate functions to the Heaviside function and the theorem on solvability of the equation (4) based on the topological degree theory. We emphasize here that the properties of existence of solutions to (4) under the described transition and continuous dependence of these solutions on the firing rate steepness do not depend on the stability\instability of the solution to (4) with the Heaviside firing rate function. The latter remark can be illustrated by comparison of the results of the papers [25], [18], and [5] to the corresponding three special cases of (4), considered in the present research:

1. Symmetric single bump solution to (4), $m = k = 1$.
2. Symmetric double bump solution to (4), $m = k = 1$.
3. Radially symmetric single bump solution to (4), $m = k = 2$.

We also stress that our results, in particular, mean that the approximation of the Heaviside function by piecewise linear firing rate functions yields continuous dependence of the solutions to the corresponding neural field equations. This property has particular importance for the theory of the heterogeneous neural fields as the transition from the heterogeneous model (3) to the homogenized model (2) can be justified for the piecewise linear firing rate functions, but not for their Heaviside limit (see [24], [25]). Thus, our results justify the usage of the Heaviside firing rate function in the frameworks of [25], [18], and [5].

The paper is organized in the following way. In Section 2 we explain our notations and state lemmas from functional analysis, which we refer to in the subsequent sections. In Section 3 we study existence and continuous dependence of the stationary solutions to (4) under the transition from continuous firing rate functions to the Heaviside function, and formulate and prove the corresponding two main theorems. Based on these theorems we investigate in Section 4 the corresponding properties of the following types of solutions to (4):

1. Symmetric single bump solutions in 1-D (Subsection 4.1).

2. Symmetric double bump solutions in 1-D (Subsection 4.2).

3. Radially symmetric single bump solutions in 2-D (Subsection 4.3).

Section 5 provides concluding remarks and outlook.

2. Preliminaries

In this section we provide an overview of the notation, introduce the main definitions and formulate the main theorems we refer to.

For a metric space $\mathfrak{M}$ with the distance $\rho_{\mathfrak{M}}$, and arbitrary $\mathfrak{S} \subset \mathfrak{M}$, $\varepsilon > 0$, we denote $B_{\mathfrak{M}}(\mathfrak{S}, \varepsilon) = \bigcup\limits_{\mathfrak{s} \in \mathfrak{S}} \{\mathfrak{m} \in \mathfrak{M} \mid \rho_{\mathfrak{M}}(\mathfrak{m}, \mathfrak{s}) < \varepsilon\}$.

Definition 2.1. Let $\mathfrak{S}$ be an arbitrary subset of the metric space $\mathfrak{M}$. The set $\mathfrak{E}$ is called *ϵ-net* for $\mathfrak{S}$ if for any $\mathfrak{s} \in \mathfrak{S}$, one can find such $\mathfrak{e} \in \mathfrak{E}$ that $\rho_{\mathfrak{M}}(\mathfrak{e}, \mathfrak{s}) \leq \epsilon$, see [14].

Let $\mathfrak{B}$ be a real Banach space equipped with the norm $\| \cdot \|_{\mathfrak{B}}$ and D be an arbitrary open bounded subset of $\mathfrak{B}$. We denote by ∂D and $\overline{D}$ the boundary and the closure of D in $\mathfrak{B}$, respectively. We denote by $\deg(\Phi, D, \mathfrak{b}_0)$ and $\operatorname{ind}(\Phi, D)$ the degree and the topological index of an arbitrary operator $\Phi : \overline{D} \to \mathfrak{B}$, respectively (if they are well-defined).

Let μ be the Lebesgue measure on R^m, Ω be a compact subset of R^m, $\Xi \subseteq R^m$, then:

$L^q(\Xi, \mu, R)$ be the space of functions $\eta : \Xi \to R$ with Lebesgue integrable q-th power of the absolute value and the following norm $\|\eta\|_{L^q(\Xi,\mu,R)} = \left(\int\limits_{\Xi} |\eta(x)|^q dx \right)^{1/q}$, $1 \leq q < \infty$.

Let $C^k(\Omega, R)$ be the space of functions $\zeta : \Omega \to R$, whose first k derivatives $\zeta^{(n)}$ $(n = 0, \ldots, k,\ \zeta^{(0)} = \zeta)$ are continuous, equipped with the norm $\|\zeta\|_{C^k(\Omega,R)} = \sum\limits_{n=0}^{k} \max\limits_{x \in \Omega} |\zeta^{(n)}(x)|$.

Let $C^k(R^m, R)$ be a locally convex space of functions $\zeta : R^m \to R$, whose first k derivatives $\zeta^{(n)}$ $(n = 0, \ldots, k,\ \zeta^{(0)} = \zeta)$ are continuous, equipped with the topology of uniform convergence of $\sum\limits_{n=0}^{k} \max |\zeta^{(n)}|$ on compact subsets of R^m.

We will not indicate $q = 1$ and $k = 0$ in the corresponding space notations.

Lemma 2.1. Let D be a open bounded subset of a real Banach space $\mathfrak{B}$, Λ be a compact subset of R, and an operator $T : \Lambda \times \overline{D} \to \mathfrak{B}$ be continuous with respect to both variables and collectively compact (i.e., $T(\Lambda, \overline{D})$ is a pre-compact set in $\mathfrak{B}$). Assume that $\lambda_n \to \lambda_0$ and $T(\lambda_n, \mathfrak{b}_n) = \mathfrak{b}_n$. Then the equation $T(\lambda_0, \mathfrak{b}) = \mathfrak{b}$ has at least one solution. Moreover, any limit point of the sequence $\{\mathfrak{b}_n\}$ is a solution of this equation, i.e., if $\mathfrak{b}_n \to \mathfrak{b}_0$ then $T(\lambda_0, \mathfrak{b}_0) = \mathfrak{b}_0$, see [21].

Definition 2.2. Let D be an open bounded subset of a real Banach space $\mathfrak{B}$. The family $\{h_t\}$, $(t \in [0,1])$ of operators acting from $\overline{D}$ to $\mathfrak{B}$ is called *homotopy* if $h_t(\mathbf{b})$ is continuous with respect to $(t, \mathfrak{b})$ on $[0,1] \times \overline{D}$, see [12].

Lemma 2.2. (Homotopy invariance) Let D be an open bounded subset of a real Banach space $\mathfrak{B}$. Suppose that $\{h_t\}$ is a homotopy of operators $h_t : \overline{D} \to \mathfrak{B}$ and $h_t - I$ is compact for each $t \in [0,1]$. If $h_t\mathfrak{b} \neq \mathfrak{b}_0$ for any $\mathfrak{b} \in \partial D$ and $t \in [0,1]$, then $\deg(h_t, D, \mathfrak{b}_0)$ is independent of t, see [12].

Definition 2.3. Let D be an open bounded subset of $\mathfrak{D}$, where $\mathfrak{D}$ is an absolute neighborhood retract (see, e.g. [10]), $\mathfrak{D} \subset \mathfrak{B}$. The continuous mapping $\psi : D \to \mathfrak{D}$ is called *admissible* provided that the fixed point set of ψ is compact in $\mathfrak{B}$, see [10].

Lemma 2.3. (Topological invariance) Let $\psi : D \to \mathfrak{D}$ be an admissible compact mapping and $\phi : \mathfrak{D} \to \mathfrak{D}'$ be a homeomorphism. Then $\phi \circ \psi \circ \phi^{-1} : \phi(D) \to \mathfrak{D}'$ is also an admissible compact mapping and

$$\mathrm{ind}(\psi, D) = \mathrm{ind}(\phi \circ \psi \circ \phi^{-1}, \phi(D)),$$

see [10].

3. Main results

In this section we study existence and continuous dependence of stationary solutions to (4) when approximating the Heaviside activation function by continuous functions. In order to do that, we consider the following homogenized Amari neural field equation

$$\begin{gathered}\partial_t u(t,x) = -u(t,x) + \int\limits_{\Xi} \int\limits_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}} - y_{\mathrm{f}}) f_\beta(u(t,y)) dy_{\mathrm{f}} dy, \\ t > 0,\ x \in \Xi \subseteq R^m,\ x_{\mathrm{f}} \in \mathcal{Y} \subset R^k,\end{gathered} \tag{5}$$

parameterized by $\beta \in [0, \infty)$.

We assume that the functions involved in (5) satisfy the following assumptions:

($\mathbf{A1}$) For any $x_{\rm f} \in \mathcal{Y}$, the connectivity kernel $\omega(\cdot, x_{\rm f}) \in C^2(\Xi, R)$.

($\mathbf{A2}$) For any $x \in R$, the connectivity kernel $\omega(x, \cdot) \in L(\mathcal{Y}, \mu, R)$.

($\mathbf{A3}$) For $\beta = 0$, the activation function is represented by the Heaviside unit step function

$$f_0(u) = \begin{cases} 0, & u \le \theta, \\ 1, & u > \theta \end{cases}$$

with some threshold value θ.

($\mathbf{A4}$) For $\beta > 0$, functions of the family $f_\beta : R \to [0, 1]$ are non-decreasing, continuous, and satisfying the following convergence conditions with respect to the parameter β:

(i) $f_\beta \to f_{\widehat{\beta}}$ uniformly on R as $\beta \to \widehat{\beta}$, $\widehat{\beta} \in (0, \infty)$;

(ii) for any $\varepsilon > 0$, $f_\beta \to f_0$ uniformly on $R \setminus B_R(\theta, \varepsilon)$ as $\beta \to 0$.

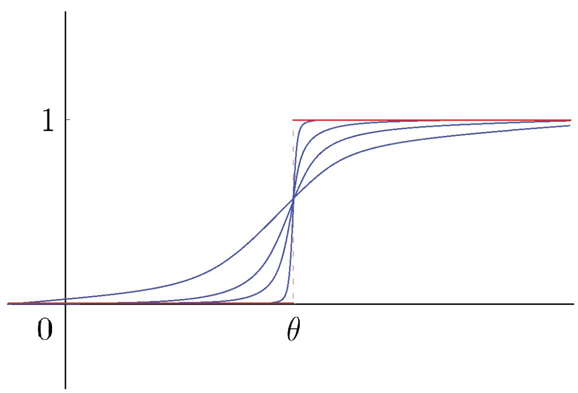

Figure 1: Approximation of the Heaviside firing rate function (red) by continuous functions (blue).

So, if the stationary solution to (5) exists, it satisfies the following equation

$$\begin{gathered} u(x) = \int\limits_{\Xi} \langle \omega \rangle (x - y) f_\beta(u(y)) dy, \\ \langle \omega \rangle (x) = \int\limits_{\mathcal{Y}} \omega(x, x_{\rm f}) dx_{\rm f}, \\ x \in \Xi \subseteq R^m, x_{\rm f} \in \mathcal{Y}. \end{gathered} \tag{6}$$

We are interested here in one particular type of solutions, which possesses the following properties.

Definition 3.1. Let $\theta > 0$ be fixed. We say that $u \in C^1(\Xi, R)$ satisfies the *θ-condition* if

(**B1**) there is a finite set of open bounded domains $\Theta_i \subset \Xi$ such that $u(x) > \theta$ on $\Theta = \bigcup_{i=1}^{N} \Theta_i$;

(**B2**) for any point x of the boundary $\mathcal{B} = \bigcup_{i=1}^{N} \mathcal{B}_i$ of Θ, it holds true that $u'(x) \neq 0$;

(**B3**) there exist $\sigma > 0$ and $r > 0$ such that $u(x) < \theta - \sigma$ for all $x \in \Xi \setminus B_{R^m}(\Theta, r)$.

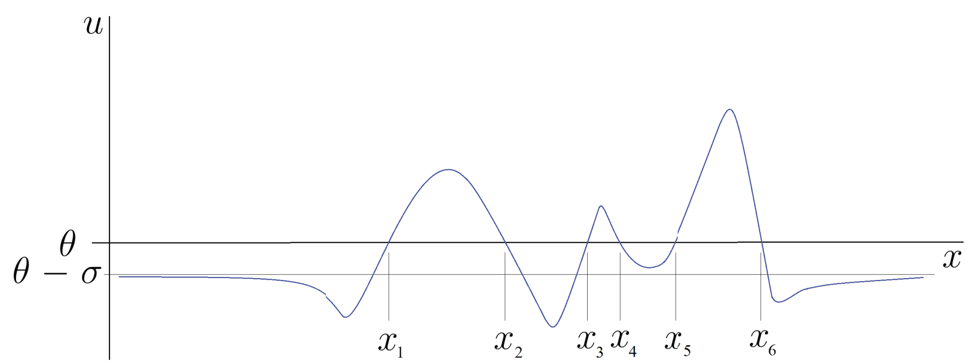

Figure 2: Example of function $U \in C^1(R, R)$ satisfying *θ-condition*. Here $\Theta = (x_1, x_2) \cup (x_3, x_4) \cup (x_5, x_6)$, $\mathcal{B} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

Remark 3.1. Definition 3.1 implies $\mathcal{B}_i \bigcap \mathcal{B}_j = \emptyset$ for any $i, j = 1, \ldots, N$, $i \neq j$.

In this section we assume existence of the stationary solution $U \in C^1(R^m, R)$ to (6), ($\Xi = R^m$), which corresponds to $\beta = 0$ and satisfies θ-condition. We are interested here in conditions, which guarantee existence of solutions u_β to (6) for $\beta > 0$ (i.e. in the case of continuous function f_β) and convergence of these solutions to U as $\beta \to 0$.

The following theorem provides conditions for convergence of the solutions

u_β to (6), $\beta > 0$, (if these solutions exist) to the stationary solution U to (6) at $\beta = 0$.

Theorem 3.1. (Continuous dependence) Let the assumptions (**A1**)$-$(**A4**) hold true, $\theta > 0$ be fixed and $U \in C^1(R^m, R)$ satisfies the θ-condition. Then there exists $\varepsilon > 0$ such that for any (sufficiently large) closed $\Omega \subset R^m$, if we assume existence of solutions $u_\beta \in B_{C^1(\Omega,R)}(U, \varepsilon)$ to the equation (6) for any $\beta \in (0, 1]$ ($\Xi = \Omega$), then there exist a solution to (6) at $\beta = 0$ and it is a limit point of the set $\{u_\beta\}$. Moreover, if the solution of (6) at $\beta = 0$ ($\Xi = \Omega$), say u_0, is unique then $\|u_\beta - u_0\|_{C^1(\Omega,R)} \to 0$.

Proof. We are going to apply Lemma 2.1, so we represent (6) in terms of the parameterized operator equation

$$u = F_\beta u,$$

where

$$F_\beta = \mathcal{W} \circ \mathcal{N}_\beta. \tag{7}$$

Here, for any $\beta \in [0, \infty)$, the Nemytskii operator

$$(\mathcal{N}_\beta u)(x) = f_\beta(u(x)), \tag{8}$$

and the linear integral operator

$$(\mathcal{W}u)(x) = \int\limits_{\Xi} \langle \omega \rangle (x - y) u(y) dy. \tag{9}$$

We introduce some important notations. For an arbitrary $\varepsilon > 0$, we denote the open sets $\Theta^{+\varepsilon} \subset R^m$ and $\Theta^{-\varepsilon} \subset R^m$ such that $U(x) > \theta + \varepsilon$ on $\Theta^{+\varepsilon} = \bigcup\limits_{i=1}^{N^{+\varepsilon}} \Theta_i^{+\varepsilon}$ and $U(x) > \theta - \varepsilon$ on $\Theta^{-\varepsilon} = \bigcup\limits_{i=1}^{N^{-\varepsilon}} \Theta_i^{-\varepsilon}$, respectively. The boundaries of these sets we denote as $\mathcal{B}^{+\varepsilon} = \bigcup\limits_{i=1}^{N^{+\varepsilon}} \mathcal{B}_i^{+\varepsilon}$ and $\mathcal{B}^{-\varepsilon} = \bigcup\limits_{i=1}^{N^{-\varepsilon}} \mathcal{B}_i^{-\varepsilon}$, respectively.

By the virtue of the conditions (**B1**) $-$ (**B3**) imposed on $U \in C^1(R^m, R)$ and Remark 3.1, there exists $\varepsilon_0 \in (0, \sigma/2)$ such that

$$N^{+\varepsilon_0} = N^{-\varepsilon_0} = N, \ \mathcal{B} \subset \Theta^{-\varepsilon_0} \setminus \Theta^{+\varepsilon_0},$$

$$\mathcal{B}_i^{-\varepsilon_0} \bigcap \mathcal{B}_j^{-\varepsilon_0} = \emptyset \text{ for any } i, j = 1, \ldots, N, \ i \neq j.$$

Choosing an arbitrary compact Ω, $\Theta^{-\varepsilon_0} \subset \Omega$, for any $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$, we get the conditions (**B1**), (**B2**) fulfilled and the following condition holding true instead of (**B3**):

($\mathbf{B3_{(\Omega)}}$) $u(x) < \theta - \sigma/2$ for all $x \in \Omega \setminus \Theta^{-\varepsilon_0}$.

Now we show that $\mathcal{N}_\beta : B_{C^1(\Omega,R)}(U, \varepsilon_0) \to L(\Omega, \mu, R)$ defined by (8) is continuous at any $\widehat{\beta} \in [0, \infty)$ uniformly on $B_{C^1(\Omega,R)}(U, \varepsilon_0)$. For $\widehat{\beta} \in [0, \infty)$, and $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$, we estimate $\|N_\beta u - N_{\widehat{\beta}} u\|_{L(\Omega,\mu,R)}$, as $\beta \to \widehat{\beta}$. The case $\widehat{\beta} \in (0, \infty)$ is trivial, as by the virtue of (**A4**), we immediately get

$$\int\limits_{\Omega} |f_\beta(u(x)) - f_{\widehat{\beta}}(u(x))| dx \to 0, \ \beta \to \widehat{\beta}$$

uniformly with respect to $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$. So, we focus on the more involved case $\widehat{\beta} = 0$.

$$\int\limits_{\Omega} |f_\beta(u(x)) - f_0(u(x))| dx =$$

$$= \int\limits_{\Theta^{+\varepsilon_0} \bigcup (\Omega \setminus \Theta^{-\varepsilon_0})} |f_\beta(u(x)) - f_0(u(x))| dx + \int\limits_{\Theta^{-\varepsilon_0} \setminus \Theta^{+\varepsilon_0}} |f_\beta(u(x)) - f_0(u(x))| dx. \tag{10}$$

For all $x \in \Theta^{+\varepsilon_0} \bigcup (\Omega \setminus \Theta^{-\varepsilon_0})$ and any $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$, $u(x)$ belongs to $R \setminus B_R(\theta, \varepsilon_0)$. Taking into account (**A4**), we get the first summand on the right-hand side of (10) converging to 0 uniformly on $B_{C^1(\Omega,R)}(U, \varepsilon_0)$, as $\beta \to 0$. Next,

$$\int\limits_{\Theta^{-\varepsilon_0} \setminus \Theta^{+\varepsilon_0}} |f_\beta(u(x)) - f_0(u(x))| dx < \frac{1}{c_0} \int\limits_{-\|U\|_{C^1(\Omega,R)}}^{\|U\|_{C^1(\Omega,R)}} |f_\beta(s) - f_0(s)| ds,$$

where $0 < c_0 < |u'(x)|$ for all $x \in \Theta^{+\varepsilon_0} \bigcup (\Omega \setminus \Theta^{-\varepsilon_0})$ and any $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$ (We assume here that $\varepsilon_0 < \min\limits_{x \in \Theta^{-\varepsilon_0} \setminus \Theta^{+\varepsilon_0}} |U'(x)|$, otherwise we repeat the procedure above with the new $\varepsilon_0 = \varepsilon_1 < \min\limits_{x \in \Theta^{-\varepsilon_1} \setminus \Theta^{+\varepsilon_1}} |U'(x)|$). Finally, we notice that assumption (**A4**) guarantees convergence to 0 of the expression on the right-hand side of the latter inequality, as $\beta \to 0$.

Thus, for any compact $\Omega \subset R^m$, $\mathcal{N}_\beta : B_{C^1(\Omega,R)}(U,\varepsilon_0) \to L(\Omega,\mu,R)$ is continuous at any $\widehat{\beta} \in [0,\infty)$ uniformly on $B_{C^1(\Omega,R)}(U,\varepsilon_0)$, which means that for all $\beta \in [0,\infty)$, the Nemytskii operator $\mathcal{N}_\beta$ is a bounded mapping from $B_{C^1(\Omega,R)}(U,\varepsilon_0)$ to $L(\Omega,\mu,R)$. We also notice that the operator $\mathcal{W}$ defined by (9) ($\Xi = \Omega$) is a linear and continuous mapping from $L(\Omega,\mu,R)$ to $C^1(\Omega,R)$ provided that assumptions (**A1**) and (**A2**) hold true.

Thus, for any $\beta \in [0,\infty)$, $F_\beta : B_{C^1(\Omega,R)}(U,\varepsilon_0) \to C^1(R^m,R)$ and

$$\|F_\beta u - F_{\widehat{\beta}}\widehat{u}\|_{C^1(\Omega,R)} \to 0,\ \beta \to \widehat{\beta},\ \|u-\widehat{u}\|_{C^1(\Omega,R)} \to 0, \text{ where } \widehat{u} \in B_{C^1(\Omega,R)}(U,\varepsilon_0).$$

Next, we prove that $F_\beta : B_{C^1(\Omega,R)}(U,\varepsilon_0) \to C^1(\Omega,R)$ ($\beta \in [0,\infty)$) are collectively compact.

By the virtue of (**A3**), (**A4**), it suffices to show that for an arbitrary $\epsilon > 0$, the set $\{\int\limits_\Omega \langle\omega\rangle(x-y)\kappa dy,\ \kappa \in [0,1]\}$ possesses a finite ϵ-net in $C^1(\Omega,R)$. We represent $\langle\omega\rangle = (\langle\omega\rangle_l)$, where $\langle\omega\rangle_l \in C^2(\Omega_l,R)$, Ω_l is the orthogonal projection of Ω to the axis OX_l ($l = 1,\dots,m$).

Choose an arbitrary $\widehat{l}$. Suppose that $\Omega_{\widehat{l}} = [a,b]$,

$$\int\limits_{[a,b]} \langle\omega\rangle_{\widehat{l}}(a-s)ds = A,$$

$$\int\limits_{[a,b]} \langle\omega\rangle'_{\widehat{l}}(a-s)ds = A',$$

$$\max_{t\in[a,b]} \int\limits_{[a,b]} \langle\omega\rangle''_{\widehat{l}}(t-s)ds = M.$$

Then, for example, the set

$$\Big\{\alpha_i + \kappa_j t,\ \alpha_i = i\frac{A + (b-a)(A' + (b-a)M))}{[(A + (b-a)(A' + (b-a)M)))/\epsilon] + 1},$$

$$\kappa_j = j\frac{A' + (b-a)M}{[(A' + (b-a)M)/\epsilon] + 1},$$

$$i = 0,1,\dots,[(A + (b-a)(A' + (b-a)M))/\epsilon] + 1,$$

$$j = 0,1,\dots,[(A' + (b-a)M)/\epsilon] + 1,\ t \in [a,b]\Big\}$$

serves as the ϵ-net for $\{\int\limits_{\Omega}\langle\omega\rangle_{\widehat{l}}(x-y)\kappa dy,\ \kappa\in[0,1]\}$ ($[z]$ denotes here the integer part of $z\in R$). Due to arbitrary choice of the component $\int\limits_{\Omega}\langle\omega\rangle_{\widehat{l}}(x-y)dy$ of $\int\limits_{\Omega}\langle\omega\rangle(x-y)dy$ ($l=1,\dots,m$), we proved collective compactness of the whole composition $F_\beta=\mathcal{W}\circ\mathcal{N}_\beta$ ($\beta\in[0,\infty)$) as acting from $B_{C^1(\Omega,R)}(U,\varepsilon_0)$ to $C^1(\Omega,R)$.

Now, if we keep in mind the properties proved and put $T(\lambda,\mathfrak{b})=F_\beta u$, $\Lambda=[0,1]$, $D=\overline{B_{C^1(\Omega,R)}(U,\varepsilon_1)}$, $\varepsilon_1<\varepsilon_0$, by using Lemma 2.1, we complete the proof. $\square$

It is often easier to study existence of solutions satisfying θ-condition to (6) when $\beta=0$. The corresponding closed form expressions for the particular types of solutions (satisfying θ-condition) to special cases of (6) can be found e.g. in [1, 17, 22, 18, 25, 5].

The next theorem provides a tool for proving existence of solutions to (6) for $\beta\in(0,\infty)$ using some knowledge about the solution to (6) at $\beta=0$.

Theorem 3.2. (Existence) Let the conditions of Theorem 3.1 be satisfied, the set Ω and the constant ε_1 be taken from Theorem 3.1. Assume that there exists solution $U\in C^1(R^m,R)$ of (6) at $\beta=0$, which satisfies θ-condition and which is unique in $\overline{B_{C^1(\Omega,R)}(U,\varepsilon_2)}$ ($\varepsilon_2<\varepsilon_1$), and $\deg(I-F_0,B_{C^1(\Omega,R)}(U,\varepsilon_2),0)\neq 0$, where the operator $F_0:B_{C^1(\Omega,R)}(U,\varepsilon_1)\to C^1(\Omega,R)$ is given by (7). Then for any $\beta\in(0,1]$, there exists solution $u_\beta\in B_{C^1(\Omega,R)}(U,\varepsilon_2)$ to the equation (6).

Proof. We prove that the family $\{h_\beta\}$, $\beta\in[0,1]$,

$$h_\beta=I-F_\beta \tag{11}$$

is homotopy. Continuity of $h_{(\cdot)}(\cdot)$ on $[0,1]\times B_{C^1(\Omega,R)}(U,\varepsilon_1)$ follows from the proof of Theorem 3.1. It remains to prove that $h_\beta(u)\neq 0$ for any $\beta\in[0,1]$ and $u\in\partial B_{C^1(\Omega,R)}(U,\varepsilon_2)$.

Collective compactness of $F_\beta:B_{C^1(\Omega,R)}(U,\varepsilon_1)\to C^1(\Omega,R)$ ($\beta\in[0,\infty)$), shown in the proof of Theorem 3.1, imply the following two possibilities for any sequence $\{u_{\beta_n}\}\subset B_{C^1(\Omega,R)}(U,\varepsilon_1)$ ($\beta_n\to 0$) of solutions to (6):

1) u_{β_n} converges to U, as $\beta_n\to 0$;

2) there exists such $\widehat{n}$ that for any $n>\widehat{n}$, $\|u_{\beta_n}-U\|_{C^1(\Omega,R)}>\varepsilon_2$ (without loss of generality we can assume that $\beta_{\widehat{n}}>1$).

This proves that $(I-F_\beta)(u)\neq 0$ for any $\beta\in[0,1]$ and $u\in\partial B_{C^1(\Omega,R)}(U,\varepsilon_1)$.

Finally, we apply Lemma 2.2 to the homotopy (11) and get existence of solutions to (6) for any $\beta\in(0,1]$. $\square$

Remark 3.2. The choice of the space $C^1(\Omega, R)$ as a basic functional space in this research is caused by the fact that even in the space of absolutely continuous functions, any ball, centered at a function satisfying θ-condition, contains functions, which do not satisfy θ-condition. The corresponding example can be found in [21], in the proof of Lemma 3.7.

4. Bumps in neural field models

In this section we apply the theory developed to the stationary bump solutions to the neural field model (5) in the following three special cases:

1. Symmetric single bump in 1-D.
2. Symmetric double bump in 1-D.
3. Radially symmetric single bump in 2-D.

Each subsection concludes with a theorem on existence and continuous dependence of the stationary solutions of the corresponding type to the equation (5) when approximating the Heaviside activation function by continuous functions.

4.1. Symmetric single bump in 1-D

We consider here the one-dimensional homogenized Amari model, i.e. the model (5) with $m = k = 1$:

$$\begin{gathered}\partial_t u(t,x,x_{\mathrm{f}}) = -u(t,x,x_{\mathrm{f}}) + \int\limits_{\Xi}\int\limits_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f_\beta(u(t,y,y_{\mathrm{f}}))dy_{\mathrm{f}}dy, \\ t>0,\ x\in\Xi\subseteq R.\end{gathered} \tag{12}$$

Here $\mathcal{Y}$ is some one-dimensional torus, the family of functions $f_\beta : R \to [0,1]$ satisfies assumptions (**A3**), (**A4**), and the function ω is typically decomposed in the following way (see e.g. [25], [18]):

$$\omega(x, x_{\mathrm{f}}) = \frac{1}{\sigma(x_{\mathrm{f}})}\chi\Big(\frac{|x|}{\sigma(x_{\mathrm{f}})}\Big), \tag{13}$$

where the function $\sigma \in C(\mathcal{Y}, (0,\infty))$ is $\mathcal{Y}$-periodic and the function $\chi \in C^2([0,\infty), R) \bigcap L([0,\infty), \mu, R)$ satisfies the property:

$$\lim_{x\to\infty} \chi(x) = 0. \tag{14}$$

Thus, assumptions (**A1**), (**A2**) are also satisfied. We emphasize here that the class of connectivity functions ω described above is rather wide. It contains all typical connectivity functions in use in the neural field theory (see e.g. [25], [18] for the heterogeneous media case, and the review [2] for the homogeneous media case).

Definition 4.1.1. Let $\theta > 0$ be fixed. We define a *symmetric single bump solution* to (12) to be a stationary solution $U \in C^1(\Xi, R)$ to (12), satisfying the following properties:

- $U(x) = U(-x)$ for all $x \in R$;
- the equation $U(x) = \theta$ has exactly two solutions $x = -a,\ x = a,\ a > 0$;
- $U(x) > \theta$ for all $x \in (-a, a)$ and $U(x) < \theta$ for all $x \in \Xi \setminus [-a, a]$.

The stationary symmetric single bump solution to (12) in the case $\beta = 0$ can be determined by the following expression (see e.g. [25]):

$$U(x) = W(x + a) - W(x - a), \tag{15}$$

where

$$W(x) = \int_0^x \langle\omega\rangle(y)dy,$$
$$\langle\omega\rangle(x) = \int_{\mathcal{Y}} \omega(x, x_{\mathrm{f}})dx_{\mathrm{f}}.$$

Due to the assumptions on the functions $\chi \in C^2(R, R) \bigcap L(R, \mu, R)$ and $\sigma \in C(\mathcal{Y}, (0, \infty))$, and the corresponding properties of the connectivity ω defined by (13), we get the following condition fulfilled:

$$\lim_{|x|\to\infty} \langle\omega\rangle(x) = 0. \tag{16}$$

Using the latter expression, we easily obtain

$$\lim_{|x|\to\infty} U(x) = 0.$$

Thus, the bump solution U satisfies θ-condition.

We investigate existence and continuous dependence of stationary bump solutions to (12), which are symmetric with respect to the ordinate axis, when approximating the Heaviside activation function in (12) (the case $\beta = 0$) by continuous functions ($\beta > 0$). Indeed, due to the translational invariance of the integration kernel ω with respect to the spatial variable x, the corresponding operators F_β ($\beta \in [0,1]$) defined by (7) map even functions to even functions. We, thus, consider solutions belonging to the space $C_e^1(\Xi, R) = \{u \in C^1(\Xi, R),\ u(x) = u(-x) \text{ for all } x \in \Xi\}$.

Lemma 4.1.1. Let the following condition be satisfied:

$$\langle \omega \rangle (2a) \neq 0. \tag{17}$$

Then for any compact set Ω, $\Omega \in R$, there exists such $\varepsilon > 0$ that the symmetric single bump U defined by (15) is a unique solution to (12) in $B_{C_e^1(\Omega,R)}(U, \varepsilon)$ when $\beta = 0$.

Proof. From the definition of the single bump solution it follows that

$$W(2a) = \theta.$$

Thus, the condition (17) guarantees uniqueness of the solution U in $B_{C_e^1(\Omega,R)}(U, \varepsilon)$ for some $\varepsilon > 0$. $\Box$

We emphasize that U is not an isolated solution to (12) in $C^1(\Xi, R)$ due to the translation invariance of bumps in the homogenized neural field (12).

We now express (15) in terms of operator equality just as it was done in Section 3:

$$U = F_0 U.$$

In order to apply Theorem 3.2, we need to calculate $\deg(I - F_0, B_{C_e^1(\Omega,R)}(U, \varepsilon), 0)$. By the definition of the topological fixed point index, we get

$$\deg(I - F_0, B_{C_e^1(\Omega,R)}(U, \varepsilon), 0) = \operatorname{ind}(F_0, B_{C_e^1(\Omega,R)}(U, \varepsilon)).$$

Without loss of generality we assume that the fixed point U of the operator F_0 is unique in $\overline{B_{C_e^1(\Omega,R)}(U, \varepsilon)}$. Thus, F_0 maps $\overline{B_{C_e^1(\Omega,R)}(U, \varepsilon)}$ into some manifold $\mathcal{M} \subset C^1(\Omega, R)$, $\mathcal{M} = \{v = W(\cdot + c) - W(\cdot - c),\ c \in \mathrm{M} \subset \Omega\}$, where compact set M is chosen in a such way that it contains c_u for all $u \in \overline{B_{C_e^1(\Omega,R)}(U, \varepsilon)}$ (One can e.g. choose M to be a segment). We define the mapping $\phi : \mathrm{M} \to \mathcal{M}$ as

$$\phi(c) = v(x),\ v(x) = W(x + c) - W(x - c),\ x \in \Omega. \tag{18}$$

Lemma 4.1.2. The mapping $\phi : \mathrm{M} \to \mathcal{M}$ defined by (18) is a homeomorphism, and $\mathcal{M}$ is an absolute neighborhood retract.

Proof. First, we note that $\phi : \mathrm{M} \to \mathcal{M}$ is a surjection by definition. In order to prove that $\phi : \mathrm{M} \to \mathcal{M}$ is an injection, we use the expression for the Frechet derivative of ϕ taken at an arbitrary $c \in \mathrm{M}$:

$$\phi'(c) = \langle\omega\rangle(\cdot + c) - \langle\omega\rangle(\cdot - c).$$

For sufficiently large set $\Omega = [-X, X]$, $X \gg a$, the condition (16) implies the following relation:

$$\max_{x\in[X-2a,X]} |\langle\omega\rangle(x)| < \max_{x\in[0,2a]} |\langle\omega\rangle(x)|. \tag{19}$$

Thus, we have $\phi'(a) \neq 0$, because assuming the contrary, we get $\langle\omega\rangle(x + a) - \langle\omega\rangle(x - a) = 0$, for all $x \in \Omega$, which contradicts with (19). Summarizing the described above properties of ϕ, we conclude that $\phi : \mathrm{M} \to \mathcal{M}$ is a homeomorphism. We also note that the set M is an absolute neighborhood retract, since it is a compact convex subset of R. Thus, by properties of homeomorphism, $\mathcal{M} = \phi(\mathrm{M})$ is an absolute neighborhood retract, too. $\square$

We now define $\mathcal{F}$ to be the restriction of F_0 on $\mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)}$, i.e.

$$\mathcal{F} = F_0|_{\mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)}},$$
$$\mathcal{F} : \mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)} \to \mathcal{M}.$$

Due to its definition, the mapping $\mathcal{F} : \mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)} \to \mathcal{M}$ is compact and admissible. Using the properties of the topological fixed point index (see e.g. [10]), we get

$$\mathrm{ind}(F_0, B_{C^1_e(\Omega,R)}(U,\varepsilon)) = \mathrm{ind}(\mathcal{F}, \mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon)).$$

Next, we apply Lemma 2.3 and obtain

$$\mathrm{ind}(\mathcal{F}, \mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon)) = \mathrm{ind}(\phi^{-1}\circ\mathcal{F}\circ\phi, \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))).$$

Lemma 4.1.3. There exists such $\delta > 0$ that the operator $\Psi = \phi^{-1}\circ\mathcal{F}\circ\phi$ maps $\overline{B_R(a,\delta)}$ to M.

Proof. Let $u(x) = W(x + c) - W(x - c)$, $c \in \mathrm{M}$. Using the mean value theorem, we estimate

$$\|u - U\|_{C^1(\Omega,R)} \leq 4\|\langle\omega\rangle\|_{C^1(\Omega,R)}|c - a| < \varepsilon$$

for all $c \in \overline{B_R(a,\delta)}$, where $\delta < \varepsilon/4\|\langle\omega\rangle\|_{C^1(\Omega,R)}$. From the latter estimate we conclude that

$$\overline{B_R(a,\delta)} \subset \phi^{-1}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))$$

which, in turn, implies

$$\mathcal{M}_\delta = \{v = W(\cdot + c) - W(\cdot - c),\ c \in \overline{B_R(a,\delta)}\} \subset \mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon)).$$

Thus, we finally get

$$\phi^{-1}(\mathcal{M}_\delta) = \overline{B_R(a,\delta)} \subset \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))),$$

which concludes the proof. □

It is easy to see that a is a fixed point of the operator $\Psi : \overline{B_R(a,\delta)} \to$ M. Moreover, a is an isolated fixed point of Ψ due to the fact that U is an isolated fixed point of $\mathcal{F}$ and topological invariance property of the index. The topological index of a finite dimensional map can be calculated as

$$\mathrm{ind}(\Psi, \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))) = \mathrm{sgn}(1 - \Psi'(a)),$$

see e.g. [16].

It follows from the definition of the operator $\Psi = \phi^{-1}\circ\mathcal{F}\circ\phi$ that

$$W(\Psi(c) + c) - W(\Psi(c) - c) = \theta \ \text{ for all } \ c \in \overline{B_R(a,\delta)}.$$

Using the implicit function theorem and the chain rule for differentiation, we get

$$\Psi'(a) = \frac{\langle\omega\rangle(0) + \langle\omega\rangle(2a)}{\langle\omega\rangle(0) - \langle\omega\rangle(2a)}.$$

Thus, $\deg(I - F_0, B_{C^1_e(\Omega,R)}(U,\varepsilon), 0) \neq 0$ as soon as the following inequality takes place:

$$\frac{\langle\omega\rangle(0) + \langle\omega\rangle(2a)}{\langle\omega\rangle(0) - \langle\omega\rangle(2a)} \neq 1.$$

Summarizing the results above and using Theorem 3.2 and Theorem 3.1, we get the main result of the subsection.

Theorem 4.1.1. Let the family of functions $f_\beta : R \to [0,1]$ ($\beta \in [0,\infty)$) satisfy assumptions (**A3**) and (**A4**). Let also the connectivity kernel ω be given by (13), where the function $\sigma \in C(\mathcal{Y}, (0,\infty))$ is $\mathcal{Y}$-periodic and the even function $\chi \in C^2(R,R) \bigcap L(R,\mu,R)$ satisfies (14). Finally, let the inequality (17) be fulfilled. Then, for any sufficiently large Ω, $\Omega \subset R$, and for each $\beta \in (0,\infty)$, there exists solution $u_\beta \in C^1_e(\Omega,R))$ to (12) ($\Xi = \Omega$). Moreover, $\|u_\beta - U\|_{C^1(\Omega,R))} \to 0$, as $\beta \to 0$, where $U \in C^1_e(R,R))$ is the stationary bump solution to (12) ($\Xi = R$, $\beta = 0$), defined by (15).

4.2. Symmetric double bump in 1-D

We keep here the modeling framework (12) under the same assumptions on the functions involved as in the previous subsection.

Definition 4.2.1. Let $\theta > 0$ be fixed. We define a *symmetric double bump solution* to (12) to be a stationary solution $U \in C^1(\Xi, R)$ to (12), satisfying the following properties:

- $U(x) = U(-x)$ for all $x \in R$;
- the equation $U(x) = \theta$ has exactly four solutions $x = -b$, $x = -a$, $x = a$, $x = b$, $b > a > 0$;
- $U(x) > \theta$ for all $x \in (-b, -a) \bigcup (a, b)$ and $U(x) < \theta$ for all $x \in (-a, a) \bigcup \Xi \setminus [-b, -a] \setminus [a, b]$.

The stationary (symmetric) double bump solution to (12) ($\beta = 0$) can be written as

$$U(x) = W(x + b) - W(x + a) + W(x - a) - W(x - b), \tag{20}$$

(see e.g. [18]).

Using the expression (16), we obtain

$$\lim_{|x|\to\infty} U(x) = 0.$$

It is easy to see now that the double bump solution U satisfies θ-condition.

Just as in the previous subsection, we investigate here existence and continuous dependence on the steepness of the function $f_\beta : R \to [0, 1]$ of the stationary double bump solutions to (12) belonging to $C_e^1(\Xi, R)$.

Lemma 4.2.1. Let the following condition be satisfied:

$$\begin{cases} \langle\omega\rangle(b - a) - \langle\omega\rangle(2a) \neq 0, \\ \langle\omega\rangle(b - a) + \langle\omega\rangle(b + a) \neq 0. \end{cases} \tag{21}$$

Then for any compact set Ω, $\Omega \in R$, there exists such $\varepsilon > 0$ that the symmetric double bump U defined by (20) is a unique solution to (12) in $B_{C_e^1(\Omega,R)}(U, \varepsilon)$ when $\beta = 0$.

Proof. From the definition of the single bump solution it follows that

$$\begin{cases} W(b - a) - W(b + a) + W(2b) = \theta, \\ W(2b) + W(2a) - 2W(b + a) = 0. \end{cases} \tag{22}$$

Differentiation of this expression with respect to the parameter a gives us

$$\begin{cases} \langle\omega\rangle(b-a) - \langle\omega\rangle(b+a) = 0, \\ \langle\omega\rangle(2a) - \langle\omega\rangle(b+a) = 0, \end{cases}$$

from where we get

$$\langle\omega\rangle(b-a) - \langle\omega\rangle(2a) = 0.$$

Differentiating (22) with respect to the parameter b, we obtain

$$\begin{cases} \langle\omega\rangle(b-a) - \langle\omega\rangle(b+a) + 2\langle\omega\rangle(2b) = 0, \\ \langle\omega\rangle(2b) - \langle\omega\rangle(b+a) = 0, \end{cases}$$

which implies

$$\langle\omega\rangle(b-a) + \langle\omega\rangle(b+a) = 0.$$

Thus, the condition (21) guarantees uniqueness of the solution U in $B_{C_e^1(\Omega,R)}(U,\varepsilon)$ for some $\varepsilon > 0$. $\square$

We express (20) in terms of the operator equality

$$U = F_0 U.$$

Without loss of generality we assume that the fixed point U of the operator F_0 is unique in $\overline{B_{C_e^1(\Omega,R)}(U,\varepsilon)}$. Thus, F_0 maps $\overline{B_{C_e^1(\Omega,R)}(U,\varepsilon)}$ into some manifold $\mathcal{M} \subset C^1(\Omega, R)$, $\mathcal{M} = \{v = W(x+d) - W(x+c) + W(x-c) - W(x-d),\ (c,d) \in \mathrm{M} \subset R^2\}$, where compact set M is chosen in a such way that it contains the points (c_u, d_u) for all $u \in \overline{B_{C_e^1(\Omega,R)}(U,\varepsilon)}$ (One can e.g. choose M to be a rectangle). We define the mapping $\phi : \mathrm{M} \to \mathcal{M}$ as

$$\begin{gathered} \phi((c,d)) = v(x), \\ v(x) = W(x+d) - W(x+c) + W(x-c) - W(x-d),\ x \in \Omega. \end{gathered} \tag{23}$$

Lemma 4.2.2. The mapping $\phi : \mathrm{M} \to \mathcal{M}$ defined by (23) is a homeomorphism, and $\mathcal{M}$ is an absolute neighborhood retract.

Proof. First, we note that $\phi : \mathrm{M} \to \mathcal{M}$ is a surjection by definition. In order to prove that $\phi : \mathrm{M} \to \mathcal{M}$ is an injection, we use the following expressions for the Frechet derivatives of ϕ:

$$\begin{gathered} \phi'_c((c,d)) = \langle\omega\rangle(\cdot - c) - \langle\omega\rangle(\cdot + c) \\ \phi'_d((c,d)) = \langle\omega\rangle(\cdot + d) - \langle\omega\rangle(\cdot - d) \end{gathered}$$

Assuming $\phi'_c((a,b)) = 0$, we get $\langle\omega\rangle(x-a) - \langle\omega\rangle(x+a) = 0$, for all $x \in \Omega$, which contradicts with (19). We, thus, have $\phi'_c((a,b)) \neq 0$. By the same way we obtain $\phi'_d((a,b)) \neq 0$, which concludes the proof of the fact that $\phi : \mathrm{M} \to \mathcal{M}$ is a homeomorphism. As the set M is an absolute neighborhood retract, then by properties of homeomorphism, the set $\mathcal{M} = \phi(\mathrm{M})$ is an absolute neighborhood retract, too. $\square$

Just as in the previous subsection, we define

$$\mathcal{F} = F_0|_{\mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)}},$$
$$\mathcal{F} : \mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)} \to \mathcal{M}.$$

The mapping $\mathcal{F} : \mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)} \to \mathcal{M}$ is compact and admissible by its definition. Using the properties of the topological fixed point index, we get

$$\mathrm{ind}(F_0, B_{C^1_e(\Omega,R)}(U,\varepsilon)) = \mathrm{ind}(\mathcal{F}, \mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon)).$$

Applying Lemma 2.3, we obtain

$$\mathrm{ind}(\mathcal{F}, \mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon)) = \mathrm{ind}(\phi^{-1}\circ\mathcal{F}\circ\phi, \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))).$$

Lemma 4.2.3. There exists such $\delta > 0$ that the operator $\Psi = \phi^{-1}\circ\mathcal{F}\circ\phi$ maps $\overline{B_{R^2}((a,b),\delta)}$ to M.

Proof. Let $u(x) = W(x+d) - W(x+c) + W(x-c) - W(x-d)$, $(c,d) \in \mathrm{M}$. Using the mean value theorem, we estimate

$$\|u - U\|_{C^1(\Omega,R)} \leq 4\|\langle\omega\rangle\|_{C^1(\Omega,R)}(|c-a| + |d-b|) < \varepsilon$$

for all $(c,d) \in \overline{B_{R^2}((a,b),\delta)}$, where $\delta < \varepsilon/8\sqrt{2}\|\langle\omega\rangle\|_{C^1(\Omega,R)}$. From the latter estimate we conclude that

$$\overline{B_{R^2}((a,b),\delta)} \subset \phi^{-1}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))$$

which implies

$$\mathcal{M}_\delta \subset \mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))$$
$$\mathcal{M}_\delta = \{v = W(\cdot + d) - W(\cdot + c) + W(\cdot - c) - W(\cdot - d),$$
$$(c,d) \in \overline{B_{R^2}((a,b),\delta)}\}.$$

Thus, we finally get

$$\phi^{-1}(\mathcal{M}_\delta) = \overline{B_{R^2}((a,b),\delta)} \subset \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))),$$

which concludes the proof. $\square$

Due to the fact that U is an isolated fixed point of $\mathcal{F}$ and topological invariance property of the index, (a,b) is an isolated fixed point of Ψ. Thus, we get

$$\begin{gathered}\Psi((a,b))=(\Psi_1((a,b))\Psi_2((a,b))),\\ \Psi((a,b))=W(\Psi_2((a,b))+b)-W(\Psi_1((a,b))+a)+\\ +W(\Psi_1((a,b))-a)-W(\Psi_2((a,b))-b).\end{gathered}$$

We calculate the topological index of a two-dimensional mapping as

$$\begin{gathered}\operatorname{ind}(\Psi,\phi^{-1}(\mathcal{F}(\mathcal{M}\bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon)))=\\ =\operatorname{sgn}\left(\det\begin{pmatrix}(\Psi_1)'_a((a,b))-1 & (\Psi_1)'_b((a,b))\\ (\Psi_2)'_a((a,b)) & (\Psi_2)'_b((a,b))-1\end{pmatrix}\right).\end{gathered}$$

The definition of the operator $\Psi=\phi^{-1}\circ\mathcal{F}\circ\phi$ yields

$$W(\Psi_2((c,d))+d)-W(\Psi_1((c,d))+c)+W(\Psi_1((c,d))-c)-W(\Psi_2((c,d))-d)=\theta$$

for all $(c,d)\in\overline{B_R(a,\delta)}$. We use the expressions

$$(U(a))'_a=0,\ (U(a))'_b=0,\ (U(b))'_a=0,\ (U(b))'_b=0.$$

Applying the implicit function theorem and the chain rule for differentiation, we get

$$\begin{aligned}(\Psi_1)'_a((a,b))&=\frac{\langle\omega\rangle(2a)+\langle\omega\rangle(0)}{\langle\omega\rangle(b+a)-\langle\omega\rangle(2a)+\langle\omega\rangle(0)-\langle\omega\rangle(b-a)};\\ (\Psi_1)'_b((a,b))&=\frac{-\langle\omega\rangle(b+a)-\langle\omega\rangle(b-a)}{\langle\omega\rangle(b+a)-\langle\omega\rangle(2a)+\langle\omega\rangle(0)-\langle\omega\rangle(b-a)};\\ (\Psi_2)'_a((a,b))&=\frac{\langle\omega\rangle(b+a)+\langle\omega\rangle(b-a)}{\langle\omega\rangle(2b)-\langle\omega\rangle(b+a)+\langle\omega\rangle(b-a)-\langle\omega\rangle(0)};\\ (\Psi_2)'_b((a,b))&=\frac{-\langle\omega\rangle(2b)-\langle\omega\rangle(0)}{\langle\omega\rangle(2b)-\langle\omega\rangle(b+a)+\langle\omega\rangle(b-a)-\langle\omega\rangle(0)}.\end{aligned}$$

Thus, $\deg(I-F_0,B_{C^1_e(\Omega,R)}(U,\varepsilon),0)\neq 0$ if the following inequality takes place:

$$\frac{2\langle\omega\rangle(b+a)\langle\omega\rangle(b-a)-2\langle\omega\rangle(2a)\langle\omega\rangle(2b)+\Big(\langle\omega\rangle(2a)+\langle\omega\rangle(2b)\Big)\Big(\langle\omega\rangle(b+a)-\langle\omega\rangle(b-a)\Big)}{\Big(\langle\omega\rangle(b+a)-\langle\omega\rangle(2a)+\langle\omega\rangle(0)-\langle\omega\rangle(b-a)\Big)\Big(\langle\omega\rangle(2b)-\langle\omega\rangle(b+a)+\langle\omega\rangle(b-a)-\langle\omega\rangle(0)\Big)}\neq 0. \tag{24}$$

The following statement is obtained by summarizing the results above and by using then Theorem 3.2 and Theorem 3.1.

Theorem 4.2.1. Let the family of functions $f_\beta : R \to [0,1]$ ($\beta \in [0,\infty)$) satisfy assumptions (**A3**) and (**A4**). Let also the connectivity kernel ω be given by (13), where the function $\sigma \in C(\mathcal{Y},(0,\infty))$ is $\mathcal{Y}$-periodic and the even function $\chi \in C^2(R,R) \bigcap L(R,\mu,R)$ satisfies (14). Finally, let the inequalities (21) and (24) be fulfilled. Then, for any sufficiently large Ω, $\Omega \subset R$, and for each $\beta \in (0,\infty)$, there exists solution $u_\beta \in C^1_e(\Omega,R))$ to (12) ($\Xi = \Omega$). Moreover, $\|u_\beta - U\|_{C^1(\Omega,R))} \to 0$, as $\beta \to 0$, where $U \in C^1_e(R,R))$ is the stationary double bump solution to (12) ($\Xi = R$, $\beta = 0$), defined by (20).

4.3. Radially symmetric single bump in 2-D

We now consider the two-dimensional homogenized Amari model, i.e. the model (5) with $m = k = 2$:

$$\partial_t u(t,x,x_{\mathrm{f}}) = -u(t,x,x_{\mathrm{f}}) + \int_{\Xi}\int_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f_\beta(u(t,y,y_{\mathrm{f}}))dy_{\mathrm{f}}dy, \\ t > 0,\ x \in \Xi \subseteq R^2. \tag{25}$$

Here $\mathcal{Y}$ is some two-dimensional torus, the family of functions $f_\beta : R \to [0,1]$ satisfies assumptions (**A3**), (**A4**), and the connectivity function $\omega : R^2 \times \mathcal{Y} \to R$ is decomposed in the following way (see e.g. [5]):

$$\omega(x,x_{\mathrm{f}}) = \frac{1}{\sigma(x_{\mathrm{f}})}\chi\Big(\frac{|x|}{\sigma(x_{\mathrm{f}})}\Big), \tag{26}$$

where $\sigma \in C(\mathcal{Y},(0,\infty))$ is $\mathcal{Y}$-periodic and $\chi \in C^2([0,\infty),R) \bigcap L([0,\infty),\mu,R)$. Thus, assumptions (**A1**) and (**A2**) are also satisfied.

Definition 4.3.1. Let $\theta > 0$ be fixed. We define a *radially symmetric single bump solution* to (25) to be a stationary solution $U \in C^1(\Xi,R)$ to (25), satisfying the following properties:

- $U(x) = U(|x|)$, where $x \in R^2$, $x = |x|\exp(i\alpha)$, $\alpha \in [0,2\pi)$;
- the equation $U(x) = \theta$ has only the solutions belonging to the set $\{x,\ |x| = \mathrm{r}\}$ for some $\mathrm{r} > 0$;
- $U(x) > \theta$ for all $x \in B_{R^2}(0,\mathrm{r})$ and $U(x) < \theta$ for all $x \in \Xi \setminus \overline{B_{R^2}(0,\mathrm{r})}$.

The stationary radially symmetric single bump solution of the radius a to (25) in the case $\beta = 0$ can be determined by the following expression (see e.g. [5]):

$$U(x) = 2\pi a \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|x|r) J_1(ar)dr, \tag{27}$$

where $\widehat{\langle\omega\rangle}$ is the Hankel transform (of order 0) of $\langle\omega\rangle$,

$$\langle\omega\rangle(x) = \int_{\mathcal{Y}} \omega(x, x_{\mathrm{f}})dx_{\mathrm{f}},$$

J_n is the Bessel function of the first kind of order n.

Let us assume that the following condition is satisfied:

$$\int_0^\infty |\widehat{\langle\omega\rangle}(r)| r^2 dr < \infty. \tag{28}$$

For an arbitrary $\gamma > 0$, using the properties of J_n, we have

$$|U(x)| \leq 2\pi a \int_0^\gamma |\widehat{\langle\omega\rangle}(r)|dr + 2\pi a \Big| \int_\gamma^\infty \widehat{\langle\omega\rangle}(r) J_0(|x|r) J_1(ar)dr \Big|.$$

Due to the assumptions on the functions $\chi \in C^2(R^2, R) \bigcap L(R^2, \mu, R)$ and $\sigma \in C(\mathcal{Y}, (0, \infty))$, and the corresponding properties of the connectivity function ω defined by (26), for an arbitrary $\epsilon > 0$, we obtain:

$$2\pi a \int_0^{\gamma(\epsilon)} |\widehat{\langle\omega\rangle}(r)|dr < \epsilon/2$$

for some $\gamma(\epsilon) > 0$. By the properties of the Bessel function J_0, for any $\gamma > 0$, we have $J_0(sr) \to 0$ uniformly with respect to $r \in [\gamma, \infty)$, as $s \to \infty$. Using these facts and the estimate (28), we finally get

$$|U(x)| \leq 2\pi a \int_0^{\gamma(\epsilon)} |\widehat{\langle\omega\rangle}(r)|dr + 2\pi a \Big| \int_{\gamma(\epsilon)}^\infty \widehat{\langle\omega\rangle}(r) J_1(ar)dr \Big| |J_0(|x|r)| < \epsilon$$

for some $\gamma(\epsilon) > 0$ and sufficiently large $|x| \in R$. Thus, we obtain

$$\lim_{|x|\to\infty} U(x) = 0, \tag{29}$$

which means that the radially symmetric single bump solution U satisfies θ-condition.

Remark 4.3.1. For the proof of (29) it is sufficient to assume that

$$\int_0^\infty \widehat{\langle\omega\rangle}(r) J_1(ar)dr < \infty.$$

instead of the more strict condition (28). However, we will need the condition (28) in the proofs below. We also stress here, that (28) is fulfilled for all typical connectivity functions used in neural field modeling.

We introduce the space

$$C^1_{rs}(\Xi, R) = \{u \in C^1(\Xi, R),\ u(x) = u(|x|) \text{ for all } x \in \Xi\}.$$

Lemma 4.3.1. Let the following condition be satisfied:

$$\int_0^\infty \widehat{\langle\omega\rangle}(r)\Big(J_0(ar)J_1(ar) + \frac{ar}{2}(J_0^2(ar) - 2J_1^2(ar) - J_0(ar)J_2(ar))\Big)dr \neq 0. \tag{30}$$

Then for an arbitrary sufficiently large compact set Ω, $\Omega \subset R^2$, there exists such $\varepsilon > 0$ that the symmetric single bump U defined by (27) is a unique solution to (25) in $B_{C^1_{rs}(\Omega,R)}(U, \varepsilon)$ when $\beta = 0$.

Proof. From the definition of the radially symmetric single bump solution it follows that

$$2\pi a \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(ar)J_1(ar)dr = \theta.$$

Thus, the condition (30) guarantees uniqueness of the solution U in $B_{C^1_{rs}(\Omega,R)}(U, \varepsilon)$ for some $\varepsilon > 0$. $\Box$

We now express (27) in terms of operator equality just as it was done in Section 3:

$$U = F_0 U.$$

In order to apply Theorem 3.2, we calculate $\deg(I - F_0, B_{C^1_{rs}(\Omega,R)}(U,\varepsilon), 0)$. By the definition of the topological fixed point index, we get

$$\deg(I - F_0, B_{C^1_{rs}(\Omega,R)}(U,\varepsilon), 0) = \mathrm{ind}(F_0, B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)).$$

Without loss of generality we assume that the fixed point U of the operator F_0 is unique in $\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}$. Thus, F_0 maps $\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}$ into some manifold $\mathcal{M} \subset C^1(\Omega, R)$,

$$\mathcal{M} = \{v = 2\pi c \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(\cdot\, r) J_1(cr) dr,\ c \in \mathrm{M} \subset R\},$$

where compact set M is chosen in a such way that it contains c_u for all $u \in \overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}$ (One can e.g. choose M to be a segment). We define the mapping $\phi : \mathrm{M} \to \mathcal{M}$ as

$$\phi(c) = v(x),\ v(x) = 2\pi c \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|x|r) J_1(cr) dr,\ x \in \Omega. \tag{31}$$

Lemma 4.3.2. Let the following condition be satisfied:

$$\int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(\cdot\, r)\Big(J_1(ar) + \frac{ar}{2}(J_0(ar) - J_2(ar))\Big) dr \not\equiv 0. \tag{32}$$

Then the mapping $\phi : \mathrm{M} \to \mathcal{M}$ defined by (31) is a homeomorphism, and $\mathcal{M}$ is an absolute neighborhood retract.

Proof. First, we note that $\phi : \mathrm{M} \to \mathcal{M}$ is a surjection by definition. Injectivity of $\phi : \mathrm{M} \to \mathcal{M}$ follows from the expression for the Frechet derivative of ϕ taken at an arbitrary $c \in \mathrm{M}$:

$$\phi'(c) = 2\pi \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(\cdot\, r)\Big(J_1(cr) + \frac{cr}{2}(J_0(cr) - J_2(cr))\Big) dr$$

and the condition (32). We also note that the set M is an absolute neighborhood retract, since it is a compact convex subset of R. Thus, by properties of homeomorphism, $\mathcal{M} = \phi(\mathrm{M})$ is an absolute neighborhood retract, too. $\square$

We define $\mathcal{F}$ to be the restriction of F_0 on $\mathcal{M}\bigcap\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}$, i.e.

$$\mathcal{F}=F_0|_{\mathcal{M}\bigcap\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}},$$
$$\mathcal{F}:\mathcal{M}\bigcap\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}\to\mathcal{M}.$$

Due to its definition, the mapping $\mathcal{F}:\mathcal{M}\bigcap\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}\to\mathcal{M}$ is compact and admissible. We use the properties of the topological fixed point index and get

$$\operatorname{ind}(F_0,B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))=\operatorname{ind}(\mathcal{F},\mathcal{M}\bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)).$$

Next, we apply Lemma 2.3 and obtain

$$\operatorname{ind}(\mathcal{F},\mathcal{M}\bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))=\operatorname{ind}(\phi^{-1}\circ\mathcal{F}\circ\phi,\phi^{-1}(\mathcal{F}(\mathcal{M}\bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))).$$

Lemma 4.3.3. Let the condition (28) be satisfied. Then there exists such $\delta>0$ that the operator $\Psi=\phi^{-1}\circ\mathcal{F}\circ\phi$ maps $\overline{B_R(a,\delta)}$ to M.

Proof. Let

$$u(x)=2\pi c\int\limits_0^\infty\widehat{\langle\omega\rangle}(r)J_0(|x|r)J_1(cr)dr,\ c\in\mathrm{M}.$$

Using the mean value theorem and the properties of the Bessel function J_1, we estimate

$$\|u-U\|_{C^1(\Omega,R)}\le$$
$$2\pi\Big\|c\int\limits_0^\infty\widehat{\langle\omega\rangle}(r)J_0(|\cdot|r)J_1(cr)dr-a\int\limits_0^\infty\widehat{\langle\omega\rangle}(r)J_0(|\cdot|r)J_1(ar)dr\Big\|_{C(\Omega,R)}+$$
$$2\pi\Big\|-c\int\limits_0^\infty\widehat{\langle\omega\rangle}(r)rJ_1(|\cdot|r)J_1(cr)dr+a\int\limits_0^\infty\widehat{\langle\omega\rangle}(r)rJ_1(|\cdot|r)J_1(ar)dr\Big\|_{C(\Omega,R)}\le$$

$$2\pi\Big\|\int\limits_0^\infty\widehat{\langle\omega\rangle}(r)J_0(|\cdot|r)(c\frac{r}{2}(J_0(\xi r)-J_2(\xi r))+aJ_1(ar))dr(a-c)\Big\|_{C(\Omega,R)}+$$
$$2\pi\Big\|\int\limits_0^\infty\widehat{\langle\omega\rangle}(r)rJ_1(|\cdot|r)(c\frac{r}{2}(J_0(\xi r)-J_2(\xi r))+aJ_1(ar))dr(a-c)\Big\|_{C(\Omega,R)},$$

where $\xi \in B_R(a, |a-c|)$. The condition (28) implies that

$$\|u - U\|_{C^1(\Omega,R)} \leq \mathfrak{N}|c-a| < \varepsilon$$

for some $\mathfrak{N} \in R$ and all $c \in \overline{B_R(a,\delta)}$, where $\delta < \varepsilon/\mathfrak{N}$. From the latter estimate we conclude that

$$\overline{B_R(a,\delta)} \subset \phi^{-1}(\mathcal{M} \bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))$$

which, in turn, implies

$$\mathcal{M}_\delta \subset \mathcal{F}(\mathcal{M} \bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))$$

$$\mathcal{M}_\delta = \Big\{ v = 2\pi c \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|\cdot|r) J_1(cr) dr,\ c \in \overline{B_R(a,\delta)} \Big\}.$$

Thus, we finally get

$$\phi^{-1}(\mathcal{M}_\delta) = \overline{B_R(a,\delta)} \subset \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))),$$

which concludes the proof. $\square$

Remark 4.3.2. The condition (28) is redundant for the the proof of the statement in Lemma 4.3.3. However, the condition it can be relaxed to is more cumbersome and harder to check.

It is easy to see that a is a fixed point of the operator $\Psi : \overline{B_R(a,\delta)} \to$ M. Moreover, a is an isolated fixed point of Ψ due to the fact that U is an isolated fixed point of $\mathcal{F}$ and topological invariance property of the index. The topological index of a finite dimensional map can be calculated as

$$\mathrm{ind}(\Psi, \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))) = \mathrm{sgn}(1 - \Psi'(a)).$$

The definition of the operator $\Psi = \phi^{-1} \circ \mathcal{F} \circ \phi$ implies that

$$2\pi c \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(\Psi(c)r) J_1(cr) dr = \theta \quad \text{for all} \quad c \in \overline{B_R(a,\delta)}.$$

Using the implicit function theorem and the chain rule for differentiation, we get

$$\int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(ar) J_1(ar) + ar\Big(J'_{0a}(ar) J_1(ar) \Psi'(a) + J_0(ar) J'_{1a}(ar)\Big) dr = 0.$$

From the latter expression we obtain the following sufficient condition for $\Psi'(a) \neq 1$:

$$\int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(ar)J_1(ar) + a\Big(J_0(ar)J_1(ar)\Big)'_a dr \neq 0. \qquad (33)$$

Thus, $\deg(I - F_0, B_{C^1_{rs}(\Omega,R)}(U,\varepsilon), 0) \neq 0$ provided that the inequality (33) is fulfilled.

Summarizing the results above and using Theorem 3.2 and Theorem 3.1, we get the main result of the subsection.

Theorem 4.3.1. Let the family of functions $f_\beta : R \to [0,1]$ ($\beta \in [0,\infty)$) satisfy assumptions (**A3**) and (**A4**). Let also the connectivity kernel ω be given by (26), where the function $\sigma \in C(\mathcal{Y},(0,\infty))$ is $\mathcal{Y}$-periodic and the function $\chi \in C^2(R^2,R)\bigcap L(R,\mu,R)$ is radially symmetric. Finally, let the conditions (28), (30), (32), and (33) be fulfilled. Then, for any sufficiently large Ω, $\Omega \subset R$, and for each $\beta \in (0,\infty)$, there exists solution $u_\beta \in C^1_{rs}(\Omega,R))$ to (25) ($\Xi = \Omega$). Moreover, $\|u_\beta - U\|_{C^1(\Omega,R))} \to 0$, as $\beta \to 0$, where $U \in C^1_{rs}(R^2,R))$ is the stationary bump solution to (25) ($\Xi = R^2$, $\beta = 0$), defined by (27).

5. Conclusions and outlook

Using the methods of functional analysis and topological degree theory, we proved theorems on existence and continuous dependence of the stationary solutions to nonlinear operator equation with the operator of the Hammerstein type on the steepness of the Hammerstein nonlinearity. We applied the theorems obtained to the m-dimensional homogenized Amari neural field model (4) and proved theorems on existence and continuous dependence of its stationary solutions under the transition from continuous firing rate functions to the discontinuous Heaviside limit. These results serve as a justification of the transition from the heterogeneous model (3) to the homogenized model (4) in the case of the Heaviside firing rate function. We investigated the following three types of stationary solutions to (4): symmetric single bump solution in 1-D, symmetric double bump solution in 1-D, and radially symmetric single bump solution in 2-D in the respect of their existence and dependence on the firing rate steepness.

The present research can be considered as an extension to m-dimensional homogenized neural field models of the results of the paper by Oleynik et al [21]. This extension was achieved by generalization of the model keeping the methods of proofs similar to the ones used in [21]. The main distinction in the proofs foundations is the choice of the basic spaces: we employ the spaces of continuous functions on compact domains instead of the spaces of integrable functions on R used by Oleynik et al. Our choice of the basic spaces was conditioned by the possibility to facilitate and shorten the proofs required and to obtain at the same time the results of [21] concerning single bump solutions as a special case of our theorems.

The models of mathematical biology, in particular, the models arising in genetics, incorporate approximation of the rapid switching between two states of the model elements. This approximation is often modeled by means of Heaviside function. Extension of the methods suggested in Section 3 to other problems of mathematical biology can be considered as a further development of the present study.

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