Methods of functional analysis in homogenized neural field theory

Funksjonalanalytiske metoder i homogenisert nevrofelt teori

Philosophiae Doctor (PhD) Thesis

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Abstract

One of the major challenges in neurobiology remains understanding the relationship between complex neural network dynamics underlying spatially structured activity states and the corresponding neural circuitry for which the electromagnetic field is macroscopically measurable via electroencephalogram (EEG) or local field potentials. Such macroscopic electrical activity in the neocortex is naturally studied in the framework of cortical networks. However, since the number of neurons and synapses in even a small piece of cortex is immense, a suitable modeling approach is to take a continuum limit of the neural networks and, thus, consider so-called neural field models of the brain cortex. This modeling framework involves integro-differential equations or Volterra integral equations and goes back to the seminal papers by Wilson, Cowan and Amari in the 1970's. In recent years, such neural fields have been used to model a wide range of neurobiological phenomena, including orientation tuning in primary visual cortex, short term working memory, control of head direction, motion perception, geometric visual hallucinations, EEG rhythms, and wave propagation in cortical slices and *in vivo*.

The aforementioned framework, however, does not take into account the heterogeneity in the cortical structure. Recent works in neuroscience have drawn attention to homogenized neural field models where the brain heterogeneity is captured by a special parameter. Such models are obtained from heterogeneous neural field models by means of homogenization: the two scale convergence method developed by Nguetseng. These investigations have been restricted to a one-dimensional case though. We take a step in the direction of considering a more realistic two-dimensional variation of the homogenized neural field model. We use pinning function technique and spectral properties of Hilbert–Schmidt integral operators to establish existence and stability of localized stationary activity states.

Various approximations and numerical approaches, which are frequently used in the mathematical neuroscience, need to be justified rigorously. Using the fixed point theorems and convergence techniques in functional spaces, we investigate the well-posedness aspects of the homogeneous and homogenized neural field models, thus justifying implementation of numerical schemes. We also justify the approximations of continuous neural fields by network models, thus, proving the validity of various disctetization methods. Using compactness in functional spaces and topological degree theory, we justify the approximation of smooth activation functions by the Heaviside unit step function in the case of localized stationary solutions for the n-dimensional homogenized neural field model. The latter result is of particular importance in the aforementioned homogenization procedure.

The present thesis illustrates that methods of functional analysis employed in mathematical neuroscience may be very beneficial.

Sammendrag

En av hovedutfordringene i nevrobiologi består i å forstå sammenhengen mellom den komplekse nettverks-dynamikken som ligger under de romlige aktivitetstilstandene i hjernebarken og makroskopiske målinger av den korresponderende elektriske kretsverk aktiviteten ved hjelp av elektroencefalogram (EEG) og lokale felt potensialer. Slik makroskopisk elektrisk aktivitet in neocortex beskrives gjerne ved hjelp av fyringsrate modeller. Men, siden antallet nevroner og synapser i selv en svært liten del av hjernebarken er enormt stort, så er det naturlig å ta kontinuumsgrensen av disse fyringsrate modellene. Dette betyr at en studerer fyringsaktiviteten i hjernebarken ved hjelp av såkalte nevrofelt modeller. Slike rammeverk for modellering baseres på integro-differensial likninger eller Volterra integral likninger. Disse rammeverkene går tilbake til banebrytende arbeider av Wilson, Cowan og Amari på 1970–tallet. I de senere årene har en brukt nevrofeltmodeller til å beskrive ett vidt sett av nevrobiologiske fenomener, som f. eks. inkludering av orientering tuning i den primære visuelle hjernebarken, korttids hukommelse, kontroll av hode retning, persepsjon, visuelle hallusinasjoner, EEG rytmer og bølgeforplantning i snitt av hjernebarken og i levende vev.

En svakhet med mange nevrofelt-modeller er at de ikke tar hensyn til heterogeniteten som er til stede i den kortikale strukturen. I noen nylig publiserte arbeider i nevrovitenskap tar en hensyn til heterogeniteten ved hjelp av en spesiell parameter. Slike modeller er utledet fra heterogene nevrofelt modeller ved hjelp av en homogeniseringsmetode basert på to-skala konvergens metoden til Nguetseng. Disse studiene er imidlertid begrenset til en rom dimensjon. I denne avhandlingen ser vi på en realistisk to-dimensjonal situasjon for en en-populasjon homogenisert nevrofelt modell. Vi bruker pinning funksjonsteknikken til å avgjøre eksistens av romlig lokaliserte tilstander og spektral egenskapene til Hilbert–Schmidt integral operatorer til å bestemme stabiliteten til disse tilstandene. Det er viktig å rettferdiggjøre de ulike approksimasjonene og de numeriske skjemaene som brukes i matematisk nevrovitenskap rigorøst. Ved å bruke fikspunkt teoremer of konvergensteknikker i funksjonsrom, studerer vi velformulerthet av homogene og homogeniserte nevrofelt modeller. Vi rettferdiggjør også implementering av numeriske skjemaer. Vi begrunner også approksimasjonen av kontinuerlige nevrofelt modeller ved hjelp av diskrete nettverks modeller, hvilket innebærer at vi rettferdiggjør ulike diskretiseringsmetoder. Ved å bruke kompakthet resultat for funksjonsrom og gradteori, rettferdiggjør vi approksimasjonen av glatte fyringsrate funksjoner med Heaviside-step funksjon når vi studerer lokaliserte stasjonære løsninger av den n dimensjonale homogeniserte nevrofelt modellen. Det sistnevnte resultatet er spesielt viktig i den tidligere nevnte homogeniseringsprosedyren.

Den foreliggende avhandlingen viser at det er svært fordelaktig å bruke funksjonalanalytiske metoder i matematisk nevrovitenskap.

List of papers

1. E. Burlakov, J. Wyller, and A. Ponosov, *Two-dimensional Amari neural field model with periodic microstructure: Rotationally symmetric bump solutions*, Communications in Nonlinear Science and Numerical Simulation 32 (2016) 81–88.

2. E. Burlakov, A. Ponosov, and J. Wyller, *Stationary solutions of continuous and discontinuous neural field equations*, Journal of Mathematical Analysis and Applications (In Press) doi:10.1016/j.jmaa.2016.06.021.

3. E. Burlakov, E. Zhukovskiy, A. Ponosov, and J. Wyller, *On wellposedness of generalized neural field equations with delay*, Journal of Abstract Differential Equations and Applications 6(1) (2015) 51–80.

4. E. Burlakov, E. Zhukovskiy, A. Ponosov, and J. Wyller, *Existence, uniqueness and continuous dependence on parameters of solutions to neural field equations*, Memoirs on Differential Equations and Mathematical Physics 65 (2015) 35–55.

List of contents

1. Introduction

The human brain cortex is the top layer of the hemispheres, of 2–4 mm thick, involving about 10^9 neurons having 60×10^{12} connections [1]. The brain cortex is responsible for such higher functions of the human brain as e.g. memory, reasoning, thought, and language [2], [3]. The basic unit of the brain cortex is the neuron. It consists of dendrites, cell body (soma), and axon. The dendrites receive electrical signals from other neurons and propagate them to the soma. If the total sum of the input electrical potential in the soma exceeds a certain threshold value, the neuron produces the burst of the output electrical signal (fires an action potential), which then propagates along the axon to other neurons. Thus, a natural way (see e.g [4]) of studying electrical activity in the neocortex is the framework of cortical networks.

The most well-known representative of such models is the Hopfield network model [5]. A generalized version of that model is given by

$$\begin{gathered} \dot{z}_i(t) = -z_i(t) + \sum_{j=1}^{N} \omega_{ij} f\Big(z_j\big(t - \tau_{ij}(t)\big)\Big), \\ t \geq 0,\ i = 1, ..., N, \end{gathered} \tag{1}$$

see e.g. [6]. The delayed Hopfield model 1 takes into account the finite speed of the electrical signal propagation in the cortical network. Here z_i is the electrical activity of the i-th neuron in the network, ω_{ij} is the connection strength between the i-th and j-th neurons, the non-negative function f gives the firing rate $f(z)$ of a neuron with activity z, and τ_{ij} is a non-negative function denoting the time it takes for the signal to reach the j-th neuron from the i-th neuron. The classical Hopfield network model has $\tau_{ij} = 0$ for all $i, j = 1, \ldots, N$.

However, since the number of neurons and synapses in even a small piece of cortex is immense, a suitable modeling approach is to take a continuum limit of the neural networks and, thus, consider so-called neural field models of the brain cortex (rigorous justification of this limit procedure using the notion of parameterized measure is given in Paper IV). The most well-known and simplest model describing the macro-level neural field dynamics is the Amari model [7]

$$\begin{gathered} \partial_t u(t, x) = -u(t, x) + \int_{\Omega} \omega(x - y) f(u(t, y)) dy, \\ t \geq 0,\ x \in \Omega \subseteq R^n. \end{gathered} \tag{2}$$

Here $u(t, x)$ denotes the activity of a neural element u at time t and position x. The connectivity function ω determines the coupling strength between the elements and the non-negative function f gives the firing rate $f(u)$ of a neuron with activity u. Neurons at a position x and time t are said to be active if $f(u(t, x)) > 0$. Typically f is a smooth function that has sigmoidal shape. Well-posedness of (2) was proved in [8]. Well-posedness of (2) for the case, when the spatial domain is a Riemannian space, was investigated in [9]. Faugeras et al [10] proved existence and uniqueness of the stationary solution to (2) as well as obtained conditions for this solution to be absolutely stable, for the case of a bounded Ω. The local and global structure of stationary solutions to neural field equations (2) on a bounded Ω was studied in [11]. Well-posedness of the following delayed Amari model

$$\begin{gathered} \partial_t u(t,x) = -u(t,x) + \int\limits_{\Omega} \omega(t,x,y) f(u(t-\tau(x,y),y))dy, \\ t \geq 0, \ x \in \Omega \end{gathered} \tag{3}$$

in the space of square integrable functions was proved in [12].

A common simplification of (2) consists of replacing a smooth firing rate function by the Heaviside function with some activation threshold $\theta > 0$

$$H(u) = \begin{cases} 0, & u \leq \theta, \\ 1, & u > \theta. \end{cases} \tag{4}$$

This replacement simplifies numerical investigations of the model as well as allows to obtain closed form expressions for some important types of solutions (see e.g. [7], [13], [14], [15]). Particular attention in the neural field theory is usually given to the localized stationary, i.e., time-independent, solutions (so-called "bumps"). It is caused by the fact that steady localized activity states in the cortex are prevalent during the normal functioning of the brain, encoding visual stimuli [16], representing head direction [17], and maintaining persistent activity states in working memory [18], [19].

It is usually tacitly assumed that the approximation of a smooth firing rate function f, which is sufficiently steep between the activation threshold value θ and the "saturation value" $\theta_{sat} = \inf\{u, f(u) = 1\}$, by the Heaviside function (4) preserves all properties of the corresponding solutions. However, no rigorous mathematical justification of the passage from a smooth to discontinuous firing rate functions in the framework of neural field models was given until the work by Oleynik et al [20], where continuous dependence of the 1-bump stationary solution to (2) under the transition from a smooth firing rate function to the Heaviside function was proved in the 1-D case.

1.1. One-dimensional Amari model

Amari [7] found analytical expressions for bump solutions and showed that there exist stable and unstable bumps in the framework of the one-dimensional model (2) with the Heaviside firing rate function (4). Later, Kishimoto and Amari [21] proved the existence of stable bumps for the same model but with a firing rate function given as

$$f(u) = \begin{cases} 0, \ u \leq \theta, \\ \varphi(u), \ \theta < u < \theta_{sat}, \\ 1, \ u \geq \theta_{sat}, \end{cases} \tag{5}$$

where $\varphi : [\theta, \theta_{sat}] \to [0, 1]$ is an arbitrary differentiable increasing normalized function such that $\varphi(\theta) = 0$, $\varphi(\theta_{sat}) = 1$. It was also shown in [21] that bump solutions to the Amari model (2) with the firing rate function (5) have no closed form analytical representation. Coombes and Schmidt [22] suggested an iterative scheme for construction of these bumps. They, however, did not give a mathematical verification of their approach. This verification was carried out in [23], where two iterative schemes for construction of such bumps were introduced and the convergence of the schemes was proved.

The linear stability of bump solutions to the one-dimensional Amari model with the Heaviside firing rate function is usually assessed by the Evans functions technique (see e.g. [24], [25], [26], [27]).

The one-dimensional Volterra formulation of (3)

$$u(t, x) = \int_{-\infty}^{t} \eta(t - s) \int_{R} \omega(x - y) f(u(s - |x - y|/v, y)) dy ds, \quad t \in R, \ x \in R \tag{6}$$

has been investigated by Venkov et al [28] in the study of axonal delay effects on Turing–Hopf instabilities and pattern formation. Here the memory function (temporal convolution kernel) $\eta(t)$ with $\eta(t) \equiv 0$ for $t < 0$ represents synaptic processing of signals within the network, and the delayed temporal argument to u in the spatial integral represents the axonal delay effect arising from the finite speed (denoted here by v) of signal propagation between points x and y.

1.2. Two-dimensional Amari model

Though most works are restricted to one spatial dimension, a more realistic modeling framework of the electrical activity in cortical tissue makes use of neural field models in two spatial dimensions. Yet, these models have been only occasionally studied in the literature.

Rotationally symmetric bump solutions to the two-dimensional Amari model were first considered by Taylor [29]. Laing and Troy [30] introduced PDE methods to study symmetry-breaking of rotationally symmetric bumps and the formation of multiple bump solutions. However, such methods can only be applied to connectivity kernels ω for which the Fourier transforms are rational functions of the square of the radius. Stability of rotationally symmetric bump solutions with respect to radial perturbations was examined in [29], [31]. However, as shown by Folias and Bressloff [26], [32], and Owen et al [14], in order to determine correctly the linear stability of radially symmetric solutions, it is necessary to take into account all possible perturbations of the circular boundary. The resulting spectral problem can be solved using e.g. Fourier methods. Existence and stability of the solutions of the ring type were examined in [14]. The works [26], [32], and [14] involve connectivity functions ω that can be represented as a sum of modified Bessel functions. The advantage of such representation is the possibility to use analytical expressions for the Hankel transform of the connectivity kernel and its integrals, which appreciably facilitates the model analysis.

Faye et al [33] extended the results of the work [26] to Amari equations on a Riemannian space, making them applicable to studying the electrical activity in the primary visual brain cortex.

Numerical investigations of bump solutions in the aforementioned two-dimensional frameworks involve Heaviside firing rate functions. A first step towards a rigorous study of stationary radially symmetric solutions of neural field equations with smooth firing rate function was taken in [34], where existence and stability of these solutions were examined.

1.3. Two-population Amari model

In the above models, both excitation and inhibition were incorporated into a one-population neural field model. However, a two-population model, where excitatory and inhibitory neurons are modeled separately may serve as a better approximation of excitation and inhibition processes in the cerebral cortex (see e.g. [15], [35] and the references therein). The two-population Amari model involves $u(t,x)$ as a vector from R^2 with the components corresponding to the excitatory and the inhibitory populations of neurons. Consequently, the firing rate function in (2) also has two components and the function ω is represented by a $2{\times}2$ functional matrix reflecting the interactions between the populations. In the case of stationary solutions, the one-population model captures the basic pattern forming instability. However, the two-population model supports a wider range of dynamics and, in particular, can undergo a Turing–Hopf instability leading to the formation of oscillatory patterns (see [36], [37], [35] and [38]).

1.4. Amari model with microstructure

The modeling framework (2) and its extensions cited above are proposed to capture the features of the brain activity on the macroscopic level. However, they do not take into account the heterogeneity in the cortical structure. In order to take into account the microstructure of the brain media it is usually assumed that the connectivity kernel is represented as $\omega_\varepsilon = \omega(\cdot, \cdot/\varepsilon)$, where the microstructure heterogeneity is parameterized by $\varepsilon > 0$ (see e.g. [39], [40], [41]). Thus, (2) takes the form

$$\begin{gathered}\partial_t u_\varepsilon(t,x) = -u_\varepsilon(t,x) + \int\limits_\Omega \omega_\varepsilon(x-y) f(u_\varepsilon(t,y))dy, \\ t \geq 0,\ \ x \in \Omega.\end{gathered} \tag{7}$$

The powerful two-scale convergence method (see e.g. [42]) based on the theory of Banach algebras with mean values has been applied by Svanstedt et al [40] to the neural field models with spatial microstructure. It was shown [40] that if the microstructure is periodic, then, as the heterogeneity parameter $\varepsilon \to 0$, the solutions to (7) two-scale weakly converge to the solution of the following homogenized problem:

$$\begin{gathered}\partial_t u(t,x,x_{\mathrm{f}}) = -u(t,x,x_{\mathrm{f}}) + \int\limits_\Omega \int\limits_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}} - y_{\mathrm{f}}) f(u(t,y,y_{\mathrm{f}}))dy_{\mathrm{f}}dy, \\ t \geq 0,\ \ x \in \Omega,\ \ x_{\mathrm{f}} \in \mathcal{Y},\end{gathered} \tag{8}$$

where x_{f} is the fine-scale variable, belonging to some torus $\mathcal{Y}$. Non-periodicity of the microstructure in (7) leads to non-Lebesgue measure $d\mu(y_{\mathrm{f}})$ in (8) [40].

The one-dimensional model (7) with periodic microstructure has extensively been studied. The waves that travel trough a neural field with a periodically modulated microstructure were described in [43] and [39]. By using an interface dynamics approach, it was showed (see [39]) that growth of the medium heterogeneity leads to the wave propagation failure in the neural field. Existence and stability of the single bump and double bump stationary solutions to (8) in 1-D were investigated in [41] and [44], respectively, for the case of the Heaviside firing rate function. Numerical construction of these bump solutions by using the iteration scheme technique was carried out in [45]. Existence and stability of the radially symmetric single bump stationary solutions to (8) in 2-D were investigated in Paper I. For the case of bump solutions, numerical analysis showed that excitation/vanishing and splitting/merging of bumps as well as switching between their stability/instability through continuous change of the heterogeneity parameter takes place (see [41], [45] and Paper I).

2. Paper summaries

1.1. Paper I

We consider radially symmetric stationary single bump solutions to the two-dimensional homogenized Amari model (8). It is assumed that the firing rate function is approximated by means of the unit step function and that the solutions are independent of periodic micro-variable. The existence of the solutions is carried out by pinning function technique. We develop a stability method for the bump solutions obtained based on the spectral properties of the Hilbert–Schmidt integral operators. The whole stability assessment then concludes with a study of maximal growth rate of the perturbations imposed on the bumps state, corresponding to the operator norm of the Hilbert–Schmidt operator. We demonstrate the bumps construction procedure and the stability assessment in detail by considering a concrete example of the connectivity kernel, which is typically used in the neural field modeling.

1.2. Paper II

We extend the results of [20] to the n-dimensional homogenized Amari model (8) and, in addition to the single bump solutions in 1-D, consider double bump solutions in 1-D and single bump solutions in 2-D. We study existence and continuous dependence of the stationary solutions to (8) under the transition from continuous firing rate functions to the Heaviside function, and formulate and prove the corresponding two main theorems: the theorem on continuous dependence of the stationary solutions to (8) under the transition from continuous firing rate functions to the Heaviside function and the theorem on solvability of the equation (8) based on the topological degree theory. We apply the theory developed to the following three types of solutions to (8):

a) Symmetric single bump solution in 1-D.

b) Symmetric double bump solution in 1-D.

c) Radially symmetric single bump solution in 2-D.

1.3. Paper III

We consider the following non-local integro-differential equations:

$$u(t,x) = \int_{-\infty}^{t} \int_{\Omega} W(t,s,x,y) f(u(s-\tau(s,x,y),y)) dy ds, \qquad t \in R,\ x \in \Omega \subseteq R^m \tag{9}$$

and

$$\begin{gathered} u(t,x) = \int_{a}^{t} \int_{\Omega} W(t,s,x,y) f(u(s-\tau(s,x,y),y)) dy ds, \\ t \in [a,\infty), x \in \Omega; \\ u(\xi,x) = \varphi(\xi,x),\ \xi \le a, x \in \Omega, \end{gathered} \tag{10}$$

which generalize all homogeneous neural field models listed in the introduction and the heterogeneous models with periodic microstructure. We define the notions of local, maximally extended and global solutions to (9) and (10). We first investigate well-posedness of an abstract Volterra operator equation. Based on this theory, we establish conditions for existence of unique global or maximally extended solutions to (9) and (10), study continuous dependence of these solutions on the spatiotemporal integration kernel, delay effects, firing rate and prehistory functions, and formulate the corresponding theorems. We consider two special cases, which are highly relevant for the neural field theory, where the assumptions of the main theorems can be appreciably relaxed. We also stress that the case of the equation (9) requires more restrictions on the functions involved. The validity of these restrictions is supported by an example, where a special case of (9) has infinitely many solutions.

1.4. Paper IV

We utilize the theory of well-posedness of an abstract Volterra operator equation developed in Paper III. We apply it to the following parameterized integro-differential equation involving integration with respect to an arbitrary measure:

$$u(t,x,\lambda) = \int_{-\infty}^{t} ds \int_{\Omega} W(t,s,x,y,\lambda) f(u(s-\tau(s,x,y,\lambda),y,\lambda),\lambda) \nu(dy,\lambda), \qquad t > a,\ x \in \Omega,\ \lambda \in \Lambda \tag{11}$$

with the initial (prehistory) condition

$$u(\xi, x, \lambda) = \varphi(\xi, x, \lambda), \quad \xi \leq a, \ x \in \Omega, \ \lambda \in \Lambda. \tag{12}$$

We show, that, in addition to the homogeneous neural field models and the heterogeneous models with periodic microstructure, it covers the models with non-periodic media heterogeneity. We obtain conditions for existence and uniqueness of solution to (11) – (12). We study continuous dependence of this solution on the spatiotemporal integration kernel, delay effects, firing rate and measure. We construct connection between the delayed Amari model and the delayed Hopfield network model (1). In addition, we offer a mathematical justification of two known discretization schemes used e.g. in [12] and [46]. We also suggest an approach for investigation of the solutions to (7) with a non-periodic perturbation of the periodic connectivity kernel.

3. Discussion

3.1. Contribution

In the present thesis we have applied methods of functional analysis for investigation of the properties of the models in use in the neural field theory. Particular attention has been paid to the recent modeling approach taking into account the brain medium microstructure: the homogenized neural field models. Such models serve as a powerful tool for studying electrical activity in the brain cortex possessing fine microstructure.

Within the mathematical neuroscience community, well-posedness aspect of the models under investigation is often tacitly assumed to hold true, even though no rigorous mathematical justification is given for this assumption. Thus, it is of interest to study the impact of model parameters on the well-posedness issue of these models i.e. existence, uniqueness and continuous dependence on input data. Using the fixed point theorems and convergence techniques in functional spaces, in Papers III and IV we established conditions for existence of unique solutions to generalized neural field models and studied continuous dependence of these solutions on all functions involved in the models. These generalized models contain most of the models in use in the neural field theory as their special cases.

We also reckon that one needs to justify rigorously various approximations and numerical approaches, which are frequently used in the mathematical neuroscience, as a lack of such justification may lead to ill-conditioning, numerical instabilities, or even divergence. In Paper IV we justified the approximations of continuous neural fields by network models using the notion of parameterized measure and, thus, proved the validity of various disctetization schemes. Using compactness in functional spaces and topological degree theory, in Paper II we justified the approximation of smooth firing rate functions by the Heaviside unit step function in the case of localized stationary solutions for the n-dimensional homogenized neural field model.

In Paper I we supplemented the research on solutions to homogenized neural field models, which was mostly restricted to 1-D (see [39], [41], [43], [44], [47] – [49]), with investigation of existence and stability of the single bump stationary solution to the two-dimensional homogenized neural field model. The study of the existence was carried out by pinning function technique and the stability was examined by estimating the growth/decay rates of the perturbations imposed on the bumps state, corresponding to the operator norm of the Hilbert–Schmidt operator.

3.2. Future perspectives

In the future work the two-dimensional homogenized Amari neural field model can be used as a starting point for studying the existence and stability of multi-bump and ring solutions as well as traveling waves and fronts in 2-D.

In Paper IV we shed light on the possibility of using our knowledge about the heterogeneous neural field models possessing periodic microstructure when investigating the models with microstructure, which is close to periodic in some sense. The detailed investigation of this possibility incorporates the theory of Banach algebras with mean values and requires a separate research.

In Paper II we suggested an approach to the problem of existence and continuous dependence of solutions to Amari neural field equation under the transition from continuous nonlinearities in the corresponding Hammerstein operators to the Heaviside nonlinearity. The approach involved compactness of the corresponding operators and methods of topological degree theory. Another possible way to treat such problems is representing the Heaviside function as a multi-valued mapping. As the theory of multi-valued mappings is rather well-developed (see, e.g. [50], [51]), we can expect that application of continuous dependence techniques, or topological degree methods to the multi-valued mapping obtained will improve the results of Paper II (e.g. allow to extend them to unbounded spatial domains).

Extension of the methods suggested in Paper II for studying stability of solutions to homogeneous and homogenized neural field equations under the transition from continuous firing rate functions to the Heaviside function can be considered as another further development of our studies.

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PAPER I

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Two-dimensional Amari neural field model with periodic microstructure: Rotationally symmetric bump solutions

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ABSTRACT

We investigate existence and stability of rotationally symmetric bump solutions to a homogenized two-dimensional Amari neural field model with periodic micro-variations built in the connectivity strength and by approximating the firing rate function with unit step function. The effect of these variations is parameterized by means of one single parameter, called the degree of heterogeneity. The bumps solutions are assumed to be independent of the micro-variable. We develop a framework for study existence of bumps as a function of the degree of heterogeneity as well as a stability method for the bumps. The former problem is based on the pinning function technique while the latter one uses spectral theory for Hilbert–Schmidt integral operators. We demonstrate numerically these procedures for the case when the connectivity kernel is modeled by means of a Mexican hat function. In this case the generic picture consists of one narrow and one broad bump. The radius of the narrow bumps increases with the heterogeneity. For the broad bumps the radius increases for small and moderate values of the activation threshold while it decreases for large values of this threshold. The stability analysis reveals that the narrow bumps remain unstable while the broad bumps are destabilized when the degree of heterogeneity exceeds a certain critical value.

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1. Introduction

Cortical networks are often investigated in the framework of firing rate neural field models. The most well-known and simplest model describing the coarse grained dynamics of such a network is the Amari model [1]

$$\begin{aligned}\partial_t u(t,x) = -u(t,x) + \int_R \omega(x-x')f(u(t,x'))dx' \\ t \geq 0, x \in R,\end{aligned} \tag{1}$$

where the function $u(t, x)$ denotes the activity of a neural element at time t and position x. The connectivity function (spatial convolution kernel) $\omega(x)$ determines the coupling between the elements and the non-negative function $f(u)$ gives the firing rate of a neuron with activity u. Neurons at a position x and time t are said to be active if $f(u(t, x)) > 0$. Particular attention is usually given to the localized stationary, i.e. time-independent, solutions to (1) (so-called "bumps"), as they are expected to correspond to normal brain functioning. Existence and stability of these solutions have been investigated in numerous papers (see e.g. [1–4]).

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Most works on bumps are restricted to one spatial dimension, however. A more realistic modeling framework of the coarse grained activity in cortical tissue makes use of neural field models in two spatial dimensions. Yet, these models have been only occasionally studied in the literature. For example, rotationally symmetric bump solutions to the two-dimensional Amari model

$$\partial_t u(t,x) = -u(t,x) + \int_{R^2} \omega(x-x') f(u(t,x'))dx' \qquad t \geq 0, x \in R^2, \tag{2}$$

were first examined in [5], [6]. Rigorous analysis of these solutions involving conditions for their existence and stability was given in [7] and [8] for the case when the connectivity function ω is expressed as a sum of modified Bessel functions.

The modeling framework (1) and its extensions are proposed to capture the features of the brain activity on the macroscopic level. However, they do not take into account the heterogeneity in the cortical structure. The first step in that direction has been taken by Coombes et al. [9]. In that paper the heterogeneous nonlocal framework

$$\partial_t u_\varepsilon(t,x) = -u_\varepsilon(t,x) + \int_{R} \omega_\varepsilon(x-x') f(u_\varepsilon(t,x'))dx', \qquad t > 0, \ x \in R, \tag{3}$$

in one spatial dimension was chosen as a starting point, where the connectivity kernel $\omega_\varepsilon(x) = \omega(x, x/\varepsilon)$ by assumption is periodic in the second variable. The powerful two-scale convergence method (see e.g. [10]) has been applied by Svanstedt et al. [11] to the neural field models with spatial microstructure. It allows one to reduce (as $\varepsilon \to 0$) the integro-differential equation (3) with the heterogeneous connectivity kernel to

$$\partial_t u(t,x,y) = -u(t,x,y) + \int_R \int_{[0,1)} \omega(x-x', y-y') f(u(t,x',y'))dy'dx', \qquad t > 0, \ x \in R, \tag{4}$$

where y is the periodic fine-scale variable. This limit procedure is known as the homogenization procedure and the corresponding equation (4) is usually referred to as the *homogenized Amari equation*. Later on, this approach was applied in Svanstedt et al. [12] and Malyutina et al. [13] to the investigation of existence and stability of the single-bump and symmetric two-bump solutions, respectively, to the model (3).

This serves as a background and motivation for the present work. We consider the two-dimensional homogenized Amari model analogue of (4). We first develop a framework for studying the existence of the rotationally symmetric single-bump stationary solutions of this model. In the construction procedure we proceed in a way analogous to the method outlined in [12] and [13]: It is assumed that the firing rate function is approximated by means of the unit step function and that the solutions are independent of periodic microvariable. Next, we develop a stability method for the bumps based on the spectral properties of the Hilbert–Schmidt integral operators, also by following ideas of Svanstedt et al. [12] and Malyutina et al. [13]. The whole stability assessment then boils down to a study of maximal growth rate of the perturbations imposed on the bumps state, corresponding to the operator norm of the Hilbert–Schmidt operator. We demonstrate the bumps construction procedure and the stability assessment in detail when the connectivity kernel is modeled by means of Mexican hat function. The main challenge in this study was the complexity of the numerical simulations caused both by the problem of dimensionality and the fact that we were not able to use analytical expressions for the Hankel transform of the connectivity kernel (due to its heterogeneity) and, consequently, of its integrals, as it was done in Folias et al. [14] and Owen et al. [8].

This paper is organized in the following way. In Section 2 we develop the framework for construction of the rotationally symmetric single bumps solutions to the two-dimensional homogenized model with the unit step firing rate function and outline the stability method for such structures. In Section 3 we illustrate the theory developed with the concrete example of the Amari equation where the connectivity is modeled by the Mexican hat function. Concluding remarks and outlook are given in Section 4.

2. General theory

2.1. Existence of single bumps

The heterogeneous Amari neural field model

$$\partial_t u_\varepsilon(t,x) = -u_\varepsilon(t,x) + \int_{R^2} \omega_\varepsilon(x-x') f(u_\varepsilon(t,x'))dx', \qquad t > 0, \ x \in R^2, \tag{5}$$

in 2*D* serves as a starting point for our study. Here $u_\varepsilon(t,x)$ is the electrical activity at the time t and the point x of the neural field, f is the firing rate function, $\omega_\varepsilon(x) = \omega(x, x/\varepsilon)$ is the connectivity kernel which by assumption is continuous, vanishing at infinity with respect to the first argument and Y-periodic even function of the second argument $y = x/\varepsilon$ ($Y = [0,1)^2$). Proceeding in the way analogous to Svanstedt et al. [12], we get the following homogenized equation

$$\partial_t u(t,x,y,\gamma) = -u(t,x,y,\gamma) + \int_{R^2} \int_{[0,1)^2} \omega(x-x', y-y', \gamma) f(u(t,x',y',\gamma))dy'dx', \qquad t > 0, \ x \in R^2, \tag{6}$$

in the limit $\varepsilon \to 0$ where y is the fine-scale variable. The heterogeneity is parameterized by $\gamma \in \Gamma$. Here Γ is some admissible parameter set. Let us introduce polar coordinates (r, α) i.e. $x = (x_1, x_2) = (r\cos(\alpha), r\sin(\alpha))$. We are interested in existence and stability of solutions U of (6) that are radially symmetric, independent of the fine - scale variable y and time - independent. In polar coordinates this type of solution satisfies the following equation

$$U(r,\gamma) = \int_0^\infty \int_0^{2\pi} \int_{[0,1)^2} \omega(x - x', y', \gamma) f(U(r,\gamma)) dy' d\alpha' dr',$$

$$r \in [0,\infty),\ \gamma \in \Gamma,\ x' = (r'\cos(\alpha'), r'\sin(\alpha')).$$

In addition, we assume that the firing rate function is given by the unit step Heaviside function with the activation threshold h i.e. $f(u) = H(u - h)$. Moreover, we study stationary solutions U for which $U(r,\gamma) > h$ for $r < a$ and $U(r,\gamma) < h$ for $r > a$, where the bump radius a is determined by the equality $U(a,\gamma) = h$. These solutions are referred to as single bump solutions. The formal expression for these solutions is given by

$$U(r,\gamma) = \int_0^a \int_0^{2\pi} \omega\rangle(x - x', \gamma) r' d\alpha' dr', \tag{7}$$

where $\langle\omega\rangle$ is the mean value

$$\omega\rangle(x,\gamma) = \int_{[0,1)^2} \omega(x, y, \gamma) dy$$

of the connectivity kernel over the period of the second variable y. We calculate the double integral in (7) using the two-dimensional Fourier transform of the radially symmetric function $\langle\omega\rangle(r,\gamma)$, expressed in polar coordinates,

$$\omega\rangle(r,\gamma) = \int_0^\infty \omega\rangle(\rho,\gamma)\rho J_0(r\rho) d\rho,$$

where J_ν is the Bessel function of the first kind of order ν and $\langle\omega\rangle$ denotes the Hankel transform of $\langle\omega\rangle$. See Bochner et al [15] for details. Following the procedure implemented in Folias et al.[14], we finally get the formal expression

$$U(r,\gamma) = 2\pi a \int_0^a \omega\rangle(r',\gamma) J_0(rr') J_1(ar') dr' \tag{8}$$

for the bump solution. The bump radius a is determined by the threshold intersection condition

$$U(a,\gamma) = h \tag{9}$$

where

$$U(a,\gamma) = 2\pi a \int_0^a \omega\rangle(r',\gamma) J_0(ar') J_1(ar') dr' \tag{10}$$

The function $U(a,\gamma)$ given by the expression (10) is called the *pinning function* while Eq. (9) is referred to as the *pinning equation*. Hence, for a given threshold value of h, Eq. (10) defines a level curve in the a, γ - plane, showing the variation of the γ - dependent bumps radius a. For each γ, one inserts the corresponding bumps radius a into the expression (8) for the bump. In Section 3 we investigate this construction procedure when the connectivity function ω is expressed in terms of Mexican hat function.

2.2. Stability of single bumps

We study stability of the stationary bump state (8) in the standard way, i.e. by perturbing the stationary solution

$$u(t,x,y,\gamma) = U(r,\gamma) + \Phi(t,x,y,\gamma),$$

where $\Phi(t,x,y,\gamma) = \varphi(x,y,\gamma)e^{\lambda t}$ (see e.g. [8], [13]). Expanding to first order in φ, we obtain

$$\varphi(x,y,\gamma) = \frac{a}{(\lambda+1)|\partial_r U(r,\gamma)|_{r=a}|} \int_0^{2\pi} \int_{[0,1)^2} \omega(|x - \bar{a}|, y - y', \gamma)\varphi(\bar{a}, y', \gamma) dy' d\theta,$$

$$\bar{a} = (a,\theta).$$

By inserting $r = a$ in the above expression and introducing

$$\mu = (\lambda + 1)|\partial_r U(r,\gamma)|_{r=a}|,$$

we get the following operator equation

$$\mu\varphi = \mathbb{H}(a,\gamma)\varphi, \tag{11}$$

where

$$\varphi = \varphi((a,\alpha), y, \gamma), \quad \mathbb{H}(a,\gamma)\varphi((a,\alpha), y) = a\int_0^{2\pi} \int_{[0,1)^2} \omega(\sqrt{2a^2 - 2a^2\cos(\alpha - \theta)}, y - y', \gamma)\varphi((a,\theta), y') dy' d\theta.$$

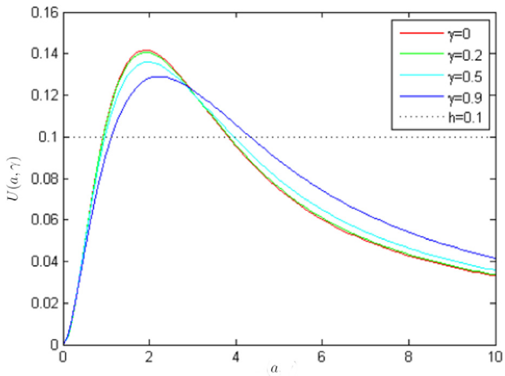

Fig. 1. The graph of the pinning function (10) in the case of the Mexican hat connectivity function (13) for different values of the degree of heterogeneity γ. The activation threshold is kept constant and within the range of admissible values.

For each $a \in (0, \infty)$, $\gamma \in \Gamma$, the operator $\mathbb{H}(a, \gamma)$ is self-adjoint on the space $L_2([0, 2\pi] \times [0, 1)^2)$ with the norm

$$\|\psi\|_{L_2} = \sqrt{\langle\psi, \psi\rangle},$$
$$\langle\psi, \phi\rangle = \int_0^{2\pi}\int_{[0,1)^2} \psi((a, \alpha), y)\phi((a, \alpha), y)dyd\alpha.$$

Indeed, for each $a \in (0, \infty)$, $\gamma \in \Gamma$, and any $\phi, \psi \in L_2([0, 2\pi] \times [0, 1)^2)$, using the properties of the connectivity function together with an interchange of the integration order, we have

$$\begin{aligned}\langle\mathbb{H}(a, \gamma)\phi, \psi\rangle &= \int_0^{2\pi}\int_{[0,1)^2} a \int_0^{2\pi}\int_{[0,1)^2} \omega(\sqrt{2a^2 - 2a^2\cos(\alpha - \alpha')}, y - y', \gamma) \\ &\quad \times \phi(\alpha', y')\psi(\alpha, y)dy'd\alpha'dyd\alpha \\ &= \int_0^{2\pi}\int_{[0,1)^2} a \int_0^{2\pi}\int_{[0,1)^2} \omega(\sqrt{2a^2 - 2a^2\cos(\alpha' - \alpha)}, y' - y, \gamma) \\ &\quad \times \psi(\alpha, y)\phi(\alpha', y')dyd\alpha dy'd\alpha' = \langle\phi, \mathbb{H}(a, \gamma)\psi\rangle.\end{aligned}$$

In addition, for any $a \in (0, \infty)$, $\gamma \in \Gamma$, the operator $\mathbb{H}(a, \gamma)$ is compact as the integral operator having bounded continuous kernel. Thus, as it follows from Hilbert–Schmidt's theorem (see e.g. [16]), we have the following expressions for the eigenvalues $\boldsymbol{\mu_n}$ and the corresponding growth/decay rates, respectively:

$$\mu_n = a\int_0^{2\pi}\int_{[0,1)^2} \omega(\sqrt{2a^2 - 2a^2\cos(\alpha - \theta)}, y - y', \gamma)dy'cos(2n\theta)d\theta,$$

$$\max_{\forall n}\{\mu_n\} = \|\mathbb{H}(a, \gamma)\|_{L_2},$$
$$\max_{\forall n}\{\lambda_n\} = \lambda_{max} = \frac{\|\mathbb{H}(a, \gamma)\|_{L_2}}{|\partial_r U(r, \gamma)|_{r=a}|} - 1. \tag{12}$$

The stability of the single bumps (8)–(10) can thus be assessed by means of the operator norm $\|\mathbb{H}(a, \gamma)\|_{L_2}$: When $\lambda_{\max} < 0 (> 0)$, then the bump is stable (unstable).

3. Example: Mexican hat connectivity function

In this section we illustrate the theory developed in the previous section by letting the connectivity kernel be given as

$$\omega(x, y, \gamma) = \frac{1}{\sigma(y, \gamma)}\chi\left(\frac{x}{\sigma(y, \gamma)}\right).$$

with

$$\sigma(y, \gamma) = 1 + \gamma\cos(2\pi y_1)\cos(2\pi y_2), \; y = (y_1, y_2), \; \gamma \in \Gamma = [0, 1).$$

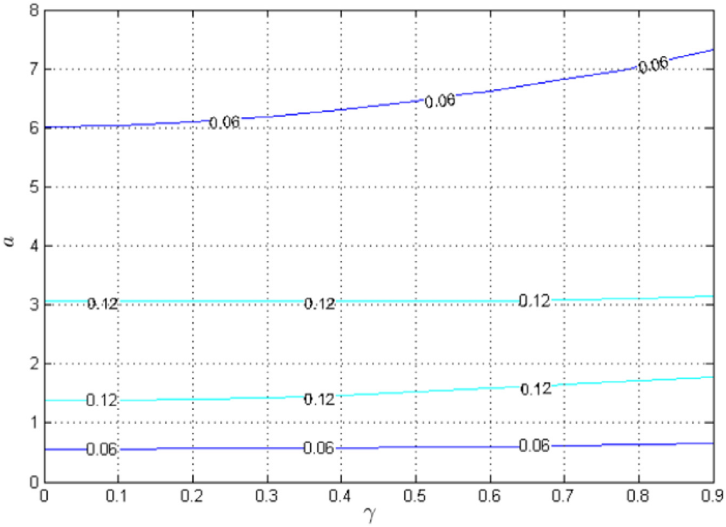

Fig. 2. Level curves (9)–(10) in the case of the Mexican hat connectivity function (13) for different values of the activation threshold values. The curves are labeled with these values.

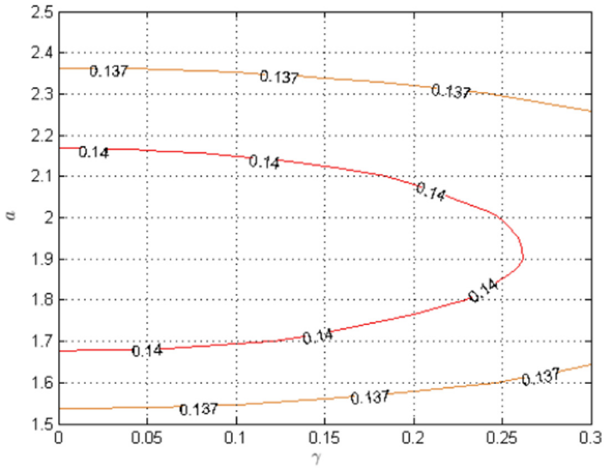

Fig. 3. Magnification of the level curve description in Fig. 2 where the broad and the narrow bumps merge together. The curves are labeled with activation threshold values.

and

$$\chi(x) = \frac{1}{2\pi}\ \ \frac{\exp(-|x|)}{2} - \frac{\exp(-|x|/2)}{4}\ \ . \tag{13}$$

This connectivity kernel is referred to as the Mexican hat function. The bump radius a is then found by solving the pinning equation (10) numerically. In Fig. 1 the graph of the pinning function is shown for selected values of the heterogeneity parameter γ i.e. $\gamma = 0,\ 0.2,\ 0.5,\ 0.9$. The intersection between the fixed threshold value h and the graph of the pinning function yields the bumps radius. In the figure we have put $h = 0.1$. From this plot we infer the following result: The generic picture consists of one narrow and one broad bumps for each admissible activation threshold value, in a way analogous to single bumps in the 1D case. Moreover, we also observe that the bumps radius of both the narrow and the broad bump increases with the degree of heterogeneity for the selected value of the threshold value. We finally notice that for the translationally invariant case ($\gamma = 0$), our plot resembles the results obtained in Owen et al. [8]. In order to study the variation of the bumps radius with the degree of

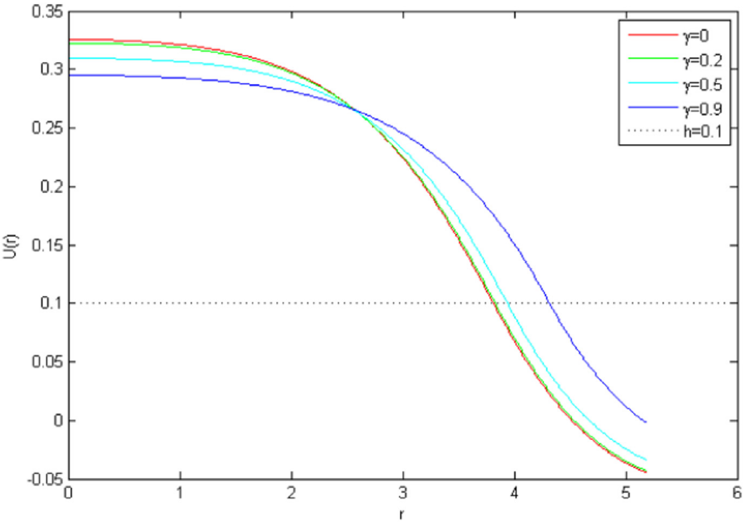

Fig. 4. The variation of the broad bump shape with the heterogeneity parameter γ.

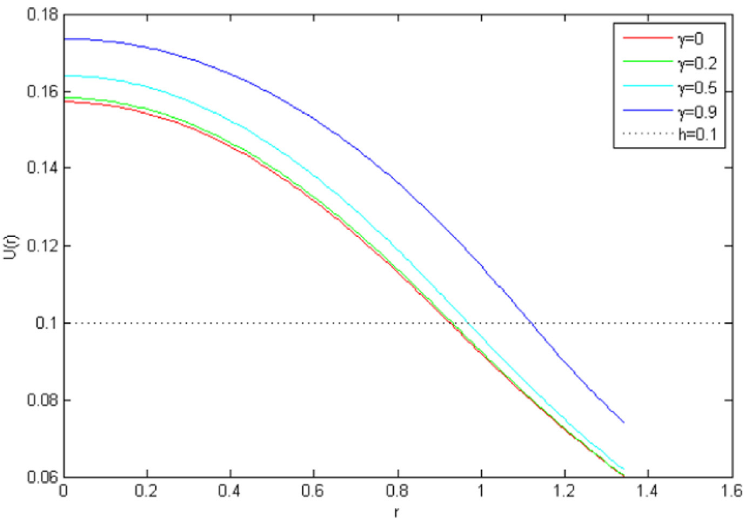

Fig. 5. The variation of the narrow bump shape with the heterogeneity parameter γ.

heterogeneity in some detail, we conveniently make use of the level curve description (9)–(10). The result of this investigation is summarized in Figs. 2 and 3. Figs. 2 and 3 support the conclusion that bumps radius a of the narrow bump increases with the degree of heterogeneity γ. The bump radius for broad bump increases for small and moderate values of the activation threshold h, while it decreases with γ for larger values of h. Variation of the broad and the narrow bump shapes with the degree of heterogeneity parameter is shown in Figs. 4 and 5, respectively.

In order to investigate stability of the stationary solutions to (6) with the connectivity given by (13), we study the maximal growth rate (12) as function of the threshold value h for different values of the degree of heterogeneity. In order to do that, we need to estimate numerically the operator norm $\|\mathbb{H}(a, \gamma)\|_{L_2}$ in (12). The result of this investigation is summarized in Fig. 6.

One readily observes that the narrow bumps remain unstable for all values of the degree of heterogeneity. For the broad bumps an increase in the degree of heterogeneity decreases the interval of activation threshold h for which the bumps are

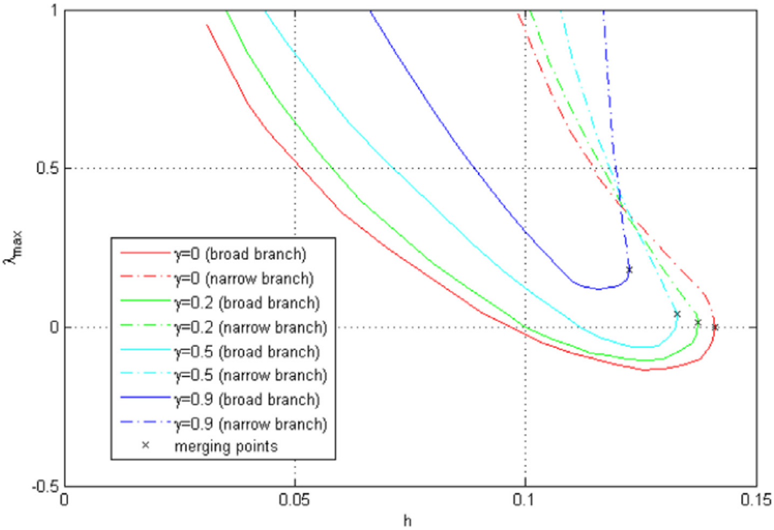

Fig. 6. The maximal growth rate of the perturbation as a function of the activation threshold for different values of the degree of heterogeneity parameter γ.

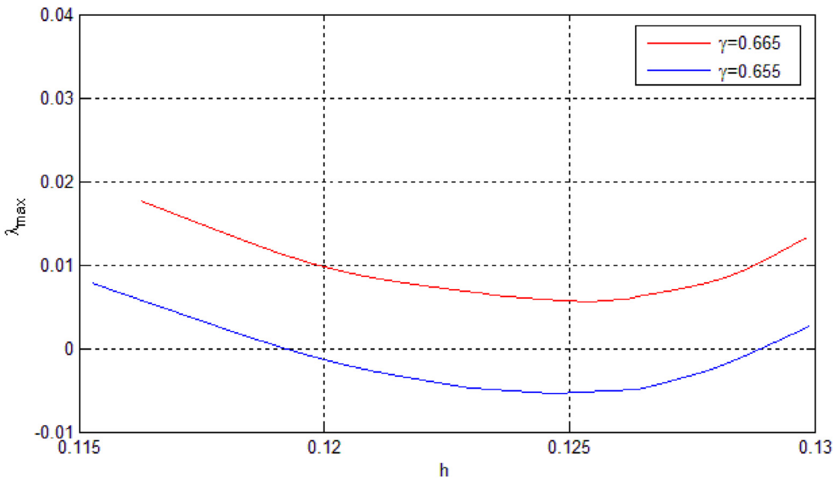

Fig. 7. The destabilization regime of the broad bump solution for the case of Mexican hat connectivity function.

stable. When the degree of heterogeneity exceeds a certain threshold value, the bumps will be unstable for all values of h. The destabilization process is further detailed in Fig. 7. Notice that Fig. 6 (namely, the case $\gamma = 0$) reproduces qualitatively the same results as in Owen et al. [8].

4. Conclusions and outlook

We have investigated the existence and stability of bump solutions in 2D of the homogenized Amari model. The starting point of this study is the homogenized Amari neural field equation. This model has previously been obtained as the limit of the parameterized heterogeneous neural field models by using the two-scale convergence technique.

The bumps solutions are assumed to be independent of the periodic microvariable and the firing rate function is modeled by the Heaviside function. We use the pinning function technique to study the existence of the bumps while the stability method is based on spectral theory for Hilbert–Schmidt integral operators. The stability can be inferred from the maximal growth rate which in turn depends on the operator norm of the actual integral operator.

We apply these procedures to the case when the connectivity kernel is modeled by means of a Mexican hat function. The outcome of this analysis can be summarized as follows: The generic picture consists of one narrow and one broad bump for the set of admissible threshold values. The bumps radius of the narrow bump increases with the degree of heterogeneity γ. In the case of broad bumps the bumps radius increases for small and moderate values of the activation threshold h, while it decreases with γ for larger values of h. Numerical analysis in this example indicates that increase of the degree of heterogeneity acts to destabilize the broad bumps while the narrow bumps always remain unstable.

In future works we aim at proving existence and continuous dependence of the stationary bump solutions under transition from the Heaviside to Lipschitz continuous firing rate functions. The transition to piecewise-linear firing rate functions is of particular importance for the theory of neural fields possessing microstructure. The aforementioned continuous dependence results link the neural field homogenization theory developed in Svanstedt et al. [11] for the case of convex firing rate functions to the numerical results obtained for the Heaviside firing rate in e.g. [12,13], and also in the present study.

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PAPER II

Stationary solutions of continuous and discontinuous neural field equations

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Abstract

We study existence and continuous dependence of the solutions to the Hammerstein operator equation under the transition from continuous nonlinearities in the Hammerstein operator to the Heaviside nonlinearity in a vicinity of the solution, corresponding to the discontinuous nonlinearity case. We apply these results to corresponding problems arising in the neural activity modeling.

Keywords: Discontinuous Hammerstein equations, solvability, continuous dependence
47H30, 46T99, 47H11, 92B99

1. Introduction

We consider a special case of nonlinear operator equation with the Hammerstein operator, the nonlinear part of is either represented by the Heaviside unit step function, or by a bounded continuous function. We are studying existence and continuous dependence of the solutions to the Hammerstein operator equation under the transition from continuous nonlinearities in the Hammerstein operator to the Heaviside nonlinearity. To do this, we choose an appropriate topology, where the Hammerstein operator with the Heaviside nonlinearity becomes continuous in a vicinity of the solution, corresponding to the case of the discontinuous Hammerstein operator nonlinearity. Then we use methods of functional analysis and topological degree theory to establish

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the results needed. This study is strongly motivated by applications of some problems arising in the neural activity modeling. Below we give a detailed descriptions of these problems.

It is well-known (see e.g. [11], [9]) that electrical activity in the neocortex is naturally studied in the framework of cortical networks. However, since the number of neurons and synapses in even a small piece of cortex is immense, a suitable modeling approach is to take a continuum limit of the neural networks and, thus, consider so-called neural field models of the brain cortex (rigorous justification of this limit procedure can be found in e.g. [4]). The simplest model describing the macro-level neural field dynamics is the Amari model [1]

$$\partial_t u(t,x) = -u(t,x) + \int\limits_{\Xi} \omega(x-y) f(u(t,y))dy, \quad t \geq 0,\, x \in \Xi \subseteq R^m. \qquad (1)$$

Here $u(t,x)$ denotes the activity of a neural element u at time t and position x. The connectivity function ω determines the coupling strength between the elements and the non-negative function $f(u)$ gives the firing rate of a neuron with activity u. Neurons at a position x and time t are said to be active if $f(u(t,x)) > 0$. Typically f is a smooth function that has sigmoidal shape. Solvability of (1) in the case of a smooth firing rate function was proved in [23], [3]. Particular attention in the neural field theory is usually given to the localized stationary, i.e., time-independent, solutions to (1) (so-called "bump solutions", or simply "bumps"), as they correspond to normal brain functioning (see e.g [26]). Faugeras et al [8] proved existence and uniqueness of the stationary solution to (1) as well as obtained conditions for this solution to be absolutely stable, for the case of a bounded Ξ.

A common simplification of (1) consists of replacing a smooth firing rate function by the Heaviside function. This replacement simplifies numerical investigations of the model as well as allows to obtain closed form expressions for some important types of solutions (see e.g. [1], [22] [17]). Existence of the solution to (1) in the case of Heaviside firing rate function was proved by Potthast et al [23]. Stability of the stationary solutions to (1) is usually assessed by the Evans function approach (see e.g. [6], [22]). The analysis of existence and stability of localized stationary solutions for a special class of the firing rate functions, the functions that are "squeezed" between two unit step functions, was carried out in [13], [20], [15]. This analysis served as a connection between stability\instability properties of the solutions to the models with the "squeezing" Heaviside firing rate functions and the solution to the model with the "squeezed" smooth firing rate function. However, no rigorous mathematical justification of the passage from a smooth to discontinuous firing rate functions in the framework of neural field models was given until the work by Oleynik et al [21], where continuous dependence of the 1-bump stationary solution to (1) under the transition from a smooth firing rate function to the Heaviside function was proved in the 1-D case.

On the other hand, more advanced neural field models have not been studied in this respect. One example is the homogenized Amari model describing the neural field dynamics on both macro- and micro- levels

$$\begin{gathered}\partial_t u(t,x,x_{\mathrm{f}}) = -u(t,x,x_{\mathrm{f}}) + \int_{\Xi}\int_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f(u(t,y,y_{\mathrm{f}}))dy_{\mathrm{f}}dy, \\ t \geq 0,\ x \in \Xi,\ x_{\mathrm{f}} \in \mathcal{Y} \subset R^k,\end{gathered} \tag{2}$$

which was introduced in the pioneering work by Coombes et al [7]. Here x_{f} is the fine-scale spatial variable and $\mathcal{Y}$ is an elementary domain of periodicity in R^k. As it was shown in [24], the solution to (2) is a weak two-scale limit of solutions to the following family of heterogeneous neural field models

$$\begin{gathered}\partial_t u(t,x) = -u(t,x) + \int_{\Xi} \omega^{\varepsilon}(x-y) f(u(t,y))dy, \\ \omega^{\varepsilon}(x) = \omega(x, x/\varepsilon),\ 0 < \varepsilon \ll 1, \\ t \geq 0,\ x \in \Xi,\end{gathered} \tag{3}$$

as $\varepsilon \to 0$, where ε corresponds to the medium heterogeneity.

The starting point for the investigation of the solutions to (2) was assuming these solutions to be independent of the fine-scale variable, i.e. solutions to the equation

$$\begin{gathered}\partial_t u(t,x) = -u(t,x) + \int_{\Xi}\int_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f(u(t,y))dy_{\mathrm{f}}dy, \\ t > 0,\ x \in \Xi \subseteq R^m,\ x_{\mathrm{f}} \in \mathcal{Y}.\end{gathered} \tag{4}$$

This assumption was also supported by numerical evidence of non-existence of the fine-scale-dependent solutions to (2) given in [19].

Existence and stability of the single bump and double bump stationary solutions to (4) in 1-D were investigated in [25] and [18], respectively, for the case of the Heaviside firing rate function. Existence and stability of the radially symmetric single bump stationary solutions to (4) in 2-D when f is represented by the Heaviside unit step function were investigated in [5].

In the present research we extend the results of [21] to the homogenized Amari model and, in addition to the single bump solutions in 1-D, consider symmetric double bump solutions in 1-D and radially symmetric bump solutions in 2-D. We formulate the following two main theorems: the theorem on continuous dependence of the stationary solutions to (4) under the transition from continuous firing rate functions to the Heaviside function and the theorem on solvability of the equation (4) based on the topological degree theory. We emphasize here that the properties of existence of solutions to (4) under the described transition and continuous dependence of these solutions on the firing rate steepness do not depend on the stability\instability of the solution to (4) with the Heaviside firing rate function. The latter remark can be illustrated by comparison of the results of the papers [25], [18], and [5] to the corresponding three special cases of (4), considered in the present research:

1. Symmetric single bump solution to (4), $m = k = 1$.
2. Symmetric double bump solution to (4), $m = k = 1$.
3. Radially symmetric single bump solution to (4), $m = k = 2$.

We also stress that our results, in particular, mean that the approximation of the Heaviside function by piecewise linear firing rate functions yields continuous dependence of the solutions to the corresponding neural field equations. This property has particular importance for the theory of the heterogeneous neural fields as the transition from the heterogeneous model (3) to the homogenized model (2) can be justified for the piecewise linear firing rate functions, but not for their Heaviside limit (see [24], [25]). Thus, our results justify the usage of the Heaviside firing rate function in the frameworks of [25], [18], and [5].

The paper is organized in the following way. In Section 2 we explain our notations and state lemmas from functional analysis, which we refer to in the subsequent sections. In Section 3 we study existence and continuous dependence of the stationary solutions to (4) under the transition from continuous firing rate functions to the Heaviside function, and formulate and prove the corresponding two main theorems. Based on these theorems we investigate in Section 4 the corresponding properties of the following types of solutions to (4):

1. Symmetric single bump solutions in 1-D (Subsection 4.1).

2. Symmetric double bump solutions in 1-D (Subsection 4.2).

3. Radially symmetric single bump solutions in 2-D (Subsection 4.3).

Section 5 provides concluding remarks and outlook.

2. Preliminaries

In this section we provide an overview of the notation, introduce the main definitions and formulate the main theorems we refer to.

For a metric space $\mathfrak{M}$ with the distance $\rho_{\mathfrak{M}}$, and arbitrary $\mathfrak{S} \subset \mathfrak{M}$, $\varepsilon > 0$, we denote $B_{\mathfrak{M}}(\mathfrak{S}, \varepsilon) = \bigcup\limits_{\mathfrak{s} \in \mathfrak{S}} \{\mathfrak{m} \in \mathfrak{M} \mid \rho_{\mathfrak{M}}(\mathfrak{m}, \mathfrak{s}) < \varepsilon\}$.

Definition 2.1. Let $\mathfrak{S}$ be an arbitrary subset of the metric space $\mathfrak{M}$. Choose some $\epsilon > 0$. The set $\mathfrak{E}$ is called *ϵ-net* for $\mathfrak{S}$ if for any $\mathfrak{s} \in \mathfrak{S}$, one can find such $\mathfrak{e} \in \mathfrak{E}$ that $\rho_{\mathfrak{M}}(\mathfrak{e}, \mathfrak{s}) \leq \epsilon$, see [14].

Let $\mathfrak{B}$ be a real Banach space equipped with the norm $\|\cdot\|_{\mathfrak{B}}$ and D be an arbitrary open bounded subset of $\mathfrak{B}$. We denote by ∂D and $\overline{D}$ the boundary and the closure of D in $\mathfrak{B}$, respectively. We denote by $\deg(\Phi, D, \mathfrak{b}_0)$ and $\operatorname{ind}(\Phi, D)$ the degree and the topological index of an arbitrary operator $\Phi : \overline{D} \to \mathfrak{B}$, respectively (if they are well-defined).

Let μ be the Lebesgue measure on R^m, Ω be a compact subset of R^m, $\Xi \subseteq R^m$, then:

$L^q(\Xi, \mu, R)$ be the space of functions $\eta : \Xi \to R$ with Lebesgue integrable q-th power of the absolute value and the following norm $\|\eta\|_{L^q(\Xi,\mu,R)} = \Big(\int\limits_{\Xi} |\eta(x)|^q dx\Big)^{1/q}$, $1 \leq q < \infty$.

Let $C^k(\Omega, R)$ be the space of functions $\zeta : \Omega \to R$, whose first k derivatives $\zeta^{(n)}$ ($n = 0, \ldots, k$, $\zeta^{(0)} = \zeta$) are continuous, equipped with the norm $\|\zeta\|_{C^k(\Omega,R)} = \sum\limits_{n=0}^{k} \max\limits_{x \in \Omega} |\zeta^{(n)}(x)|$.

Let $C^k(R^m, R)$ be a locally convex space of functions $\zeta : R^m \to R$, whose first k derivatives $\zeta^{(n)}$ ($n = 0, \ldots, k$, $\zeta^{(0)} = \zeta$) are continuous, equipped with the topology of uniform convergence of $\sum\limits_{n=0}^{k} \max |\zeta^{(n)}|$ on compact subsets of R^m.

We will not indicate $q = 1$ and $k = 0$ in the corresponding space notations.

Lemma 2.1. Let D be a open bounded subset of a real Banach space $\mathfrak{B}$, Λ be a compact subset of R, and an operator $T : \Lambda \times \overline{D} \to \mathfrak{B}$ be continuous with respect to both variables and collectively compact (i.e., $T(\Lambda, \overline{D})$ is a pre-compact set in $\mathfrak{B}$). Assume that $\lambda_n \to \lambda_0$ and $T(\lambda_n, \mathfrak{b}_n) = \mathfrak{b}_n$. Then the equation $T(\lambda_0, \mathfrak{b}) = \mathfrak{b}$ has at least one solution. Moreover, any limit point of the sequence $\{\mathfrak{b}_n\}$ is a solution of this equation, i.e., if $\mathfrak{b}_n \to \mathfrak{b}_0$ then $T(\lambda_0, \mathfrak{b}_0) = \mathfrak{b}_0$, see [21].

Definition 2.2. Let D be an open bounded subset of a real Banach space $\mathfrak{B}$. The family $\{h_t\}$, $(t \in [0,1])$ of operators acting from $\overline{D}$ to $\mathfrak{B}$ is called *homotopy* if $h_t(\mathbf{b})$ is continuous with respect to $(t, \mathfrak{b})$ on $[0,1] \times \overline{D}$, see [12].

Lemma 2.2. (Homotopy invariance) Let D be an open bounded subset of a real Banach space $\mathfrak{B}$. Suppose that $\{h_t\}$ is a homotopy of operators $h_t : \overline{D} \to \mathfrak{B}$ and $h_t - I$ is compact for each $t \in [0,1]$. If $h_t\mathfrak{b} \neq \mathfrak{b}_0$ for any $\mathfrak{b} \in \partial D$ and $t \in [0,1]$, then $\deg(h_t, D, \mathfrak{b}_0)$ is independent of t, see [12].

Definition 2.3. Let D be an open bounded subset of $\mathfrak{D}$, where $\mathfrak{D}$ is an absolute neighborhood retract (see, e.g. [10]), $\mathfrak{D} \subset \mathfrak{B}$. The continuous mapping $\psi : D \to \mathfrak{D}$ is called *admissible* provided that the fixed point set of ψ is compact in $\mathfrak{B}$, see [10].

Lemma 2.3. (Topological invariance) Let $\psi : D \to \mathfrak{D}$ be an admissible compact mapping and $\phi : \mathfrak{D} \to \mathfrak{D}'$ be a homeomorphism. Then $\phi \circ \psi \circ \phi^{-1} : \phi(D) \to \mathfrak{D}'$ is also an admissible compact mapping and

$$\operatorname{ind}(\psi, D) = \operatorname{ind}(\phi \circ \psi \circ \phi^{-1}, \phi(D)),$$

see [10].

3. Main results

In this section we study existence and continuous dependence of stationary solutions to (4) when approximating the Heaviside activation function by continuous functions. In order to do that, we consider the following homogenized Amari neural field equation

$$\begin{gathered}\partial_t u(t,x) = -u(t,x) + \int\limits_{\Xi} \int\limits_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}} - y_{\mathrm{f}}) f_\beta(u(t,y)) dy_{\mathrm{f}} dy, \\ t > 0,\ x \in \Xi \subseteq R^m,\ x_{\mathrm{f}} \in \mathcal{Y} \subset R^k,\end{gathered} \tag{5}$$

parameterized by $\beta \in [0, \infty)$.

We assume that the functions involved in (5) satisfy the following assumptions:

(**A1**) For any $x_\mathrm{f} \in \mathcal{Y}$, the connectivity kernel $\omega(\cdot, x_\mathrm{f}) \in C^2(\Xi, R)$.

(**A2**) For any $x \in R$, the connectivity kernel $\omega(x, \cdot) \in L(\mathcal{Y}, \mu, R)$.

(**A3**) For $\beta = 0$, the activation function is represented by the Heaviside unit step function

$$f_0(u) = \begin{cases} 0, & u \leq \theta, \\ 1, & u > \theta \end{cases}$$

with some threshold value θ.

(**A4**) For $\beta > 0$, functions of the family $f_\beta : R \to [0, 1]$ are non-decreasing, continuous, and satisfying the following convergence conditions with respect to the parameter β:

(i) $f_\beta \to f_{\widehat{\beta}}$ uniformly on R as $\beta \to \widehat{\beta}$, $\widehat{\beta} \in (0, \infty)$;

(ii) for any $\varepsilon > 0$, $f_\beta \to f_0$ uniformly on $R \setminus B_R(\theta, \varepsilon)$ as $\beta \to 0$.

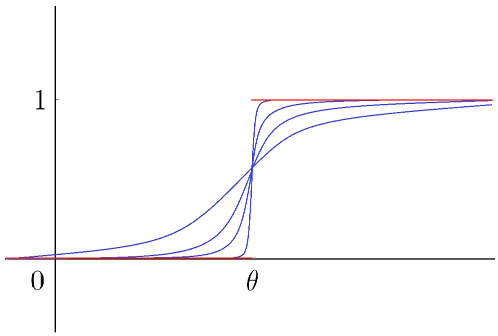

Figure 1: Approximation of the Heaviside firing rate function (red) by continuous functions (blue).

So, if the stationary solution to (5) exists, it satisfies the following equation

$$\begin{gathered} u(x) = \int\limits_{\Xi} \langle \omega \rangle (x - y) f_\beta(u(y)) dy, \\ \langle \omega \rangle (x) = \int\limits_{\mathcal{Y}} \omega(x, x_\mathrm{f}) dx_\mathrm{f}, \\ x \in \Xi \subseteq R^m, x_\mathrm{f} \in \mathcal{Y}. \end{gathered} \tag{6}$$

We are interested here in one particular type of solutions, which possesses the following properties.

Definition 3.1. Let $\theta > 0$ be fixed. We say that $u \in C^1(\Xi, R)$ satisfies the *θ-condition* if

(**B1**) there is a finite set of open bounded domains $\Theta_i \subset \Xi$ such that $u(x) > \theta$ on $\Theta = \bigcup_{i=1}^{N} \Theta_i$;

(**B2**) for any point x of the boundary $\mathcal{B} = \bigcup_{i=1}^{N} \mathcal{B}_i$ of Θ, it holds true that $u'(x) \neq 0$;

(**B3**) there exist $\sigma > 0$ and $r > 0$ such that $u(x) < \theta - \sigma$ for all $x \in \Xi \setminus B_{R^m}(\Theta, r)$.

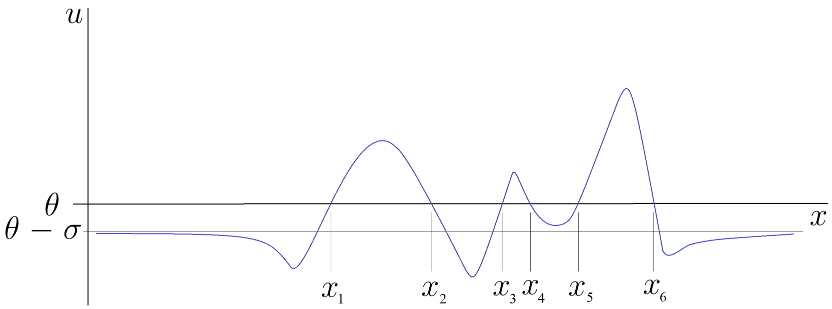

Figure 2: Example of function $U \in C^1(R, R)$ satisfying *θ-condition*. Here $\Theta = (x_1, x_2) \cup (x_3, x_4) \cup (x_5, x_6)$, $\mathcal{B} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

Remark 3.1. Definition 3.1 implies $\mathcal{B}_i \bigcap \mathcal{B}_j = \emptyset$ for any $i, j = 1, \ldots, N$, $i \neq j$.

In this section we assume existence of the stationary solution $U \in C^1(R^m, R)$ to (6), ($\Xi = R^m$), which corresponds to $\beta = 0$ and satisfies θ-condition. We are interested here in conditions, which guarantee existence of solutions u_β to (6) for $\beta > 0$ (i.e. in the case of continuous function f_β) and convergence of these solutions to U as $\beta \to 0$.

The following theorem provides conditions for convergence of the solutions

u_β to (6), $\beta > 0$, (if these solutions exist) to the stationary solution U to (6) at $\beta = 0$.

Theorem 3.1. (Continuous dependence) Let the assumptions (**A1**)–(**A4**) hold true, $\theta > 0$ be fixed and $U \in C^1(R^m, R)$ satisfies the θ-condition. Then there exists $\varepsilon > 0$ such that for any (sufficiently large) closed $\Omega \subset R^m$, if we assume existence of solutions $u_\beta \in B_{C^1(\Omega,R)}(U, \varepsilon)$ to the equation (6) for any $\beta \in (0, 1]$ ($\Xi = \Omega$), then there exist a solution to (6) at $\beta = 0$ and it is a limit point of the set $\{u_\beta\}$. Moreover, if the solution of (6) at $\beta = 0$ ($\Xi = \Omega$), say u_0, is unique then $\|u_\beta - u_0\|_{C^1(\Omega,R)} \to 0$.

Proof. We are going to apply Lemma 2.1, so we represent (6) in terms of the parameterized operator equation

$$u = F_\beta u,$$

where

$$F_\beta = \mathcal{W} \circ \mathcal{N}_\beta. \tag{7}$$

Here, for any $\beta \in [0, \infty)$, the Nemytskii operator

$$(\mathcal{N}_\beta u)(x) = f_\beta(u(x)), \tag{8}$$

and the linear integral operator

$$(\mathcal{W}u)(x) = \int\limits_{\Xi} \langle\omega\rangle(x - y)u(y)dy. \tag{9}$$

We introduce some important notations. For an arbitrary $\varepsilon > 0$, we denote the open sets $\Theta^{+\varepsilon} \subset R^m$ and $\Theta^{-\varepsilon} \subset R^m$ such that $U(x) > \theta + \varepsilon$ on $\Theta^{+\varepsilon} = \bigcup\limits_{i=1}^{N^{+\varepsilon}} \Theta_i^{+\varepsilon}$ and $U(x) > \theta - \varepsilon$ on $\Theta^{-\varepsilon} = \bigcup\limits_{i=1}^{N^{-\varepsilon}} \Theta_i^{-\varepsilon}$, respectively. The boundaries of these sets we denote as $\mathcal{B}^{+\varepsilon} = \bigcup\limits_{i=1}^{N^{+\varepsilon}} \mathcal{B}_i^{+\varepsilon}$ and $\mathcal{B}^{-\varepsilon} = \bigcup\limits_{i=1}^{N^{-\varepsilon}} \mathcal{B}_i^{-\varepsilon}$, respectively.

By the virtue of the conditions (**B1**) – (**B3**) imposed on $U \in C^1(R^m, R)$ and Remark 3.1, there exists $\varepsilon_0 \in (0, \sigma/2)$ such that

$$N^{+\varepsilon_0} = N^{-\varepsilon_0} = N, \ \mathcal{B} \subset \Theta^{-\varepsilon_0} \setminus \Theta^{+\varepsilon_0},$$

$$\mathcal{B}_i^{-\varepsilon_0} \bigcap \mathcal{B}_j^{-\varepsilon_0} = \emptyset \text{ for any } i, j = 1, \ldots, N, \ i \neq j.$$

Choosing an arbitrary compact Ω, $\Theta^{-\varepsilon_0} \subset \Omega$, for any $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$, we get the conditions (**B1**), (**B2**) fulfilled and the following condition holding true instead of (**B3**):

($\mathbf{B3}_{(\mathbf{\Omega})}$) $u(x) < \theta - \sigma/2$ for all $x \in \Omega \setminus \Theta^{-\varepsilon_0}$.

Now we show that $\mathcal{N}_\beta : B_{C^1(\Omega,R)}(U, \varepsilon_0) \to L(\Omega, \mu, R)$ defined by (8) is continuous at any $\widehat{\beta} \in [0, \infty)$ uniformly on $B_{C^1(\Omega,R)}(U, \varepsilon_0)$. For $\widehat{\beta} \in [0, \infty)$, and $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$, we estimate $\|N_\beta u - N_{\widehat{\beta}} u\|_{L(\Omega,\mu,R)}$, as $\beta \to \widehat{\beta}$. The case $\widehat{\beta} \in (0, \infty)$ is trivial, as by the virtue of (**A4**), we immediately get

$$\int\limits_{\Omega} |f_\beta(u(x)) - f_{\widehat{\beta}}(u(x))| dx \to 0, \ \beta \to \widehat{\beta}$$

uniformly with respect to $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$. So, we focus on the more involved case $\widehat{\beta} = 0$.

$$\int\limits_{\Omega} |f_\beta(u(x)) - f_0(u(x))| dx =$$

$$= \int\limits_{\Theta^{+\varepsilon_0} \bigcup (\Omega \setminus \Theta^{-\varepsilon_0})} |f_\beta(u(x)) - f_0(u(x))| dx + \int\limits_{\Theta^{-\varepsilon_0} \setminus \Theta^{+\varepsilon_0}} |f_\beta(u(x)) - f_0(u(x))| dx. \tag{10}$$

For all $x \in \Theta^{+\varepsilon_0} \bigcup (\Omega \setminus \Theta^{-\varepsilon_0})$ and any $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$, $u(x)$ belongs to $R \setminus B_R(\theta, \varepsilon_0)$. Taking into account (**A4**), we get the first summand on the right-hand side of (10) converging to 0 uniformly on $B_{C^1(\Omega,R)}(U, \varepsilon_0)$, as $\beta \to 0$. Next,

$$\int\limits_{\Theta^{-\varepsilon_0} \setminus \Theta^{+\varepsilon_0}} |f_\beta(u(x)) - f_0(u(x))| dx < \frac{1}{c_0} \int\limits_{-\|U\|_{C^1(\Omega,R)}}^{\|U\|_{C^1(\Omega,R)}} |f_\beta(s) - f_0(s)| ds,$$

where $0 < c_0 < |u'(x)|$ for all $x \in \Theta^{+\varepsilon_0} \bigcup (\Omega \setminus \Theta^{-\varepsilon_0})$ and any $u \in B_{C^1(\Omega,R)}(U, \varepsilon_0)$ (We assume here that $\varepsilon_0 < \min\limits_{x \in \Theta^{-\varepsilon_0} \setminus \Theta^{+\varepsilon_0}} |U'(x)|$, otherwise we repeat the procedure above with the new $\varepsilon_0 = \varepsilon_1 < \min\limits_{x \in \Theta^{-\varepsilon_1} \setminus \Theta^{+\varepsilon_1}} |U'(x)|$). Finally, we notice that assumption (**A4**) guarantees convergence to 0 of the expression on the right-hand side of the latter inequality, as $\beta \to 0$.

Thus, for any compact $\Omega \subset R^m$, $\mathcal{N}_\beta : B_{C^1(\Omega,R)}(U,\varepsilon_0) \to L(\Omega,\mu,R)$ is continuous at any $\widehat{\beta} \in [0,\infty)$ uniformly on $B_{C^1(\Omega,R)}(U,\varepsilon_0)$, which means that for all $\beta \in [0,\infty)$, the Nemytskii operator $\mathcal{N}_\beta$ is a bounded mapping from $B_{C^1(\Omega,R)}(U,\varepsilon_0)$ to $L(\Omega,\mu,R)$. We also notice that the operator $\mathcal{W}$ defined by (9) ($\Xi = \Omega$) is a linear and continuous mapping from $L(\Omega,\mu,R)$ to $C^1(\Omega,R)$ provided that assumptions (**A1**) and (**A2**) hold true.

Thus, for any $\beta \in [0,\infty)$, $F_\beta : B_{C^1(\Omega,R)}(U,\varepsilon_0) \to C^1(R^m,R)$ and

$$\|F_\beta u - F_{\widehat{\beta}}\widehat{u}\|_{C^1(\Omega,R)} \to 0,\ \beta \to \widehat{\beta},\ \|u-\widehat{u}\|_{C^1(\Omega,R)} \to 0, \text{ where } \widehat{u} \in B_{C^1(\Omega,R)}(U,\varepsilon_0).$$

Next, we prove that $F_\beta : B_{C^1(\Omega,R)}(U,\varepsilon_0) \to C^1(\Omega,R)$ ($\beta \in [0,\infty)$) are collectively compact.

By the virtue of (**A3**), (**A4**), it suffices to show that for an arbitrary $\epsilon > 0$, the set $\{\int_\Omega \langle\omega\rangle(x-y)\kappa dy,\ \kappa \in [0,1]\}$ possesses a finite ϵ-net in $C^1(\Omega,R)$. We represent $\langle\omega\rangle = (\langle\omega\rangle_l)$, where $\langle\omega\rangle_l \in C^2(\Omega_l,R)$, Ω_l is the orthogonal projection of Ω to the axis OX_l ($l = 1,\dots,m$).

Choose an arbitrary $\widehat{l}$. Suppose that $\Omega_{\widehat{l}} = [a,b]$,

$$\int_{[a,b]} \langle\omega\rangle_{\widehat{l}}(a-s)ds = A,$$

$$\int_{[a,b]} \langle\omega\rangle'_{\widehat{l}}(a-s)ds = A',$$

$$\max_{t\in[a,b]} \int_{[a,b]} \langle\omega\rangle''_{\widehat{l}}(t-s)ds = M.$$

Then, for example, the set

$$\Big\{\alpha_i + \kappa_j t,\ \alpha_i = i\frac{A + (b-a)(A' + (b-a)M))}{[(A + (b-a)(A' + (b-a)M)))/\epsilon] + 1},$$

$$\kappa_j = j\frac{A' + (b-a)M}{[(A' + (b-a)M)/\epsilon] + 1},$$

$$i = 0,1,\dots,[(A + (b-a)(A' + (b-a)M))/\epsilon] + 1,$$

$$j = 0,1,\dots,[(A' + (b-a)M)/\epsilon] + 1,\ t \in [a,b]\Big\}$$

serves as the ϵ-net for $\{ \int\limits_{\Omega} \langle \omega \rangle_{\hat{l}}(x-y)\kappa dy,\ \kappa \in [0,1]\}$ ($[z]$ denotes here the integer part of $z \in R$). Due to arbitrary choice of the component $\int\limits_{\Omega} \langle \omega \rangle_{\hat{l}}(x-y)dy$ of $\int\limits_{\Omega} \langle \omega \rangle (x-y)dy$ ($l = 1, \dots, m$), we proved collective compactness of the whole composition $F_\beta = \mathcal{W} \circ \mathcal{N}_\beta$ ($\beta \in [0,\infty)$) as acting from $B_{C^1(\Omega,R)}(U, \varepsilon_0)$ to $C^1(\Omega, R)$.

Now, if we keep in mind the properties proved and put $T(\lambda, \mathfrak{b}) = F_\beta u$, $\Lambda = [0,1]$, $D = \overline{B_{C^1(\Omega,R)}(U, \varepsilon_1)}$, $\varepsilon_1 < \varepsilon_0$, by using Lemma 2.1, we complete the proof. $\Box$

It is often easier to study existence of solutions satisfying θ-condition to (6) when $\beta = 0$. The corresponding closed form expressions for the particular types of solutions (satisfying θ-condition) to special cases of (6) can be found e.g. in [1, 17, 22, 18, 25, 5].

The next theorem provides a tool for proving existence of solutions to (6) for $\beta \in (0,\infty)$ using some knowledge about the solution to (6) at $\beta = 0$.

Theorem 3.2. (Existence) Let the conditions of Theorem 3.1 be satisfied, the set Ω and the constant ε_1 be taken from Theorem 3.1. Assume that there exists solution $U \in C^1(R^m, R)$ of (6) at $\beta = 0$, which satisfies θ-condition and which is unique in $\overline{B_{C^1(\Omega,R)}(U, \varepsilon_2)}$ ($\varepsilon_2 < \varepsilon_1$), and $\deg(I - F_0, B_{C^1(\Omega,R)}(U, \varepsilon_2), 0) \neq 0$, where the operator $F_0 : B_{C^1(\Omega,R)}(U, \varepsilon_1) \to C^1(\Omega, R)$ is given by (7). Then for any $\beta \in (0,1]$, there exists solution $u_\beta \in B_{C^1(\Omega,R)}(U, \varepsilon_2)$ to the equation (6).

Proof. We prove that the family $\{h_\beta\}$, $\beta \in [0,1]$,

$$h_\beta = I - F_\beta \tag{11}$$

is homotopy. Continuity of $h_{(\cdot)}(\cdot)$ on $[0,1] \times B_{C^1(\Omega,R)}(U, \varepsilon_1)$ follows from the proof of Theorem 3.1. It remains to prove that $h_\beta(u) \neq 0$ for any $\beta \in [0,1]$ and $u \in \partial B_{C^1(\Omega,R)}(U, \varepsilon_2)$.

Collective compactness of $F_\beta : B_{C^1(\Omega,R)}(U, \varepsilon_1) \to C^1(\Omega, R)$ ($\beta \in [0,\infty)$), shown in the proof of Theorem 3.1, imply the following two possibilities for any sequence $\{u_{\beta_n}\} \subset B_{C^1(\Omega,R)}(U, \varepsilon_1)$ ($\beta_n \to 0$) of solutions to (6):

1) u_{β_n} converges to U, as $\beta_n \to 0$;

2) there exists such $\widehat{n}$ that for any $n > \widehat{n}$, $\|u_{\beta_n} - U\|_{C^1(\Omega,R)} > \varepsilon_2$ (without loss of generality we can assume that $\beta_{\widehat{n}} > 1$).

This proves that $(I - F_\beta)(u) \neq 0$ for any $\beta \in [0,1]$ and $u \in \partial B_{C^1(\Omega,R)}(U, \varepsilon_1)$.

Finally, we apply Lemma 2.2 to the homotopy (11) and get existence of solutions to (6) for any $\beta \in (0,1]$. $\Box$

Remark 3.2. The choice of the space $C^1(\Omega, R)$ as a basic functional space in this research is caused by the fact that even in the space of absolutely continuous functions, any ball, centered at a function satisfying θ-condition, contains functions, which do not satisfy θ-condition. The corresponding example can be found in [21], in the proof of Lemma 3.7.

4. Bumps in neural field models

In this section we apply the theory developed to the stationary bump solutions to the neural field model (5) in the following three special cases:

1. Symmetric single bump in 1-D.
2. Symmetric double bump in 1-D.
3. Radially symmetric single bump in 2-D.

Each subsection concludes with a theorem on existence and continuous dependence of the stationary solutions of the corresponding type to the equation (5) when approximating the Heaviside activation function by continuous functions.

4.1. Symmetric single bump in 1-D

We consider here the one-dimensional homogenized Amari model, i.e. the model (5) with $m = k = 1$:

$$\partial_t u(t, x, x_{\mathrm{f}}) = -u(t, x, x_{\mathrm{f}}) + \int\limits_{\Xi}\int\limits_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f_\beta(u(t, y, y_{\mathrm{f}})) dy_{\mathrm{f}} dy, \quad t > 0, \; x \in \Xi \subseteq R. \tag{12}$$

Here $\mathcal{Y}$ is some one-dimensional torus, the family of functions $f_\beta : R \to [0, 1]$ satisfies assumptions (**A3**), (**A4**), and the function ω is typically decomposed in the following way (see e.g. [25], [18]):

$$\omega(x, x_{\mathrm{f}}) = \frac{1}{\sigma(x_{\mathrm{f}})} \chi\Big(\frac{|x|}{\sigma(x_{\mathrm{f}})}\Big), \tag{13}$$

where the function $\sigma \in C(\mathcal{Y}, (0, \infty))$ is $\mathcal{Y}$-periodic and the function $\chi \in C^2([0, \infty), R) \bigcap L([0, \infty), \mu, R)$ satisfies the property:

$$\lim_{x\to\infty} \chi(x) = 0. \tag{14}$$

Thus, assumptions (**A1**), (**A2**) are also satisfied. We emphasize here that the class of connectivity functions ω described above is rather wide. It contains all typical connectivity functions in use in the neural field theory (see e.g. [25], [18] for the heterogeneous media case, and the review [2] for the homogeneous media case).

Definition 4.1.1. Let $\theta > 0$ be fixed. We define a *symmetric single bump solution* to (12) to be a stationary solution $U \in C^1(\Xi, R)$ to (12), satisfying the following properties:

- $U(x) = U(-x)$ for all $x \in R$;
- the equation $U(x) = \theta$ has exactly two solutions $x = -a,\ x = a,\ a > 0$;
- $U(x) > \theta$ for all $x \in (-a, a)$ and $U(x) < \theta$ for all $x \in \Xi \setminus [-a, a]$.

The stationary symmetric single bump solution to (12) in the case $\beta = 0$ can be determined by the following expression (see e.g. [25]):

$$U(x) = W(x + a) - W(x - a), \tag{15}$$

where

$$W(x) = \int_0^x \langle\omega\rangle(y)dy,$$
$$\langle\omega\rangle(x) = \int_{\mathcal{Y}} \omega(x, x_{\mathrm{f}})dx_{\mathrm{f}}.$$

Due to the assumptions on the functions $\chi \in C^2(R, R) \bigcap L(R, \mu, R)$ and $\sigma \in C(\mathcal{Y}, (0, \infty))$, and the corresponding properties of the connectivity ω defined by (13), we get the following condition fulfilled:

$$\lim_{|x|\to\infty} \langle\omega\rangle(x) = 0. \tag{16}$$

Using the latter expression, we easily obtain

$$\lim_{|x|\to\infty} U(x) = 0.$$

Thus, the bump solution U satisfies θ-condition.

We investigate existence and continuous dependence of stationary bump solutions to (12), which are symmetric with respect to the ordinate axis, when approximating the Heaviside activation function in (12) (the case $\beta = 0$) by continuous functions ($\beta > 0$). Indeed, due to the translational invariance of the integration kernel ω with respect to the spatial variable x, the corresponding operators F_β ($\beta \in [0,1]$) defined by (7) map even functions to even functions. We, thus, consider solutions belonging to the space $C_e^1(\Xi, R) = \{u \in C^1(\Xi, R),\ u(x) = u(-x) \text{ for all } x \in \Xi\}$.

Lemma 4.1.1. Let the following condition be satisfied:

$$\langle\omega\rangle(2a) \neq 0. \tag{17}$$

Then for any compact set Ω, $\Omega \in R$, there exists such $\varepsilon > 0$ that the symmetric single bump U defined by (15) is a unique solution to (12) in $B_{C_e^1(\Omega,R)}(U, \varepsilon)$ when $\beta = 0$.

Proof. From the definition of the single bump solution it follows that

$$W(2a) = \theta.$$

Thus, the condition (17) guarantees uniqueness of the solution U in $B_{C_e^1(\Omega,R)}(U, \varepsilon)$ for some $\varepsilon > 0$. □

We emphasize that U is not an isolated solution to (12) in $C^1(\Xi, R)$ due to the translation invariance of bumps in the homogenized neural field (12).

We now express (15) in terms of operator equality just as it was done in Section 3:

$$U = F_0 U.$$

In order to apply Theorem 3.2, we need to calculate $\deg(I - F_0, B_{C_e^1(\Omega,R)}(U, \varepsilon), 0)$. By the definition of the topological fixed point index, we get

$$\deg(I - F_0, B_{C_e^1(\Omega,R)}(U, \varepsilon), 0) = \mathrm{ind}(F_0, B_{C_e^1(\Omega,R)}(U, \varepsilon)).$$

Without loss of generality we assume that the fixed point U of the operator F_0 is unique in $\overline{B_{C_e^1(\Omega,R)}(U, \varepsilon)}$. Thus, F_0 maps $\overline{B_{C_e^1(\Omega,R)}(U, \varepsilon)}$ into some manifold $\mathcal{M} \subset C^1(\Omega, R)$, $\mathcal{M} = \{v = W(\cdot + c) - W(\cdot - c),\ c \in \mathrm{M} \subset \Omega\}$, where compact set M is chosen in a such way that it contains c_u for all $u \in \overline{B_{C_e^1(\Omega,R)}(U, \varepsilon)}$ (One can e.g. choose M to be a segment). We define the mapping $\phi : \mathrm{M} \to \mathcal{M}$ as

$$\phi(c) = v(x),\ v(x) = W(x + c) - W(x - c),\ x \in \Omega. \tag{18}$$

Lemma 4.1.2. The mapping $\phi : \mathrm{M} \to \mathcal{M}$ defined by (18) is a homeomorphism, and $\mathcal{M}$ is an absolute neighborhood retract.

Proof. First, we note that $\phi : \mathrm{M} \to \mathcal{M}$ is a surjection by definition. In order to prove that $\phi : \mathrm{M} \to \mathcal{M}$ is an injection, we use the expression for the Frechet derivative of ϕ taken at an arbitrary $c \in \mathrm{M}$:

$$\phi'(c) = \langle\omega\rangle(\cdot + c) - \langle\omega\rangle(\cdot - c).$$

For sufficiently large set $\Omega = [-X, X]$, $X \gg a$, the condition (16) implies the following relation:

$$\max_{x\in[X-2a,X]} |\langle\omega\rangle(x)| < \max_{x\in[0,2a]} |\langle\omega\rangle(x)|. \tag{19}$$

Thus, we have $\phi'(a) \neq 0$, because assuming the contrary, we get $\langle\omega\rangle(x + a) - \langle\omega\rangle(x - a) = 0$, for all $x \in \Omega$, which contradicts with (19). Summarizing the described above properties of ϕ, we conclude that $\phi : \mathrm{M} \to \mathcal{M}$ is a homeomorphism. We also note that the set M is an absolute neighborhood retract, since it is a compact convex subset of R. Thus, by properties of homeomorphism, $\mathcal{M} = \phi(\mathrm{M})$ is an absolute neighborhood retract, too. □

We now define $\mathcal{F}$ to be the restriction of F_0 on $\mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U, \varepsilon)}$, i.e.

$$\mathcal{F} = F_0|_{\mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)}},$$
$$\mathcal{F} : \mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U, \varepsilon)} \to \mathcal{M}.$$

Due to its definition, the mapping $\mathcal{F} : \mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U, \varepsilon)} \to \mathcal{M}$ is compact and admissible. Using the properties of the topological fixed point index (see e.g. [10]), we get

$$\mathrm{ind}(F_0, B_{C^1_e(\Omega,R)}(U, \varepsilon)) = \mathrm{ind}(\mathcal{F}, \mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U, \varepsilon)).$$

Next, we apply Lemma 2.3 and obtain

$$\mathrm{ind}(\mathcal{F}, \mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U, \varepsilon)) = \mathrm{ind}(\phi^{-1}\circ\mathcal{F}\circ\phi, \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U, \varepsilon))).$$

Lemma 4.1.3. There exists such $\delta > 0$ that the operator $\Psi = \phi^{-1}\circ\mathcal{F}\circ\phi$ maps $\overline{B_R(a, \delta)}$ to M.

Proof. Let $u(x) = W(x + c) - W(x - c)$, $c \in \mathrm{M}$. Using the mean value theorem, we estimate

$$\|u - U\|_{C^1(\Omega,R)} \leq 4\|\langle\omega\rangle\|_{C^1(\Omega,R)}|c - a| < \varepsilon$$

for all $c \in \overline{B_R(a,\delta)}$, where $\delta < \varepsilon/4\|\langle\omega\rangle\|_{C^1(\Omega,R)}$. From the latter estimate we conclude that

$$\overline{B_R(a,\delta)} \subset \phi^{-1}(\mathcal{M} \bigcap B_{C_e^1(\Omega,R)}(U,\varepsilon))$$

which, in turn, implies

$$\mathcal{M}_\delta = \{v = W(\cdot + c) - W(\cdot - c),\ c \in \overline{B_R(a,\delta)}\} \subset \mathcal{F}(\mathcal{M} \bigcap B_{C_e^1(\Omega,R)}(U,\varepsilon)).$$

Thus, we finally get

$$\phi^{-1}(\mathcal{M}_\delta) = \overline{B_R(a,\delta)} \subset \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C_e^1(\Omega,R)}(U,\varepsilon))),$$

which concludes the proof. □

It is easy to see that a is a fixed point of the operator $\Psi : \overline{B_R(a,\delta)} \to$ M. Moreover, a is an isolated fixed point of Ψ due to the fact that U is an isolated fixed point of $\mathcal{F}$ and topological invariance property of the index. The topological index of a finite dimensional map can be calculated as

$$\mathrm{ind}(\Psi, \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C_e^1(\Omega,R)}(U,\varepsilon))) = \mathrm{sgn}(1 - \Psi'(a)),$$

see e.g. [16].

It follows from the definition of the operator $\Psi = \phi^{-1} \circ \mathcal{F} \circ \phi$ that

$$W(\Psi(c) + c) - W(\Psi(c) - c) = \theta \ \text{ for all } \ c \in \overline{B_R(a,\delta)}.$$

Using the implicit function theorem and the chain rule for differentiation, we get

$$\Psi'(a) = \frac{\langle\omega\rangle(0) + \langle\omega\rangle(2a)}{\langle\omega\rangle(0) - \langle\omega\rangle(2a)}.$$

Thus, $\deg(I - F_0, B_{C_e^1(\Omega,R)}(U,\varepsilon), 0) \neq 0$ as soon as the following inequality takes place:

$$\frac{\langle\omega\rangle(0) + \langle\omega\rangle(2a)}{\langle\omega\rangle(0) - \langle\omega\rangle(2a)} \neq 1.$$

Summarizing the results above and using Theorem 3.2 and Theorem 3.1, we get the main result of the subsection.

Theorem 4.1.1. Let the family of functions $f_\beta : R \to [0,1]$ $(\beta \in [0,\infty))$ satisfy assumptions (**A3**) and (**A4**). Let also the connectivity kernel ω be given by (13), where the function $\sigma \in C(\mathcal{Y}, (0,\infty))$ is $\mathcal{Y}$-periodic and the even function $\chi \in C^2(R,R) \bigcap L(R,\mu,R)$ satisfies (14). Finally, let the inequality (17) be fulfilled. Then, for any sufficiently large Ω, $\Omega \subset R$, and for each $\beta \in (0,\infty)$, there exists solution $u_\beta \in C_e^1(\Omega,R))$ to (12) $(\Xi = \Omega)$. Moreover, $\|u_\beta - U\|_{C^1(\Omega,R))} \to 0$, as $\beta \to 0$, where $U \in C_e^1(R,R))$ is the stationary bump solution to (12) $(\Xi = R,\ \beta = 0)$, defined by (15).

4.2. Symmetric double bump in 1-D

We keep here the modeling framework (12) under the same assumptions on the functions involved as in the previous subsection.

Definition 4.2.1. Let $\theta > 0$ be fixed. We define a *symmetric double bump solution* to (12) to be a stationary solution $U \in C^1(\Xi, R)$ to (12), satisfying the following properties:

- $U(x) = U(-x)$ for all $x \in R$;
- the equation $U(x) = \theta$ has exactly four solutions $x = -b$, $x = -a$, $x = a$, $x = b$, $b > a > 0$;
- $U(x) > \theta$ for all $x \in (-b, -a) \bigcup (a, b)$ and $U(x) < \theta$ for all $x \in (-a, a) \bigcup \Xi \setminus [-b, -a] \setminus [a, b]$.

The stationary (symmetric) double bump solution to (12) ($\beta = 0$) can be written as

$$U(x) = W(x + b) - W(x + a) + W(x - a) - W(x - b), \tag{20}$$

(see e.g. [18]).

Using the expression (16), we obtain

$$\lim_{|x|\to\infty} U(x) = 0.$$

It is easy to see now that the double bump solution U satisfies θ-condition.

Just as in the previous subsection, we investigate here existence and continuous dependence on the steepness of the function $f_\beta : R \to [0, 1]$ of the stationary double bump solutions to (12) belonging to $C^1_e(\Xi, R)$.

Lemma 4.2.1. Let the following condition be satisfied:

$$\begin{cases} \langle\omega\rangle(b - a) - \langle\omega\rangle(2a) \neq 0, \\ \langle\omega\rangle(b - a) + \langle\omega\rangle(b + a) \neq 0. \end{cases} \tag{21}$$

Then for any compact set Ω, $\Omega \in R$, there exists such $\varepsilon > 0$ that the symmetric double bump U defined by (20) is a unique solution to (12) in $B_{C^1_e(\Omega,R)}(U, \varepsilon)$ when $\beta = 0$.

Proof. From the definition of the single bump solution it follows that

$$\begin{cases} W(b - a) - W(b + a) + W(2b) = \theta, \\ W(2b) + W(2a) - 2W(b + a) = 0. \end{cases} \tag{22}$$

Differentiation of this expression with respect to the parameter a gives us

$$\begin{cases} \langle\omega\rangle(b-a) - \langle\omega\rangle(b+a) = 0, \\ \langle\omega\rangle(2a) - \langle\omega\rangle(b+a) = 0, \end{cases}$$

from where we get

$$\langle\omega\rangle(b-a) - \langle\omega\rangle(2a) = 0.$$

Differentiating (22) with respect to the parameter b, we obtain

$$\begin{cases} \langle\omega\rangle(b-a) - \langle\omega\rangle(b+a) + 2\langle\omega\rangle(2b) = 0, \\ \langle\omega\rangle(2b) - \langle\omega\rangle(b+a) = 0, \end{cases}$$

which implies

$$\langle\omega\rangle(b-a) + \langle\omega\rangle(b+a) = 0.$$

Thus, the condition (21) guarantees uniqueness of the solution U in $B_{C^1_e(\Omega,R)}(U,\varepsilon)$ for some $\varepsilon > 0$. □

We express (20) in terms of the operator equality

$$U = F_0 U.$$

Without loss of generality we assume that the fixed point U of the operator F_0 is unique in $\overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)}$. Thus, F_0 maps $\overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)}$ into some manifold $\mathcal{M} \subset C^1(\Omega,R)$, $\mathcal{M} = \{v = W(x+d)-W(x+c)+W(x-c)-W(x-d),\ (c,d) \in \mathrm{M} \subset R^2\}$, where compact set M is chosen in a such way that it contains the points (c_u, d_u) for all $u \in \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)}$ (One can e.g. choose M to be a rectangle). We define the mapping $\phi : \mathrm{M} \to \mathcal{M}$ as

$$\begin{gathered} \phi((c,d)) = v(x), \\ v(x) = W(x+d) - W(x+c) + W(x-c) - W(x-d),\ x \in \Omega. \end{gathered} \tag{23}$$

Lemma 4.2.2. The mapping $\phi : \mathrm{M} \to \mathcal{M}$ defined by (23) is a homeomorphism, and $\mathcal{M}$ is an absolute neighborhood retract.

Proof. First, we note that $\phi : \mathrm{M} \to \mathcal{M}$ is a surjection by definition. In order to prove that $\phi : \mathrm{M} \to \mathcal{M}$ is an injection, we use the following expressions for the Frechet derivatives of ϕ:

$$\begin{gathered} \phi'_c((c,d)) = \langle\omega\rangle(\cdot - c) - \langle\omega\rangle(\cdot + c) \\ \phi'_d((c,d)) = \langle\omega\rangle(\cdot + d) - \langle\omega\rangle(\cdot - d) \end{gathered}$$

Assuming $\phi'_c((a,b)) = 0$, we get $\langle\omega\rangle(x-a) - \langle\omega\rangle(x+a) = 0$, for all $x \in \Omega$, which contradicts with (19). We, thus, have $\phi'_c((a,b)) \neq 0$. By the same way we obtain $\phi'_d((a,b)) \neq 0$, which concludes the proof of the fact that $\phi : \mathrm{M} \to \mathcal{M}$ is a homeomorphism. As the set M is an absolute neighborhood retract, then by properties of homeomorphism, the set $\mathcal{M} = \phi(\mathrm{M})$ is an absolute neighborhood retract, too. □

Just as in the previous subsection, we define

$$\mathcal{F} = F_0|_{\mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)}},$$
$$\mathcal{F} : \mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)} \to \mathcal{M}.$$

The mapping $\mathcal{F} : \mathcal{M} \bigcap \overline{B_{C^1_e(\Omega,R)}(U,\varepsilon)} \to \mathcal{M}$ is compact and admissible by its definition. Using the properties of the topological fixed point index, we get

$$\mathrm{ind}(F_0, B_{C^1_e(\Omega,R)}(U,\varepsilon)) = \mathrm{ind}(\mathcal{F}, \mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon)).$$

Applying Lemma 2.3, we obtain

$$\mathrm{ind}(\mathcal{F}, \mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon)) = \mathrm{ind}(\phi^{-1}\circ\mathcal{F}\circ\phi, \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))).$$

Lemma 4.2.3. There exists such $\delta > 0$ that the operator $\Psi = \phi^{-1}\circ\mathcal{F}\circ\phi$ maps $\overline{B_{R^2}((a,b),\delta)}$ to M.

Proof. Let $u(x) = W(x+d) - W(x+c) + W(x-c) - W(x-d)$, $(c,d) \in \mathrm{M}$. Using the mean value theorem, we estimate

$$\|u - U\|_{C^1(\Omega,R)} \leq 4\|\langle\omega\rangle\|_{C^1(\Omega,R)}(|c-a| + |d-b|) < \varepsilon$$

for all $(c,d) \in \overline{B_{R^2}((a,b),\delta)}$, where $\delta < \varepsilon/8\sqrt{2}\|\langle\omega\rangle\|_{C^1(\Omega,R)}$. From the latter estimate we conclude that

$$\overline{B_{R^2}((a,b),\delta)} \subset \phi^{-1}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))$$

which implies

$$\mathcal{M}_\delta \subset \mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))$$
$$\mathcal{M}_\delta = \{v = W(\cdot+d) - W(\cdot+c) + W(\cdot-c) - W(\cdot-d),$$
$$(c,d) \in \overline{B_{R^2}((a,b),\delta)}\}.$$

Thus, we finally get

$$\phi^{-1}(\mathcal{M}_\delta) = \overline{B_{R^2}((a,b),\delta)} \subset \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_e(\Omega,R)}(U,\varepsilon))),$$

which concludes the proof. $\square$

Due to the fact that U is an isolated fixed point of $\mathcal{F}$ and topological invariance property of the index, (a,b) is an isolated fixed point of Ψ. Thus, we get

$$\Psi((a,b)) = (\Psi_1((a,b))\Psi_2((a,b))),$$
$$\Psi((a,b)) = W(\Psi_2((a,b))+b) - W(\Psi_1((a,b))+a)+$$
$$+W(\Psi_1((a,b))-a) - W(\Psi_2((a,b))-b).$$

We calculate the topological index of a two-dimensional mapping as

$$\operatorname{ind}(\Psi, \phi^{-1}(\mathcal{F}(\mathcal{M}\bigcap B_{C_e^1(\Omega,R)}(U,\varepsilon))) =$$
$$= \operatorname{sgn}\Bigg(\det\begin{pmatrix} (\Psi_1)'_a((a,b))-1 & (\Psi_1)'_b((a,b)) \\ (\Psi_2)'_a((a,b)) & (\Psi_2)'_b((a,b))-1 \end{pmatrix}\Bigg).$$

The definition of the operator $\Psi = \phi^{-1}\circ\mathcal{F}\circ\phi$ yields

$$W(\Psi_2((c,d))+d) - W(\Psi_1((c,d))+c) + W(\Psi_1((c,d))-c) - W(\Psi_2((c,d))-d) = \theta$$

for all $(c,d) \in \overline{B_R(a,\delta)}$. We use the expressions

$$(U(a))'_a = 0,\ (U(a))'_b = 0,\ (U(b))'_a = 0,\ (U(b))'_b = 0.$$

Applying the implicit function theorem and the chain rule for differentiation, we get

$$(\Psi_1)'_a((a,b)) = \frac{\langle\omega\rangle(2a)+\langle\omega\rangle(0)}{\langle\omega\rangle(b+a)-\langle\omega\rangle(2a)+\langle\omega\rangle(0)-\langle\omega\rangle(b-a)};$$
$$(\Psi_1)'_b((a,b)) = \frac{-\langle\omega\rangle(b+a)-\langle\omega\rangle(b-a)}{\langle\omega\rangle(b+a)-\langle\omega\rangle(2a)+\langle\omega\rangle(0)-\langle\omega\rangle(b-a)};$$
$$(\Psi_2)'_a((a,b)) = \frac{\langle\omega\rangle(b+a)+\langle\omega\rangle(b-a)}{\langle\omega\rangle(2b)-\langle\omega\rangle(b+a)+\langle\omega\rangle(b-a)-\langle\omega\rangle(0)};$$
$$(\Psi_2)'_b((a,b)) = \frac{-\langle\omega\rangle(2b)-\langle\omega\rangle(0)}{\langle\omega\rangle(2b)-\langle\omega\rangle(b+a)+\langle\omega\rangle(b-a)-\langle\omega\rangle(0)}.$$

Thus, $\deg(I-F_0, B_{C_e^1(\Omega,R)}(U,\varepsilon), 0) \neq 0$ if the following inequality takes place:

$$\frac{2\langle\omega\rangle(b+a)\langle\omega\rangle(b-a)-2\langle\omega\rangle(2a)\langle\omega\rangle(2b)+\Big(\langle\omega\rangle(2a)+\langle\omega\rangle(2b)\Big)\Big(\langle\omega\rangle(b+a)-\langle\omega\rangle(b-a)\Big)}{\Big(\langle\omega\rangle(b+a)-\langle\omega\rangle(2a)+\langle\omega\rangle(0)-\langle\omega\rangle(b-a)\Big)\Big(\langle\omega\rangle(2b)-\langle\omega\rangle(b+a)+\langle\omega\rangle(b-a)-\langle\omega\rangle(0)\Big)} \neq 0. \tag{24}$$

The following statement is obtained by summarizing the results above and by using then Theorem 3.2 and Theorem 3.1.

Theorem 4.2.1. Let the family of functions $f_\beta : R \to [0,1]$ ($\beta \in [0,\infty)$) satisfy assumptions (**A3**) and (**A4**). Let also the connectivity kernel ω be given by (13), where the function $\sigma \in C(\mathcal{Y},(0,\infty))$ is $\mathcal{Y}$-periodic and the even function $\chi \in C^2(R,R) \bigcap L(R,\mu,R)$ satisfies (14). Finally, let the inequalities (21) and (24) be fulfilled. Then, for any sufficiently large Ω, $\Omega \subset R$, and for each $\beta \in (0,\infty)$, there exists solution $u_\beta \in C^1_e(\Omega,R))$ to (12) ($\Xi = \Omega$). Moreover, $\|u_\beta - U\|_{C^1(\Omega,R))} \to 0$, as $\beta \to 0$, where $U \in C^1_e(R,R))$ is the stationary double bump solution to (12) ($\Xi = R$, $\beta = 0$), defined by (20).

4.3. Radially symmetric single bump in 2-D

We now consider the two-dimensional homogenized Amari model, i.e. the model (5) with $m = k = 2$:

$$\partial_t u(t,x,x_{\mathrm{f}}) = -u(t,x,x_{\mathrm{f}}) + \int_\Xi \int_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f_\beta(u(t,y,y_{\mathrm{f}}))dy_{\mathrm{f}}dy, \quad t>0,\ x \in \Xi \subseteq R^2. \tag{25}$$

Here $\mathcal{Y}$ is some two-dimensional torus, the family of functions $f_\beta : R \to [0,1]$ satisfies assumptions (**A3**), (**A4**), and the connectivity function $\omega : R^2 \times \mathcal{Y} \to R$ is decomposed in the following way (see e.g. [5]):

$$\omega(x,x_{\mathrm{f}}) = \frac{1}{\sigma(x_{\mathrm{f}})}\chi\Big(\frac{|x|}{\sigma(x_{\mathrm{f}})}\Big), \tag{26}$$

where $\sigma \in C(\mathcal{Y},(0,\infty))$ is $\mathcal{Y}$-periodic and $\chi \in C^2([0,\infty),R) \bigcap L([0,\infty),\mu,R)$. Thus, assumptions (**A1**) and (**A2**) are also satisfied.

Definition 4.3.1. Let $\theta > 0$ be fixed. We define a *radially symmetric single bump solution* to (25) to be a stationary solution $U \in C^1(\Xi,R)$ to (25), satisfying the following properties:

- $U(x) = U(|x|)$, where $x \in R^2$, $x = |x|\exp(i\alpha)$, $\alpha \in [0,2\pi)$;
- the equation $U(x) = \theta$ has only the solutions belonging to the set $\{x,\ |x| = \mathrm{r}\}$ for some $\mathrm{r} > 0$;
- $U(x) > \theta$ for all $x \in B_{R^2}(0,\mathrm{r})$ and $U(x) < \theta$ for all $x \in \Xi \setminus \overline{B_{R^2}(0,\mathrm{r})}$.

The stationary radially symmetric single bump solution of the radius a to (25) in the case $\beta = 0$ can be determined by the following expression (see e.g. [5]):

$$U(x) = 2\pi a \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|x|r) J_1(ar) dr, \tag{27}$$

where $\widehat{\langle\omega\rangle}$ is the Hankel transform (of order 0) of $\langle\omega\rangle$,

$$\langle\omega\rangle(x) = \int_{\mathcal{Y}} \omega(x, x_{\mathrm{f}}) dx_{\mathrm{f}},$$

J_n is the Bessel function of the first kind of order n.

Let us assume that the following condition is satisfied:

$$\int_0^\infty |\widehat{\langle\omega\rangle}(r)| r^2 dr < \infty. \tag{28}$$

For an arbitrary $\gamma > 0$, using the properties of J_n, we have

$$|U(x)| \leq 2\pi a \int_0^\gamma |\widehat{\langle\omega\rangle}(r)| dr + 2\pi a \Big| \int_\gamma^\infty \widehat{\langle\omega\rangle}(r) J_0(|x|r) J_1(ar) dr \Big|.$$

Due to the assumptions on the functions $\chi \in C^2(R^2, R) \bigcap L(R^2, \mu, R)$ and $\sigma \in C(\mathcal{Y}, (0, \infty))$, and the corresponding properties of the connectivity function ω defined by (26), for an arbitrary $\epsilon > 0$, we obtain:

$$2\pi a \int_0^{\gamma(\epsilon)} |\widehat{\langle\omega\rangle}(r)| dr < \epsilon/2$$

for some $\gamma(\epsilon) > 0$. By the properties of the Bessel function J_0, for any $\gamma > 0$, we have $J_0(sr) \to 0$ uniformly with respect to $r \in [\gamma, \infty)$, as $s \to \infty$. Using these facts and the estimate (28), we finally get

$$|U(x)| \leq 2\pi a \int_0^{\gamma(\epsilon)} |\widehat{\langle\omega\rangle}(r)| dr + 2\pi a \Big| \int_{\gamma(\epsilon)}^\infty \widehat{\langle\omega\rangle}(r) J_1(ar) dr \Big| |J_0(|x|r)| < \epsilon$$

for some $\gamma(\epsilon) > 0$ and sufficiently large $|x| \in R$. Thus, we obtain

$$\lim_{|x|\to\infty} U(x) = 0, \tag{29}$$

which means that the radially symmetric single bump solution U satisfies θ-condition.

Remark 4.3.1. For the proof of (29) it is sufficient to assume that

$$\int_0^\infty \widehat{\langle\omega\rangle}(r) J_1(ar)dr < \infty.$$

instead of the more strict condition (28). However, we will need the condition (28) in the proofs below. We also stress here, that (28) is fulfilled for all typical connectivity functions used in neural field modeling.

We introduce the space

$$C^1_{rs}(\Xi, R) = \{u \in C^1(\Xi, R),\ u(x) = u(|x|) \text{ for all } x \in \Xi\}.$$

Lemma 4.3.1. Let the following condition be satisfied:

$$\int_0^\infty \widehat{\langle\omega\rangle}(r)\Big(J_0(ar)J_1(ar) + \frac{ar}{2}(J_0^2(ar) - 2J_1^2(ar) - J_0(ar)J_2(ar))\Big)dr \neq 0. \tag{30}$$

Then for an arbitrary sufficiently large compact set Ω, $\Omega \subset R^2$, there exists such $\varepsilon > 0$ that the symmetric single bump U defined by (27) is a unique solution to (25) in $B_{C^1_{rs}(\Omega,R)}(U, \varepsilon)$ when $\beta = 0$.

Proof. From the definition of the radially symmetric single bump solution it follows that

$$2\pi a \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(ar)J_1(ar)dr = \theta.$$

Thus, the condition (30) guarantees uniqueness of the solution U in $B_{C^1_{rs}(\Omega,R)}(U, \varepsilon)$ for some $\varepsilon > 0$. $\square$

We now express (27) in terms of operator equality just as it was done in Section 3:

$$U = F_0 U.$$

In order to apply Theorem 3.2, we calculate $\deg(I - F_0, B_{C^1_{rs}(\Omega,R)}(U,\varepsilon), 0)$. By the definition of the topological fixed point index, we get

$$\deg(I - F_0, B_{C^1_{rs}(\Omega,R)}(U,\varepsilon), 0) = \mathrm{ind}(F_0, B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)).$$

Without loss of generality we assume that the fixed point U of the operator F_0 is unique in $\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}$. Thus, F_0 maps $\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}$ into some manifold $\mathcal{M} \subset C^1(\Omega, R)$,

$$\mathcal{M} = \{v = 2\pi c \int\limits_0^\infty \widehat{\langle\omega\rangle}(r) J_0(\cdot\, r) J_1(cr)dr,\ c \in \mathrm{M} \subset R\},$$

where compact set M is chosen in a such way that it contains c_u for all $u \in \overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}$ (One can e.g. choose M to be a segment). We define the mapping $\phi : \mathrm{M} \to \mathcal{M}$ as

$$\phi(c) = v(x),\ v(x) = 2\pi c \int\limits_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|x|r) J_1(cr)dr,\ x \in \Omega. \tag{31}$$

Lemma 4.3.2. Let the following condition be satisfied:

$$\int\limits_0^\infty \widehat{\langle\omega\rangle}(r) J_0(\cdot\, r)\Big(J_1(ar) + \frac{ar}{2}(J_0(ar) - J_2(ar))\Big)dr \not\equiv 0. \tag{32}$$

Then the mapping $\phi : \mathrm{M} \to \mathcal{M}$ defined by (31) is a homeomorphism, and $\mathcal{M}$ is an absolute neighborhood retract.

Proof. First, we note that $\phi : \mathrm{M} \to \mathcal{M}$ is a surjection by definition. Injectivity of $\phi : \mathrm{M} \to \mathcal{M}$ follows from the expression for the Frechet derivative of ϕ taken at an arbitrary $c \in \mathrm{M}$:

$$\phi'(c) = 2\pi \int\limits_0^\infty \widehat{\langle\omega\rangle}(r) J_0(\cdot\, r)\Big(J_1(cr) + \frac{cr}{2}(J_0(cr) - J_2(cr))\Big)dr$$

and the condition (32). We also note that the set M is an absolute neighborhood retract, since it is a compact convex subset of R. Thus, by properties of homeomorphism, $\mathcal{M} = \phi(\mathrm{M})$ is an absolute neighborhood retract, too. □

We define $\mathcal{F}$ to be the restriction of F_0 on $\mathcal{M}\bigcap\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}$, i.e.

$$\mathcal{F} = F_0|_{\mathcal{M}\bigcap\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)}},$$
$$\mathcal{F} : \mathcal{M}\bigcap\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)} \to \mathcal{M}.$$

Due to its definition, the mapping $\mathcal{F} : \mathcal{M}\bigcap\overline{B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)} \to \mathcal{M}$ is compact and admissible. We use the properties of the topological fixed point index and get

$$\mathrm{ind}(F_0, B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)) = \mathrm{ind}(\mathcal{F}, \mathcal{M}\bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)).$$

Next, we apply Lemma 2.3 and obtain

$$\mathrm{ind}(\mathcal{F}, \mathcal{M}\bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon)) = \mathrm{ind}(\phi^{-1}\circ\mathcal{F}\circ\phi, \phi^{-1}(\mathcal{F}(\mathcal{M}\bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))).$$

Lemma 4.3.3. Let the condition (28) be satisfied. Then there exists such $\delta > 0$ that the operator $\Psi = \phi^{-1}\circ\mathcal{F}\circ\phi$ maps $\overline{B_R(a,\delta)}$ to M.

Proof. Let

$$u(x) = 2\pi c \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|x|r) J_1(cr)dr, \ c \in \mathrm{M}.$$

Using the mean value theorem and the properties of the Bessel function J_1, we estimate

$$\|u - U\|_{C^1(\Omega,R)} \leq$$
$$2\pi\Big\| c\int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|\cdot|r) J_1(cr)dr - a\int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|\cdot|r) J_1(ar)dr\Big\|_{C(\Omega,R)} +$$
$$2\pi\Big\| - c\int_0^\infty \widehat{\langle\omega\rangle}(r) r J_1(|\cdot|r) J_1(cr)dr + a\int_0^\infty \widehat{\langle\omega\rangle}(r) r J_1(|\cdot|r) J_1(ar)dr\Big\|_{C(\Omega,R)} \leq$$

$$2\pi\Big\| \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|\cdot|r)(c\frac{r}{2}(J_0(\xi r) - J_2(\xi r)) + aJ_1(ar))dr(a-c)\Big\|_{C(\Omega,R)} +$$
$$2\pi\Big\| \int_0^\infty \widehat{\langle\omega\rangle}(r) r J_1(|\cdot|r)(c\frac{r}{2}(J_0(\xi r) - J_2(\xi r)) + aJ_1(ar))dr(a-c)\Big\|_{C(\Omega,R)},$$

where $\xi \in B_R(a, |a-c|)$. The condition (28) implies that

$$\|u - U\|_{C^1(\Omega,R)} \leq \mathfrak{N}|c-a| < \varepsilon$$

for some $\mathfrak{N} \in R$ and all $c \in \overline{B_R(a,\delta)}$, where $\delta < \varepsilon/\mathfrak{N}$. From the latter estimate we conclude that

$$\overline{B_R(a,\delta)} \subset \phi^{-1}(\mathcal{M} \bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))$$

which, in turn, implies

$$\mathcal{M}_\delta \subset \mathcal{F}(\mathcal{M} \bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))$$

$$\mathcal{M}_\delta = \Big\{ v = 2\pi c \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(|\cdot|r) J_1(cr) dr, \ c \in \overline{B_R(a,\delta)} \Big\}.$$

Thus, we finally get

$$\phi^{-1}(\mathcal{M}_\delta) = \overline{B_R(a,\delta)} \subset \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))),$$

which concludes the proof. □

Remark 4.3.2. The condition (28) is redundant for the the proof of the statement in Lemma 4.3.3. However, the condition it can be relaxed to is more cumbersome and harder to check.

It is easy to see that a is a fixed point of the operator $\Psi : \overline{B_R(a,\delta)} \to$ M. Moreover, a is an isolated fixed point of Ψ due to the fact that U is an isolated fixed point of $\mathcal{F}$ and topological invariance property of the index. The topological index of a finite dimensional map can be calculated as

$$\operatorname{ind}(\Psi, \phi^{-1}(\mathcal{F}(\mathcal{M} \bigcap B_{C^1_{rs}(\Omega,R)}(U,\varepsilon))) = \operatorname{sgn}(1 - \Psi'(a)).$$

The definition of the operator $\Psi = \phi^{-1} \circ \mathcal{F} \circ \phi$ implies that

$$2\pi c \int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(\Psi(c) r) J_1(cr) dr = \theta \ \text{ for all } \ c \in \overline{B_R(a,\delta)}.$$

Using the implicit function theorem and the chain rule for differentiation, we get

$$\int_0^\infty \widehat{\langle\omega\rangle}(r) J_0(ar) J_1(ar) + ar\Big(J'_{0a}(ar) J_1(ar) \Psi'(a) + J_0(ar) J'_{1a}(ar)\Big) dr = 0.$$

From the latter expression we obtain the following sufficient condition for $\Psi'(a) \neq 1$:

$$\int_0^\infty \widehat{\langle\omega\rangle}(r)J_0(ar)J_1(ar) + a\Big(J_0(ar)J_1(ar)\Big)'_a dr \neq 0. \tag{33}$$

Thus, $\deg(I - F_0, B_{C^1_{rs}(\Omega,R)}(U,\varepsilon), 0) \neq 0$ provided that the inequality (33) is fulfilled.

Summarizing the results above and using Theorem 3.2 and Theorem 3.1, we get the main result of the subsection.

Theorem 4.3.1. Let the family of functions $f_\beta : R \to [0,1]$ $(\beta \in [0,\infty))$ satisfy assumptions (**A3**) and (**A4**). Let also the connectivity kernel ω be given by (26), where the function $\sigma \in C(\mathcal{Y},(0,\infty))$ is $\mathcal{Y}$-periodic and the function $\chi \in C^2(R^2,R)\bigcap L(R,\mu,R)$ is radially symmetric. Finally, let the conditions (28), (30), (32), and (33) be fulfilled. Then, for any sufficiently large Ω, $\Omega \subset R$, and for each $\beta \in (0,\infty)$, there exists solution $u_\beta \in C^1_{rs}(\Omega,R))$ to (25) $(\Xi = \Omega)$. Moreover, $\|u_\beta - U\|_{C^1(\Omega,R))} \to 0$, as $\beta \to 0$, where $U \in C^1_{rs}(R^2,R))$ is the stationary bump solution to (25) $(\Xi = R^2,\ \beta = 0)$, defined by (27).

5. Conclusions and outlook

Using the methods of functional analysis and topological degree theory, we proved theorems on existence and continuous dependence of the stationary solutions to nonlinear operator equation with the operator of the Hammerstein type on the steepness of the Hammerstein nonlinearity. We applied the theorems obtained to the m-dimensional homogenized Amari neural field model (4) and proved theorems on existence and continuous dependence of its stationary solutions under the transition from continuous firing rate functions to the discontinuous Heaviside limit. These results serve as a justification of the transition from the heterogeneous model (3) to the homogenized model (4) in the case of the Heaviside firing rate function. We investigated the following three types of stationary solutions to (4): symmetric single bump solution in 1-D, symmetric double bump solution in 1-D, and radially symmetric single bump solution in 2-D in the respect of their existence and dependence on the firing rate steepness.

The present research can be considered as an extension to m-dimensional homogenized neural field models of the results of the paper by Oleynik et al [21]. This extension was achieved by generalization of the model keeping the methods of proofs similar to the ones used in [21]. The main distinction in the proofs foundations is the choice of the basic spaces: we employ the spaces of continuous functions on compact domains instead of the spaces of integrable functions on R used by Oleynik et al. Our choice of the basic spaces was conditioned by the possibility to facilitate and shorten the proofs required and to obtain at the same time the results of [21] concerning single bump solutions as a special case of our theorems.

The models of mathematical biology, in particular, the models arising in genetics, incorporate approximation of the rapid switching between two states of the model elements. This approximation is often modeled by means of Heaviside function. Extension of the methods suggested in Section 3 to other problems of mathematical biology can be considered as a further development of the present study.

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PAPER III

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On Wellposedness of Generalized Neural Field Equations with Delay

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Abstract. We obtain conditions for existence of unique global or maximally extended solutions to generalized neural field equations. We also study continuous dependence of these solutions on the spatiotemporal integration kernel, delay effects, firing rate and prehistory functions.

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1 Introduction

Firing rate models are used in the investigation of the properties of strongly interconnected cortical networks. In neural field models the cortical tissue has in addition been modeled as continuous lines or sheets of neurons. In such models the spatiotemporally varying neural activity is described by a single or several scalar fields, one for each neuron type incorporated in the model. These models are formulated in terms of differential, integro-differential equations and integral equations. The most well-known and simplest model in that respect is the Amari model (see e.g. [2])

$$u_t(t,x) = -u(t,x) + \int\limits_R \omega(x-y) f(u(t,y))dy + I(t,x) + h, \qquad t \geq 0, x \in R. \tag{1.1}$$

Here the function $u(t,x)$ denotes the activity of a neural element at time t and position x. The connectivity function (spatial convolution kernel) $\omega(x)$ determines the coupling between the elements and the non-negative function $f(u)$ gives the firing rate of a neuron with activity u. Neurons at a position x and time t are said to be active if $f(u(t,x)) > 0$. The function $I(t,x)$ and the parameter h represent a variable and a constant external inputs, respectively.

The literature on the Amari model (1.1) and its extensions is vast. The key issues in most of the published papers on these models are existence and stability of coherent structures like localized stationary solutions (so-called *bumps*) and traveling fronts/pulses, pattern formation as the outcome of a Turing type of instability and issues like wellposedness of the actual models. See e.g. the reviews [12], [9] and [8] (and the references therein) for more details.

For example, Blomquist *et al* [7] investigated the existence and stability of bumps within the framework of the two population neural field model

$$Au_t(t,x) = -u(t,x) + \int\limits_R \mathbb{W}(x-y) f(u(t,y))dy, \quad t \geq 0, x \in R, \tag{1.2}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \mathbb{W}(x) = \begin{pmatrix} \omega_{ee}(x) & -\omega_{ei}(x) \\ \omega_{ie}(x) & -\omega_{ii}(x) \end{pmatrix},$$

$$u(t,x) = \begin{pmatrix} u_e(t,x) \\ u_i(t,x) \end{pmatrix}, \quad f(u(t,x)) = \begin{pmatrix} f_e(u_e(t,x)) \\ f_i(u_i(t,x)) \end{pmatrix}.$$

Here $u_e(x,t)$ and $u_i(x,t)$ represent the activity of an excitatory and an inhibitory neural elements, respectively, and $\alpha \in R$ represents the inhibitory time constant (measured relative to the excitatory time constant). The model (1.2) was used as a starting point in [26] for the study of pattern formation through a Turing type of instability.

Faye *et al* [13] established conditions for the existence and uniqueness of solutions to the neural field model with delay

$$u_t(t,x) = -Lu(t,x) + \int\limits_\Omega \omega(t,x,y) f(u(t-\tau(x,y),y))dy + I(t,x), \qquad t \in [a,\infty), \; x \in \Omega \subset R^m \tag{1.3}$$

in the space of square integrable functions. Here $L = diag(l_1,\dots,l_n)$, $l_i > 0$ characterize the dynamics of the i-th population, and non-negative function $\tau(x,y)$ denotes the time it takes for the signal at the position x to reach the position y.

Although the modeling framework given by (1.1) and its extensions (including (1.2) and (1.3)) qualitatively are expected to capture the essential features of the brain activity on the macroscale level, they do not account for the heterogeneity in the cortical structure. Thus they represent a simplification of the actual situation. Hence it is a pressing need to develop mathematical tools for the study of waves and stationary activity patterns in heterogeneous media that can be used in brain modeling. One common tool which could be useful in the study of such problems is homogenization techniques (see e.g. [27]). Coombes *et al* [10], Svanstedt *et al* [19] and Malyutina *et al* [16] have taken a step in that direction by considering the parameterized Amari model

$$\begin{gathered} u_t^{\varepsilon}(t,x) = -u^{\varepsilon}(t,x) + \int\limits_{R} \omega^{\varepsilon}(x-y) f(u^{\varepsilon}(t,y))dy, \\ t \geq 0, x \in R, \end{gathered} \tag{1.4}$$

where the connectivity kernel $\omega^{\varepsilon}(x) = \omega(x, x/\varepsilon)$ by assumption is periodic in the second variable. It is proved (see e.g. [19]) by means of Nguetsengs multiscale convergence technique [17, 15, 25] that, as $\varepsilon \to 0$, the solution of the model (1.4) converges to the solution of the homogenized Amari model

$$\begin{gathered} u_t(t,x_c,x_f) = -u_0(t,x_c,x_f) + \int\limits_{R^m} \int\limits_{\mathcal{Y}} \omega(x_c - y_c, x_f - y_f) f(u(t,y_c,y_f)) dy_c dy_f, \\ t > 0,\ x_c \in R^m,\ x_f \in \mathcal{Y} \subset R^m. \end{gathered} \tag{1.5}$$

Here x_c and x_f are coarse-scale and fine-scale spatial variables, respectively, ω is a connectivity kernel which by assumption is periodic in the second variable.

The Volterra formulation

$$\begin{gathered} u(t,x) = \int\limits_{-\infty}^{t} \eta(t-s) \int\limits_{R} \omega(x-y) f(u(s-|x-y|/v, y)) dy ds, \\ t \in R,\ x \in R. \end{gathered} \tag{1.6}$$

has been investigated by Venkov *et al* [24] in the study of axonal delay effects on Turing–Hopf instabilities and pattern formation. Here the memory function (temporal convolution kernel) $\eta(t)$ with $\eta(t) \equiv 0$ for $t < 0$ represents synaptic processing of signals within the network, and the delayed temporal argument to u in the spatial integral represents the axonal delay effect arising from the finite speed (denoted here by v) of signal propagation between points x and y. Notice that we obtain (1.1) with $h = I(x,t) \equiv 0$ from (1.6) by means of the differentiation when $\eta(t) = \exp(-t), t \geq 0$ and $v \to \infty$.

A common feature of the aforementioned nonlocal neural field models is the dependence on several biologically important parameters. Within the mathematical neuroscience community, the structural stability aspect of the models under investigation is often tacitly assumed to hold true, even though no rigorous mathematical justification is given for this assumption. Thus, it is of interest to study the impact of these parameters on the wellposedness issue of these models i.e. existence, uniqueness and continuous dependence on input data. The study of continuous dependence on parameters in the solutions is indeed related to the property of structural stability of such complex systems, which is of fundamental importance in systems biology.

The question which then naturally arises is how to deal with this problem for nonlocal systems like (1.1) – (1.6). The continuous dependence of solutions to various classes of operator equations on parameters is studied in many papers (see e.g. the review [22], as well as very important results obtained for functional differential equations [4], pp. 29-40 and the references therein). These results cannot be applied directly to study of continuous dependence/structural stability of nonlocal neural field models, however.

Two basic approaches to the analysis of continuous dependence on a parameter have emerged. In the framework of the first approach it is assumed that for some value of the parameter, which is often referred to as the limit value, the modeling equation has a solution. The aim is to find conditions which guarantee that for the values of the parameter, which are sufficiently close (in a certain sense) to the limit value, the equation is also solvable and this solution is a continuous function of the parameter. This approach was implemented e.g. in [1] and [3]. In the second approach solvability for the limit value of the parameter is to be proved first, so the conditions needed are usually more strict in this case (see e.g. [28]).

The literature on Volterra equations is vast. A detailed review of the integral Volterra equations theory is given by Tsalyuk [21]. The theory of both abstract and integral Volterra operators and many useful references are presented in Corduneanu [11]. The general formulations of Volterra property for abstract operators was introduced in Zhukovskiy [28].

This serves as a background and motivation for the present study: Our aim is to establish conditions for existence and uniqueness of solutions to a nonlinear integral equation which generalizes all the models listed above as well as the continuous dependence of the solutions on parameters by using the ideas developed by Zhukovskiy [28]. We do this by studying the following equation:

$$u(t,x) = \int_{-\infty}^{t} \int_{\Omega} W(t,s,x,y) f(u(s-\tau(s,x,y),y))\,dy\,ds, \qquad t \in R,\ x \in \Omega \subseteq R^m \tag{1.7}$$

and its important truncated special case

$$u(t,x) = \int_{a}^{t} \int_{\Omega} W(t,s,x,y) f(u(s-\tau(s,x,y),y))\,dy\,ds, \qquad t \in [a,\infty), x \in \Omega; \quad u(\xi,x) \equiv 0,\ \xi \le a, x \in \Omega. \tag{1.8}$$

We do not consider external inputs $I(t,x)$ and h (unlike [2], [13]) in our models, as they do not involve any nonlinearities and, hence, only make statements and proofs more cumbersome. We stress, however, that all the results below remain valid in the presence of the external inputs as well.

Note that we get (1.2) from (1.8) by taking

$$W(t,s,x,y) = \eta(t,s)\omega(x-y)$$

with

$$\eta(t,s) = diag\Big(\exp(-(t-s)), \alpha\exp(-\alpha(t-s))\Big) \text{ and } \tau(t,x,y) \equiv 0.$$

If we neglect $I(t,x)$ in (1.3), we can obtain (1.3) from (1.8) with

$$W(t,s,x,y) = \eta(t,s)\omega(t,x,y),$$

$$\eta(t,s) = diag\Big(l_1\exp(-l_1(t-s)),\dots,l_n\exp(-l_n(t-s))\Big),\ \tau(t,x,y) = \tau(x,y).$$

Taking $\Omega = R^m \times \mathcal{Y}$ ($\mathcal{Y}$ is some m-dimensional torus [25]),

$$x = (x_c, x_f),\ y = (y_c, y_f),$$

$$W(t,s,x,y) = \exp\big(-(t-s)\big)\omega(x_c - y_c, x_f - y_f)$$

in (1.8) with

$$\tau(t,x,y) \equiv 0,$$

we get the model (1.5). Finally, with

$$W(t,s,x,y) = \eta(t-s)\omega(x-y)$$

and

$$\tau(t,x,y) = |x-y|/v$$

in (1.7), we obtain (1.6), which covers, in turn, the model (1.1) without the external inputs.

Our results generalize the results obtained by Potthast *et al* [18] and Faye *et al* [13] concerning existence of a unique solution to the Amari model (1.1) in the Banach space of continuous bounded functions and to the model (1.3) in the space of square integrable functions on a bounded domain, respectively. Here we also study dependence of solutions on the parameters.

The paper is organized in the following way. Section 2 is devoted to the study of local solvability, extendability and continuous dependence of solutions to operator Volterra equations on parameters. Building on these general results we investigate the models (1.7) and (1.8) in Section 3. Section 4 contains conclusions and an outlook.

We stress that one of the challenging parts of out study is application of the general theory of Volterra operators to the integral equations (1.7) and (1.8), which are defined on unbounded spatial and temporal domains. This general setting requires some conditions which are difficult to verify (see main theorems in Section 3). In two special cases, which are highly relevant for the neural field theory, we can however relax these conditions. The analogues of the main theorems for these special cases are formulated as remarks in Section 3 and their proofs are given in Appendix.

2 Existence, uniqueness and continuous dependence of solutions on parameters: the case of Volterra operator equations

Let us introduce the following notation:

R^n is the space of vectors consisting of n real components with the norm $|\cdot|$;

Ω is some closed subset of R^m;

$\mathcal{B}$ is some Banach space with the norm $||\cdot||_{\mathcal{B}}$;

$Y([a,b],\mathcal{B})$ is a Banach space of functions $y:[a,b]\to\mathcal{B}$ with the norm $||\cdot||_Y$;

$\mathfrak{B}(\Omega,R^n)$ is some Banach space of functions $v:\Omega\to R^n$ with the norm $||\cdot||_{\mathfrak{B}(\Omega,R^n)}$;

Λ is some metric space;

μ is the Lebesgue measure;

$L^p(\Omega,\mu,R^n)$ is the space of all measurable and integrable with p-th degree functions $\chi:\Omega\to R^n$ with the norm $||\chi||_{L^p([\Omega],\mu,R^n)} = \Big(\int\limits_{\Omega}|\chi(s)|^p ds\Big)^{1/p}$, $1\le p<\infty$;

$BC(\Omega,R^n)$ is the space of all continuous bounded functions $\vartheta:\Omega\to R^n$ with the norm $||\vartheta||_{BC(\Omega,R^n)} = \sup\limits_{x\in\Omega}|\vartheta(x)|$;

$C_0(\Omega,R^n)$ is the space of all continuous functions $\hat{\vartheta}:\Omega\to R^n$ satisfying the additional condition $\lim\limits_{|x|\to\infty}|\hat{\vartheta}(x)|=0$ in the case if Ω is unbounded, with the norm $\|\hat{\vartheta}\|_{C_0(\Omega,R^n)}=\max\limits_{x\in\Omega}|\hat{\vartheta}(x)|$;

$C([a,b],\mathfrak{B}(\Omega,R^n))$ is the space of all continuous functions $v:[a,b]\to\mathfrak{B}(\Omega,R^n)$, with the norm $\|v\|_{C([a,b],\mathfrak{B}(\Omega,R^n))}=\max\limits_{t\in[a,b]}\|v(t)\|_{\mathfrak{B}(\Omega,R^n)}$.

$C((-\infty,b],\mathfrak{B}(\Omega,R^n))$ is the space of all continuous functions $\hat{v}:(-\infty,b]\to\mathfrak{B}(\Omega,R^n)$ such that $\lim\limits_{t\to-\infty}\|\hat{v}(t)\|_{\mathfrak{B}(\Omega,R^n)}=0$, with the norm $\|\hat{v}\|_{C((-\infty,b],\mathfrak{B}(\Omega,R^n))}=\max\limits_{t\in(-\infty,b]}\|\hat{v}(t)\|_{\mathfrak{B}(\Omega,R^n)}$.

In the notation for functional spaces we will not indicate the definition domains and the image sets of functions, provided that this leads to no ambiguity.

Definition 2.1. An operator $\Psi:Y\to Y$ is said to be a *Volterra operator* (in the sense of A.N. Tikhonov [20]) if for any $\xi\in(0,b-a)$ and any $y_1,y_2\in Y$ the fact that $y_1(t)=y_2(t)$ on $[a,a+\xi]$ implies that $(\Psi y_1)(t)=(\Psi y_2)(t)$ on $[a,a+\xi]$.

In what follows we assume that in the space Y the following condition is fulfilled:

$\mathcal{V}$-condition [28]: For arbitrary $y\in Y$, $\{y_i\}\subset Y$ such that $\|y_i-y\|_Y\to 0$ and for any $\xi\in(0,b-a)$ if $y_i(t)=0$ on $[a,a+\xi]$, then $y(t,x)=0$ on $[a,a+\xi]$.

For any $\xi\in(0,b-a)$ let $Y_\xi=Y([a,a+\xi],\mathcal{B})$ denote the linear space of restrictions y_ξ of functions $y\in Y$ to $[a,a+\xi]$ which implies that for each $y_\xi\in Y_\xi$ there exists at least one extension $y\in Y$ of the function y_ξ. Then we can define the norm of Y_ξ by $\|y_\xi\|_{Y_\xi}=\inf\|y\|_Y$, where the infimum is taken over all extensions $y\in Y$ of the function y_ξ. Hence, the space Y_ξ becomes a Banach space.

For an arbitrary $\xi\in(0,b-a)$ let an operator $P_\xi:Y\to Y$ takes each $y_\xi\in Y_\xi$ to some extension $y\in Y$ of y_ξ. Moreover, we define the operators $E_\xi:Y\to Y_\xi$ by $(E_\xi y)(t)=y(t)$, $t\in[a,a+\xi]$ and $\Psi_\xi:Y_\xi\to Y_\xi$ by $\Psi_\xi y_\xi=E_\xi\Psi P_\xi y_\xi$, respectively. Note that for any Volterra operator $\Psi:Y\to Y$ the operator $\Psi_\xi:Y_\xi\to Y_\xi$ is also a Volterra operator and it is independent of the way $y=P_\xi y_\xi$ extends y_ξ.

Definition 2.2. A Volterra operator $\Psi:Y\to Y$ is called *locally contracting* if there exists $q<1$ such that for any $r>0$ one can find $\delta>0$ such that the following two conditions are satisfied for all $y_1,y_2\in Y$, such that $\|y_1\|_Y\le r$, $\|y_2\|_Y\le r$:

$\mathfrak{q}_1$) $$\|E_\delta\Psi y_1-E_\delta\Psi y_2\|_{Y_\delta}\le q\|E_\delta y_1-E_\delta y_2\|_{Y_\delta},$$

$\mathfrak{q}_2$) for any $\gamma\in(0,b-a-\delta]$, the condition $E_\gamma y_1=E_\gamma y_2$ implies that
$$\|E_{\gamma+\delta}\Psi y_1-E_{\gamma+\delta}\Psi y_2\|_{Y_{\gamma+\delta}}\le q\|E_{\gamma+\delta}y_1-E_{\gamma+\delta}y_1\|_{Y_{\gamma+\delta}}.$$

The class of locally contracting operators is rather wide. It includes not only contracting operators, but also, e.g. τ-Volterra operators.

Definition 2.3. An operator $\Psi:Y\to Y$ is called *τ-Volterra* if for any $y_1,y_2\in Y$ the condition $(\Psi y_1)(t)=(\Psi y_2)(t)$ holds true on $[a,a+\tau]$ and for any $\xi\in[0,b-a-\tau]$, if $y_1(t)=y_2(t)$ on $[a,a+\xi]$, then $(\Psi y_1)(t)=(\Psi y_2)(t)$ on $[a,a+\xi+\tau]$.

Notice that τ-Volterra operators satisfy conditions $\mathfrak{q}_1$) and $\mathfrak{q}_2$) with $q = 0$ and $\delta = \tau$, which are independent of a choice of r.

Let us now consider the equation

$$y(t) = (\Psi y)(t),\ t \in [a,b], \tag{2.1}$$

where $\Psi : Y \to Y$ is a Volterra operator.

Definition 2.4. We define a *local solution* to Eq. (2.1) on $[a,a+\gamma]$, $\gamma \in (0,b-a)$ to be a function $y_\gamma \in Y_\gamma$ that satisfies the equation $\Psi_\gamma y_\gamma = y_\gamma$ on $[a,a+\gamma]$. We define a *maximally extended solution* to Eq. (2.1) on $[a,a+\zeta)$, $\zeta \in (0,b-a]$ to be a function $y_\zeta : [a,a+\zeta) \to \mathcal{B}$, whose restriction y_γ to $[a,a+\gamma]$ is a local solution of Eq. (2.1) for any $\gamma < \zeta$ and $\lim\limits_{\gamma\to\zeta-0} \|y_\gamma\|_{Y_\gamma} = \infty$. We define a *global solution* to Eq. (2.1) to be a function $y \in Y$ that satisfies this equation on the entire interval $[a,b]$.

Let us now consider the equation

$$y(t) = (F(y,\lambda))(t),\ t \in [a,b] \tag{2.2}$$

with a parameter $\lambda \in \Lambda$, where for each $\lambda \in \Lambda$ a Volterra operator $F(\cdot,\lambda) : Y \to Y$ satisfies the property: $F(\cdot,\lambda_0) = \Psi$ for some $\lambda_0 \in \Lambda$. Our aim is to formulate conditions for existence and uniqueness of solutions to Eq. (2.2) on a certain fixed set $[a,a+\xi] \subset [a,b]$ (We, naturally, also apply Definition 4 to Eq. (2.2) at each fixed $\lambda \in \Lambda$); and convergence of these solutions to solution to Eq. (2.1) in the norm of Y_ξ as λ approaches λ_0. This means, that the problem (2.2) is wellposed.

Definition 2.5. For any $\lambda \in \Lambda_0 \subseteq \Lambda$, let the Volterra operator $F(\cdot,\lambda) : Y \to Y$ be given. This family of operators is called *uniformly locally contracting* if there exist $q \geq 0$ and $\delta > 0$, such that for each $\lambda \in \Lambda_0 \subseteq \Lambda$ the operator $F(\cdot,\lambda) : Y \to Y$ is locally contracting with the constants q and δ.

The following theorem represents our main tool to study of the wellposedness of the models (1.7) and (1.8). Minding future applications, we formulate this theorem here in a more general form than it is needed for the classical neural field theory.

Theorem 2.1. *Assume that the following two conditions are satisfied:*

1) *There is a neighborhood U_0 of λ_0 where the operators $F(\cdot,\lambda) : Y \to Y$, $\lambda \in U_0$ are uniformly locally contracting;*

2) *For arbitrary $y \in Y$, the mapping $F : Y \times \Lambda \to Y$ is continuous at (y,λ_0).*

Then for each $\lambda \in U_0$, Eq. (2.2) *has a unique global or maximally extended solution, and each local solution is a restriction of this solution.*

If Eq. (2.2) *has a global solution y_0 at $\lambda = \lambda_0$, then for each λ (sufficiently close to λ_0) it also has a global solution $y = y(\lambda)$, and $\|y(\lambda) - y_0\|_Y \to 0$ as $\lambda \to \lambda_0$.*

If Eq. (2.2) *has a maximally extended solution $y_{0\zeta}$ defined on $[a,a+\zeta)$ at $\lambda = \lambda_0$, then for any $\gamma \in (0,\zeta)$ one can find a neighborhood of λ_0 such that for any λ in this neighborhood Eq.* (2.2) *has a local solution $y_\gamma = y_\gamma(\lambda)$ defined on $[a,a+\gamma]$ and $\|y_\gamma(\lambda) - y_{0\gamma}\|_{Y_\gamma} \to 0$ as $\lambda \to \lambda_0$.*

Proof Choose a fixed $\lambda \in U_0$. Let $r > 0$, $\xi \in (0,b-a)$, $y_\xi \in Y_\xi$, $\widehat{y} \in Y$. Let $B_Y(\widehat{y},r)$ denote the set of functions $y \in Y$ such that $\|y - \widehat{y}\|_Y < r$ and $Y([a,b],\mathcal{B},y_\xi)$ denote the set of functions $y \in Y$ such that $E_\xi y = y_\xi$. Put $B_{Y([a,b],y_\xi)}(\widehat{y},r) = B_Y(\widehat{y},r) \bigcap Y([a,b],\mathcal{B},y_\xi)$.

We construct the solution in the following way. We set $r_1 = (1-q)^{-1}\|F(0,\lambda)\|_Y + 1$ and find all $\delta > 0$ that satisfy the condition 1) with $r = r_1$. For $\delta_1 = \frac{1}{2}\sup\{\delta\}$, we have

$$\|E_{\delta_1}F(y,\lambda) - E_{\delta_1}F(u,\lambda)\|_{Y_{\delta_1}} \leq q\|E_{\delta_1}y - E_{\delta_1}u\|_{Y_{\delta_1}}$$

at any $y,u \in B_Y(0,r_1)$. Then $F((B_Y(0,r)),\lambda) \subset B_Y(0,r)$ for $B_Y(0,r)$ with $r \geq r_1$. By the Banach fixed point theorem ([14], p. 43) the mapping $F_{\delta_1}(\cdot,\lambda)$ has a fixed point y_{δ_1} in the ball $B_{Y_{\delta_1}}(0,r_1)$. This fixed point is a local solution to Eq. (2.2). Using the Banach theorem, one can also prove that for arbitrary $\vartheta_1 \in (0,\delta_1)$ and any local solution $\widetilde{y}_{\vartheta_1}$ to Eq. (2.2) defined on $[a,a+\vartheta_1]$ it holds that $\widetilde{y}_{\vartheta_1}(t) = y_{\delta_1}(t)$ at all $t \in [a,a+\vartheta_1]$.

Choose $r_2 = (1-q)^{-1}\|F(P_{\delta_1}y_{\delta_1},\lambda)\|_Y + 1$ and find all possible $\delta > 0$ that satisfy the condition 1) with $r = r_2$. For $\delta_2 = \frac{1}{2}\sup\{\delta\}$ at any $y,u \in B_{Y([a,b],y_{\delta_1})}(P_{\delta_1}y_{\delta_1},r_2)$ we have

$$\|E_{\delta_1+\delta_2}F(y,\lambda)P_{\delta_1}y_{\delta_1} - E_{\delta_1+\delta_2}F(u,\lambda)\|_{Y_{\delta_1+\delta_2}} \leq q\|E_{\delta_1+\delta_2}y - E_{\delta_1+\delta_2}u\|_{Y_{\delta_1+\delta_2}}.$$

According to the Banach theorem there exists a fixed point $y_{\delta_1+\delta_2}$ of the mapping $F_{\delta_1+\delta_2}(\cdot,\lambda)$ in $B_{Y([a,a+\delta_1+\delta_2],y_{\delta_1})}(E_{\delta_1+\delta_2}P_{\delta_1}y_{\delta_1},r_2)$. This fixed point is a local solution to Eq. (2.2) defined on $[a,a+\delta_1+\delta_2]$. It is an extension of the local solution y_{δ_1}. For any $\vartheta_2 \in (0,\delta_2)$ and any local solution $\widetilde{y}_{\delta_1+\vartheta_2}$ to Eq. (2.2) defined on $[a,a+\delta_1+\vartheta_2]$, it holds that $\widetilde{y}_{\delta_1+\vartheta_2}(t) = y_{\delta_1+\delta_2}(t)$ for all $t \in [a,a+\delta_1+\vartheta_2]$. Next, let us choose $r_3 = (1-q)^{-1}\|F(P_{\delta_1+\delta_2}y_{\delta_1+\delta_2},\lambda)\|_Y + 1$, find all possible $\delta > 0$ that satisfy the condition 1) with $r = r_3$ and repeat the procedure, etc.

If the norms of the obtained local solutions are uniformly bounded by some $\mathfrak{M} \in R$, then for $r = \mathfrak{M} + 1$ due to the local contractivity of the operator $F(\cdot,\lambda) : Y \to Y$ we find δ such that $\delta_i \geq \frac{\delta}{2}$ at each of the steps described above. Therefore, in a finite number of steps we will obtain a unique global solution to Eq. (2.2). But if such $\mathfrak{M}$ does not exist, then the number of steps becomes infinite. As a result, we obtain a unique maximally extended solution to Eq. (2.2).

We now prove the continuous dependence of solutions on a parameter λ. Consider the case when, Eq. (2.2) has global solution $y_0 = y(\lambda_0) \in Y$ at $\lambda = \lambda_0$. Let us find $\delta > 0$ satisfying the condition 1) at $r = \|y_0\|_Y + 1$, and any $\lambda \in U_0$. For $k = [\frac{b-a}{\delta}] + 1$ denote $\Delta_l = l\delta$, $l = 1,2,\ldots,k$. Since the condition 2) holds true, for any $\varepsilon > 0$ one can find $\sigma_1 > 0$ and a neighborhood U_1 such that for each $\lambda \in U_1$ we have

$$\|F(u,\lambda) - F(y,\lambda_0)\|_Y < \frac{(1-q)\varepsilon}{6}$$

for all $u \in Y$ such that $\|u - y\|_Y < \sigma_1$. Assume that $\sigma_1 < \frac{(1-q)\varepsilon}{6}$. Let us find $\sigma_2 > 0$ and U_2 such that for arbitrary $\lambda \in U_2$ it holds that

$$\|F_{\Delta_{k-1}}(u_{\Delta_{k-1}},\lambda) - F_{\Delta_{k-1}}(y_{\Delta_{k-1}},\lambda_0)\|_{Y_{\Delta_{k-1}}} < \frac{(1-q)\sigma_1}{6}$$

for all $u_{\Delta_{k-1}} \in Y_{\Delta_{k-1}}$, $\|u_{\Delta_{k-1}} - y_{\Delta_{k-1}}\|_{Y\Delta_{k-1}} < \sigma_2$. Assume that $\sigma_2 < \frac{(1-q)\sigma_1}{6}$, $U_2 \subseteq U_1$. There exist $\sigma_3 > 0$ and U_3 such that for any $\lambda \in U_3$ it holds true that

$$\|F_{\Delta_{k-2}}(u_{\Delta_{k-2}},\lambda) - F_{\Delta_{k-2}}(y_{\Delta_{k-2}},\lambda_0)\|_{Y\Delta_{k-2}} < \frac{(1-q)\sigma_2}{6}$$

for any $u_{\Delta_{k-2}} \in Y_{\Delta_{k-2}}$, $\|u_{\Delta_{k-2}} - y_{\Delta_{k-2}}\|_{Y_{\Delta_{k-2}}} < \sigma_3$; $\sigma_3 < \frac{(1-q)\sigma_2}{6}$, $U_3 \subseteq U_2$ etc. We perform k iterations and at the last step find σ_k and U_k, $0 < \sigma_k < \frac{(1-q)\sigma_{k-1}}{6}$, $U_k \subseteq U_{k-1}$.

Let $y_{0\Delta_1}$ denote a local solution to Eq. (2.2) at $\lambda = \lambda_0$, that is a fixed point of the operator $F_{\Delta_1}(\cdot,\lambda_0) : Y_{\Delta_1} \to Y_{\Delta_1}$. If $\|u_{\Delta_1} - y_{0\Delta_1}\|_{Y_{\Delta_1}} < \sigma_k$, then

$$\|F_{\Delta_1}(u_{\Delta_1},\lambda) - F_{\Delta_1}(y_{0\Delta_1},\lambda_0)\|_{Y_{\Delta_1}} < \frac{(1-q)\sigma_{k-1}}{6}$$

for all $\lambda \in U_k$. Taking into account the condition 1), we get for any natural number m that

$$\|F^m_{\Delta_1}(y_{0\Delta_1},\lambda) - y_{0\Delta_1}\|_{Y_{\Delta_1}} \le \|F^m_{\Delta_1}(y_{0\Delta_1},\lambda) - F^{m-1}_{\Delta_1}(y_{0\Delta_1},\lambda)\|_{Y_{\Delta_1}} + \ldots$$

$$\ldots + \|F_{\Delta_1}(y_{0\Delta_1},\lambda) - y_{0\Delta_1}\|_{Y_{\Delta_1}} \le (q^{m-1} + \ldots + q + 1)\frac{(1-q)\sigma_{k-1}}{6} \le \frac{\sigma_{k-1}}{6}.$$

Due to the convergence of the approximations $F^m_{\Delta_1}(y_{0\Delta_1},\lambda)$ to the fixed point $y_{\Delta_1} = y_{\Delta_1}(\lambda)$ of the operator $F_{\Delta_1}(\cdot,\lambda) : Y_{\Delta_1} \to Y_{\Delta_1}$ we obtain that $\|y_{\Delta_1} - y_{0\Delta_1}\|_{Y_{\Delta_1}} \le \frac{\sigma_{k-1}}{6}$ for each $\lambda \in U_k$. Further, let $y_{0\Delta_2}$ be a local solution to Eq. (2.2) at $\lambda = \lambda_0$ defined on $[a,a+\Delta_2]\times R^n$. Then, for all $\lambda \in U_k \subseteq U_{k-1}$ and any $u_{\Delta_2} \in B_{Y([a,a+\Delta_2],y_{\Delta_1})}(y_{0\Delta_2},\sigma_{k-1})$ we get

$$\|F_{\Delta_2}(u_{\Delta_2},\lambda) - y_{0\Delta_2}\|_{Y_{\Delta_2}} = \|F_{\Delta_2}(u_{\Delta_2},\lambda) - F_{\Delta_2}(y_{0\Delta_2},\lambda_0)\|_{Y_{\Delta_2}} < \frac{(1-q)\sigma_{k-2}}{6}.$$

Then

$$\|F_{\Delta_2}(u_{\Delta_2},\lambda) - u_{\Delta_2}\|_{Y_{\Delta_2}} < \sigma_{k-1} + \frac{(1-q)\sigma_{k-2}}{6} < \frac{(1-q)\sigma_{k-2}}{3}.$$

For all $m = 1,2,\ldots$ we have

$$\|F^m_{\Delta_2}(u_{\Delta_2},\lambda) - u_{\Delta_2}\|_{Y_{\Delta_2}} \le \|F^m_{\Delta_2}(u_{\Delta_2},\lambda) - F^{m-1}_{\Delta_2}(u_{\Delta_2},\lambda)\|_{Y_{\Delta_2}} + \ldots$$

$$\ldots + \|F_{\Delta_2}(u_{\Delta_2},\lambda) - u_{\Delta_2}\|_{Y_{\Delta_2}} \le (q^{m-1} + \ldots + q + 1)\frac{(1-q)\sigma_{k-2}}{3} \le \frac{\sigma_{k-2}}{3}.$$

Taking into account the convergence of the approximations $F^m_{\Delta_2}(u_{\Delta_2},\lambda)$ to $y_{\Delta_2} = y_{\Delta_2}(\lambda)$ we obtain

$$\|y_{\Delta_2} - y_{0\Delta_2}\|_{Y_{\Delta_2}} \le \|y_{\Delta_2} - F^m_{\Delta_2}(u_{\Delta_2},\lambda)\|_{Y_{\Delta_2}} +$$

$$+ \|F^m_{\Delta_2}(u_{\Delta_2},\lambda) - u_{\Delta_2}\|_{Y_{\Delta_2}} + \|u_{\Delta_2} - y_{0\Delta_2}\|_{Y_{\Delta_2}} \le \frac{\sigma_{k-2}}{3} + \sigma_{k-1} \le \frac{\sigma_{k-2}}{2}.$$

Using the convergence of sequential approximations $F^m_{\Delta_3}(u_{\Delta_3},\lambda)$ to a fixed point $y_{\Delta_3} = y_{\Delta_3}(\lambda)$ of the operator $F_{\Delta_3}(\cdot,\lambda) : Y_{\Delta_3} \to Y_{\Delta_3}$ for any $u_{\Delta_3} \in B_{Y([a,a+\Delta_3],y_{\Delta_2})}(y_{0\Delta_3},\sigma_{k-2})$ and each $\lambda \in U_k \subseteq U_{k-1} \subseteq U_{k-2}$, we obtain the estimate $\|y_{\Delta_3} - y_{0\Delta_3}\|_{Y_{\Delta_3}} \le \frac{\sigma_{k-3}}{2}$. We, then, repeat this procedure. At the k-th step we prove in an analogous way that the inequality $\|y(\lambda) - y_0\|_Y < \varepsilon$ holds true for all $\lambda \in U_k$. Therefore, $\|y(\lambda) - y_0\|_Y \to 0$ as $\lambda \to \lambda_0$.

Let now a solution $y_{0\eta}$ to Eq. (2.2) at $\lambda = \lambda_0$ be maximally extended. Fix arbitrary $\gamma \in (0,\eta)$ and let $y_{0\gamma}$ denote the restriction of the solution $y_{0\eta}$ to $[a,a+\gamma]\times R^n$. For the equation $u_\gamma = F_\gamma(u_\gamma,\lambda_0)$ the function $y_{0\gamma} \in Y([a,a+\gamma]\times\Omega,R^n)$ is a global solution. As is shown above, for all λ from some neighborhood of λ_0 the equations $u_\gamma = F_\gamma(u_\gamma,\lambda)$ have global solutions $y_\gamma(\lambda)$, and $\|y_\gamma(\lambda) - y_{0\gamma}\|_{Y_\gamma} \to 0$ as $\lambda \to \lambda_0$. □

The proof of Theorem 1 has several corollaries which are summarized in the following remarks:

Remark 2.2. If the constant δ in the condition 1) of Theorem 2.1 is independent of r, then Eq. (2.2) has a global solution. This is the case e.g. for τ-Volterra operators.

Remark 2.3. In case of a priori boundedness of the solution, it is possible to extend the solution beyond the point b in the same way as it was done in the proof of Theorem 2.1. This will give a unique solution defined on $[a,\infty)$.

Notice that the existence of a maximally extended solution to Eq. (2.2) at $\lambda = \lambda_0$ does not guarantee the existence of maximally extended solutions to eq (2.2) at λ arbitrarily close to λ_0. The following example illustrates this fact.

Example 2.1. Let operators $\Phi(\cdot,\lambda) : L^1([0,\pi],\mu,R) \to L^1([0,\pi],\mu,R)$, $\lambda \in [0,\pi]$, be defined as

$$(\Phi(y,\lambda))(t) = \begin{cases} 0, & \text{if } t \in [0,\lambda); \\ \Big(\int\limits_0^{t-\lambda} y(s)ds\Big)^2 + 1, & \text{if } t \in [\lambda,\pi]. \end{cases}$$

These operators are Volterra operators and satisfy the condition 1) of Theorem 2.1: For $q = \frac{1}{2}$ and any $r > 0$ one can choose $\delta = \frac{1}{4r}$, and condition 1) becomes fulfilled for all $t \in [0,\pi)$ and any $\lambda \in [0,\pi]$). Condition 2) of the Theorem 2.1 is also fulfilled. The equation $y(t) = (\Phi(y,0))(t)$, $t \in [0,\pi]$ has a unique maximally extended solution $y(t) = \frac{1}{\cos^2 t}$ defined on $[0, \frac{\pi}{2})$. Now, since for any $\lambda \in (0,\pi]$ the operator $\Phi(\cdot,\lambda)$ is a τ-Volterra operator, the equation $y(t) = (\Phi(y,\lambda))(t)$, $t \in [0,\pi]$ is globally solvable for each $\lambda \in (0,\pi]$.

When analyzing Theorem 2.1, it is natural to ask the question whether the maximally extended solutions to Eq. (2.2) are defined on time intervals with arbitrarily small length. The following two remarks give answers to that question:

Remark 2.4. Let the assumptions of Theorem 2.1 be fulfilled and let there exist some neighborhood $\widehat{U} \subseteq U_0$ of λ_0 such that Eq. (2.2) has maximally extended solutions y_{ζ_λ} defined on $[a,a+\zeta_\lambda)$ for any $\lambda \in \widehat{U}$. Then $\inf\limits_{\forall\lambda\in\widehat{U}} \zeta_\lambda > 0$. Since for all $\lambda \in U_0$ operators $F(\cdot,\lambda)$ are uniformly locally contracting, we get $\inf\limits_{\forall\lambda\in\widehat{U}} \zeta_\lambda > 0$.

Remark 2.5. Let the assumptions of Theorem 2.1 be fulfilled and let for $\lambda = \lambda_0$ and some sequence $\lambda_i \subset U_0$ equation (2.2) have maximally extended solutions $y_{0\zeta}$ and y_{ζ_i} defined on $[a,a+\zeta)$ and $[a,a+\zeta_i)$, respectively. Then $\beta = \min\{\zeta, \inf\limits_{\forall i} \zeta_i\} > 0$, and either $\beta = \zeta$, or $\beta = \zeta_{i_0}$ at some i_0.

The positivity of β follows from Remark 3. Next, we choose arbitrary $\varepsilon > 0$ and a sequence $\gamma_j \in (0,\beta)$, $\gamma_j \to \beta$, $j \to \infty$. For each $\gamma_j \in (0,\beta)$ there exists a finite $\sup\limits_{\forall i} \|y_{i\gamma_j}\|_{Y_{\gamma_j}}$ otherwise $\beta = \gamma_j$. Let us associate the number γ_1 with the corresponding local solution $y_{i_1\gamma_1}$ to Eq. (2.2) at $\lambda = \lambda_{i_1}$, where i_1 is the least number such that $\max\{\|y_{0\gamma_1}\|_{Y_{\gamma_1}}, \sup\limits_{\forall i} \|y_{i\gamma_1}\|_{Y_{\gamma_1}}\} - \|y_{i_1\gamma_1}\|_{Y_{\gamma_1}} < \varepsilon$; we associate the number γ_2 with the corresponding local solution $y_{i_2\gamma_2}$ to Eq. (2.2) at $\lambda = \lambda_{i_2}$, where i_2 is the least number such that $\max\{\|y_{0\gamma_2}\|_{Y_{\gamma_2}}, \sup\limits_{\forall i} \|y_{i\gamma_2}\|_{Y[a,a+\gamma_2]}\} - \|y_{i_2\gamma_2}\|_{Y_{\gamma_2}} < \varepsilon$ etc. We obtain a subsequence $\{i_j\}$ of numbers of local solutions $y_{i\gamma_j}$ to Eq. (2.2) such that $\|y_{i_j\gamma_j}\|_{Y_{\gamma_j}} \to \infty$ as $j \to \infty$. If the subsequence $\{i_j\}$ is bounded, then one can find a number i_{j_0} such that $\lim\limits_{\gamma\to\beta-0} \|y_{i_{j_0}\gamma}\|_{Y_\gamma} = \infty$, i.e. $\zeta_{i_{j_0}} = \beta$. Otherwise, using the fact that $\|y_{i_j\gamma} - y_{0\gamma}\|_{Y_\gamma} \to 0$ as $j \to \infty$ for any $\gamma \in (0,\zeta)$ we obtain $\lim\limits_{\gamma\to\beta-0} \|y_{0\gamma}\|_{Y_\gamma} = \infty$, i.e. $\zeta = \beta$.

3 Existence, uniqueness and continuous dependence of solutions on parameters: the case of neural field equations

In this section we apply the results obtained in the previous section to a class of nonlinear integral equations, typical representatives of which can be found in the neural field theory. For the sake of convenience, we consider the following generalization of the model (1.8):

$$\begin{gathered} u(t,x) = \varphi(a,x) + \int_a^t \int_\Omega W(t,s,x,y) f(u(s-\tau(s,x,y),y))dyds, \\ t \in [a,\infty), x \in \Omega; \\ u(\xi,x) = \varphi(\xi,x),\ \xi \le a, x \in \Omega. \end{gathered} \tag{3.1}$$

under the following assumptions on the functions involved:

$(A1)$ For any $b > a$, $(t,x) \in [a,b]\times\Omega$, the function $W(t,\cdot,x,\cdot) : [a,b]\times\Omega \to R^n$ is measurable.

$(A2)$ For any $b > a$, at almost all $(s,y) \in [a,b]\times\Omega$, the function $W(\cdot,s,\cdot,y) : [a,b]\times\Omega \to R^n$ is uniformly continuous.

$(A3)$ For any $b > a$, $t \in [a,b]$, $\int_\Omega \sup_{x\in\Omega} \left|W(t,s,x,y)\right| dy \le G(s)$, where $G \in L^1([a,b],\mu,R^n)$.

$(A4)$ The function $f : R^n \to R^n$ is measurable and for any r > 0 one can find $\mathrm{f_r} > 0$, such that for all $u \in R^n$, $|u| \le r$, it holds true that $|f(u)| \le \mathrm{f_r}$.

$(A5)$ The delay function $\tau : R\times\Omega\times\Omega \to [0,\infty)$ is continuous on $R\times\Omega\times\Omega$.

$(A6)$ The prehistory function φ belongs to $C((-\infty,a],BC(\Omega,R^n))$.

The model (3.1) with $\varphi(\xi,x) \equiv 0$ can be obtained from (1.7) by taking $W(t,s,x,y) = \eta(t,s)\omega(x,y)$, where, e.g.

$$\eta(t,s) = \begin{cases} \kappa\exp(-\kappa(t-s)), & \text{if } t \ge a; \\ 0, & \text{if } t < a; \end{cases}$$

or

$$\eta(t,s) = \begin{cases} \kappa(t-s)\exp(-\kappa(t-s)), & \text{if } t \ge a; \\ 0, & \text{if } t < a \end{cases}$$

and ω can be represented by the "Mexican hat"

$$\omega(x,y) = M\exp(-m|x-y|) - K\exp(-k|x-y|)$$

or the "wizard hat"

$$\omega(x,y) = M(1-|x-y|)\exp(-m|x-y|),$$

and

$$f(u) = \begin{cases} u^\kappa/(\theta^\kappa + u^\kappa), & \text{if } u \ge 0; \\ 0, & \text{if } u < 0, \end{cases}$$

for some $\kappa > 0$, $\theta > 0$, $M > K > 0$, and $m > k > 0$. These functions satisfy the conditions $(A1)$ – $(A4)$. The condition $(A4)$ is also fulfilled e.g. for the sigmoidal functions

$$f(u) = \frac{1}{2}\Big(1 + \tanh\big(\kappa(u-\theta)\big)\Big)$$

or

$$f(u) = \frac{1}{1+\exp(-\kappa(u-\theta))}$$

with some positive κ and θ. We do not assume in $(A4)$ that function f is bounded (as in the classical neural field theory), because it allows us to obtain more general results which may have other applications. If we take the delay functions $\tau(t,x,y) = |x-y|/\upsilon$ for some positive velocity υ or $\tau(t,x,y) = d(x,y)$ with continuous function $d : R\times R \to [0,\infty)$ from [24] and [13], respectively, we find out that the condition $(A5)$ is also satisfied.

We introduce the definition of local, maximally extended and global solutions just as in the previous section (Definition 2.4).

Definition 3.1. We define a *local solution* to Eq. (3.1) on $[a,a+\gamma]\times R^n$, $\gamma \in (0,\infty)$ to be a function $u_\gamma \in C([a,a+\gamma],BC(\Omega,R^n))$ that satisfies the equation (3.1) on $[a,a+\gamma]\times R^n$. We define a *maximally extended solution* to Eq. (3.1) on $[a,a+\zeta)\times\Omega$, $\zeta \in (0,\infty)$ to be a function $u_\zeta : [a,a+\zeta)\times \Omega \to R^n$, whose restriction u_γ to $[a,a+\gamma]\times\Omega$ for any $\gamma < \zeta$ is a local solution of Eq. (3.1) and $\lim\limits_{\gamma\to\zeta-0} \|u_\gamma\|_{C([a,a+\gamma],BC(\Omega,R^n))} = \infty$. We define a *global solution* to Eq. (3.1) to be a function $u : [a,\infty)\times \Omega \to R^n$, whose restriction u_γ to $[a,a+\gamma]\times\Omega$ is its local solution for any $\gamma \in (0,\infty)$.

Theorem 3.1. *Let the assumptions* $(A1)$ – $(A6)$ *hold true. If for any* $r > 0$ *there exists* $\widetilde{f_r} \in R$ *such that for all* $u_1,u_2 \in R^n$, $|u_1| \le r$, $|u_2| \le r$, *we have* $|f(u_1)-f(u_2)| \le \widetilde{f_r}|u_1-u_2|$, *then Eq.* (3.1) *has a unique global or maximally extended solution and each local solution is a restriction of this solution.*

Proof We will use Theorem 2.1, namely, the condition 1), which is responsible for solvability of the Eq. (2.2)) and Remark 2.2 of the previous section to prove the solvability of (3.1).

First, we choose an arbitrary $b \in (a,\infty)$, define the following operator

$$(Fu)(t,x) = \varphi(a,x) + \int_a^t \int_\Omega W(t,s,x,y) f\big((S_\tau u)(s,x,y)\big)dyds, \tag{3.2}$$

$$(S_\tau^\varphi u)(t,x,y) = \begin{cases} \varphi(t-\tau(t,x,y),x), & \text{if} \quad t-\tau(t,x,y) < a; \\ u(t-\tau(t,x,y),y), & \text{if} \quad t-\tau(t,x,y) \ge a, \end{cases}$$

and show that

$$F : C([a,b],BC(\Omega,R^n)) \to C([a,b],BC(\Omega,R^n)).$$

For any $t \in [a,b]$ and $u \in C([a,b],BC(\Omega,R^n))$ we have

$$|(Fu)(t,x_1)-(Fu)(t,x_1)| \le$$

$$\le \Big|\varphi(a,x_1) + \int_a^t \int_\Omega W(t,s,x_1,y) f\big((S_\tau u)(s,x_1,y)\big)dyds-$$

$$-\varphi(a,x_2) + \int_a^t \int_\Omega W(t,s,x_2,y) f\big((S_\tau u)(s,x_2,y)\big)dyds\Big| \le$$

$$\le |\varphi(a,x_1)-\varphi(a,x_2)|+$$

$$+\int_a^t\int_\Omega \left|W(t,s,x_1,y)-W(t,s,x_2,y)\right|\left|f\big((S_\tau u)(s,x_1,y)\big)\right|dyds+$$

$$+\int_a^t\int_\Omega \left|W(t,s,x_2,y)\right|\left|f\big((S_\tau u)(s,x_1,y)\big)-f\big((S_\tau u)(s,x_2,y)\big)\right|dyds.$$

By the virtue of the assumption $(A6)$, the first term goes to 0 as $|x_1-x_2|\to 0$. The assumptions $(A2)$ – $(A4)$ and $(A6)$ guarantee convergence to 0 of the second term on the right hand side of this inequality as $|x_1-x_2|\to 0$. The superposition $f\big((S_\tau u)(s,\cdot,y)\big)$ is continuous as the assumptions $(A4)$ – $(A6)$ hold true. This fact and the assumption $(A3)$ imply convergence of the last term to 0 as $|x_1-x_2|\to 0$. This proves continuity of $(Fu)(t,\cdot)$.

For each $t\in[a,b]$ and any $u\in C([a,b],BC(\Omega,R^n))$ the function $(Fu)(t,\cdot)$ is bounded by the virtue of the assumptions (A3), (A4) and (A6).

Finally, we choose an arbitrary $u\in C([a,b],BC(\Omega,R^n))$ and, assuming that $t_2>t_1$, check that $(Fu)(\cdot,x)$ is continuous:

$$\sup_{x\in\Omega}|(Fu)(t_1,x)-(Fu)(t_2,x)|\le$$

$$\le\sup_{x\in\Omega}\Big|\int_a^{t_1}\int_\Omega W(t_1,s,x,y)f\big((S_\tau u)(s,x,y)\big)dyds-$$

$$-\int_a^{t_2}\int_\Omega W(t_2,s,x,y)f\big((S_\tau u)(s,x,y)\big)dyds\Big|\le$$

$$\le\sup_{x\in\Omega}\Big|\int_a^{t_1}\int_\Omega \big(W(t_1,s,x,y)-W(t_2,s,x,y)\big)f\big((S_\tau u)(s,x,y)\big)dyds\Big|+$$

$$+\sup_{x\in\Omega}\int_{t_1}^{t_2}\int_\Omega \Big|W(t_2,s,x,y)f\big((S_\tau u)(s,x,y)\big)dyds\Big|\le$$

$$\le\int_a^{t_1}\int_\Omega \sup_{x\in\Omega}\left|W(t_1,s,x,y)-W(t_2,s,x,y)\right|\sup_{x\in\Omega}\left|f\big((S_\tau u)(s,x,y)\big)\right|dyds+$$

$$+\int_{t_1}^{t_2}\int_\Omega \sup_{x\in\Omega}\left|W(t_2,s,x,y)\right|\sup_{x\in\Omega}\left|f\big((S_\tau u)(s,x,y)\big)\right|dyds.$$

We note that by the virtue of the assumptions $(A2)$ – $(A4)$ and $(A6)$, the first term converges to 0 as $t_1-t_2\to 0$. The second summand goes to 0 as the assumptions $(A3)$, $(A4)$ and $(A6)$ hold true and $t_1-t_2\to 0$.

Thus we proved that $F:C([a,b],BC(\Omega,R^n))\to C([a,b],BC(\Omega,R^n))$.

Next, we examine the fulfilment of Theorem 2.1 condition for the defined above operator $F:C([a,b],BC(\Omega,R^n))\to C([a,b],BC(\Omega,R^n))$. Choose an arbitrary $q_0<1$, $r>0$. Let $\gamma\in(0,b-a)$

and $u_1(t,\cdot) = u_2(t,\cdot)$, $t \in [a, a+\gamma]$, where $\|u_1\|_{C([a,b],BC(\Omega,R^n))} \le r$ and $\|u_2\|_{C([a,b],BC(\Omega,R^n))} \le r$. By assumption, we get the estimates

$$\sup_{t\in[a,a+\gamma+\delta],x\in\Omega}\Big|\int_a^t\int_\Omega W(t,s,x,y)f\Big((S_\tau^\varphi u_1)(s,x,y)\Big)dyds-$$

$$-\int_a^t\int_\Omega W(t,s,x,y)f\Big((S_\tau^\varphi u_2)(s,x,y)\Big)dyds\Big| \le$$

$$\le \sup_{t\in[a+\gamma,a+\gamma+\delta],x\in\Omega}\Big|\int_{a+\gamma}^t\int_\Omega W(t,s,x,y)\Big(f\Big((S_\tau^\varphi u_1)(s,x,y)\Big)-$$

$$-f\Big((S_\tau^\varphi u_2)(s,x,y)\Big)\Big)dyds\Big| \le$$

$$\le \sup_{t\in[a+\gamma,a+\gamma+\delta],x\in\Omega}\int_{a+\gamma}^t\int_\Omega\Big|W(t,s,x,y)\Big|\widetilde{f}_r\|u_1-u_2\|_{C([a,b],BC(\Omega,R^n))}dyds \le$$

$$\le \sup_{t\in[a+\gamma,a+\gamma+\delta],x\in\Omega}\int_{a+\gamma}^t\int_\Omega\Big|W(t,s,x,y)\Big|\widetilde{f}_r dyds\|u_1-u_2\|_{C([a,b],BC(\Omega,R^n))} \le$$

$$\le q\|u_1-u_2\|_{C([a,b],BC(\Omega,R^n))}.$$

Here

$$q = \widetilde{f}_r \sup_{t\in[a+\gamma,a+\gamma+\delta],x\in\Omega}\int_{a+\gamma}^t\int_\Omega\Big|W(t,s,x,y)\Big|dyds.$$

Thus, we can always find $\delta > 0$ such that $q \le q_0$. Hence, the property $\mathfrak{q}_2$) for the mapping F, given by (3.2), holds true. The verification of the property $\mathfrak{q}_1$) is analogous. Taking into account Remark 2.2, we prove the theorem. □

Remark 3.1. If in the Theorem 3.1 condition $\widetilde{f}_r = \widetilde{f}$ is independent of r (as e.g. in classical neural field models, where $0 \le f(u) \le 1$), then according to Remark 2.1 we will get a global solution to the Eq. (3.1). In this case, if we take $\tau(t,x,y) \equiv 0$, Theorem 3.1 becomes analogous to the results concerning solvability of the Amari model obtained by Potthast *et al.* [18].

Remark 3.2. If in Theorem 3.1 the condition $\widetilde{f}_r = \widetilde{f}$ is independent of r, Theorem 3.1 can be compared to the theorem on solvability of Eq. (1.3) in $C([a,b],L^2(\Omega,R^n))$ for any $b > a$ proved in Faye *et al.* [13] Here we obtained the same result for the more general model (3.1) in $C([a,b],BC(\Omega,R^n))$. We note that in case when the delay $\tau(t,x,y) = \tau(x,y)$ is independent of t, it is possible to prove Theorem 3.1 for the space $C([a,b],L^2(\Omega,R^n))$ using our technique as well thus getting the main theoretical result of [13].

Note that the remarks 3 and 4 on maximally extended solutions are valid for the problem (3.1) as well.

It is also worth mentioning that our approach to delayed functional-differential equations is based on the idea to include the prehistory function in the inner superposition operator. It allows us to consider the operator equation (2.1) with the operator (3.2) defined on $[a,b]$ instead of $(-\infty,b]$. The same approach to functional-differential equations with delay was implemented e.g. in [5], [6].

Next we complete the study of wellposedness of the problem (3.1) by investigating continuous dependence of solutions to the associated problem

$$\begin{gathered} u(t,x) = \varphi_\lambda(a,x) + \int\limits_a^t \int\limits_\Omega W_\lambda(t,s,x,y) f_\lambda(u(s-\tau_\lambda(s,x,y),y))dyds, \\ t \in [a,\infty), x \in \Omega; \\ u(\xi,x) = \varphi_\lambda(\xi,x),\ \xi \le a, x \in \Omega \end{gathered} \tag{3.3}$$

on a parameter $\lambda \in \Lambda$.

The assumptions $(A_\lambda 1)$ – $(A_\lambda 6)$ imposed on the functions in the model (3.3) for each $\lambda \in \Lambda$ repeat the assumptions $(A1)$ – $(A6)$, respectively.

We will naturally apply Definition 3.1 to the model (3.3) at each $\lambda \in \Lambda$.

The following theorem gives conditions that guarantee wellposedness of the problem (3.3).

Theorem 3.2. *Let the assumptions $(A_\lambda 1)$ – $(A_\lambda 6)$ hold true. Assume that the following conditions are satisfied:*

1) *There is a neighborhood U_0 of λ_0 such that for any $r > 0$ there exists $\widetilde{f_r} \in R$ (independent of $\lambda \in U_0$) such that for which $|f_\lambda(u_1) - f_\lambda(u_2)| \le \widetilde{f_r}|u_1 - u_2|$ for all $u_1, u_2 \in R^n$, $|u_1| \le r$, $|u_2| \le r$.*

For any $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ it holds true that:

2) *For any $b > a$,*

$$\sup_{t\in[a,b],\ x\in\Omega} \Big| \int\limits_a^t \int\limits_\Omega W_{\lambda_i}(t,s,x,y)dyds - \int\limits_a^t \int\limits_\Omega W_{\lambda_0}(t,s,x,y)dyds \Big| \to 0;$$

3) *For any $b > a$, if $|u_i(\cdot,\cdot) - u(\cdot,\cdot)| \to 0$ in measure on $[a,b]\times\Omega$ as $i \to \infty$, then $|f_{\lambda_i}(u_i(\cdot,\cdot)) - f_{\lambda_0}(u(\cdot,\cdot))| \to 0$ in measure on $[a,b]\times\Omega$ as $i \to \infty$;*

4) *For any $b > a$, $\sup\limits_{x\in\Omega} |\tau_{\lambda_i}(\cdot,x,\cdot) - \tau_{\lambda_0}(\cdot,x,\cdot)| \to 0$ in measure on $[a,b]\times\Omega$;*

5) $\|\varphi_{\lambda_i} - \varphi_{\lambda_0}\|_{C((-\infty,a],BC(\Omega,R^n))} \to 0.$

Then there is a neighborhood U of λ_0, such that for each element $\lambda \in U$, Eq. (3.3) *has a unique global or maximally extended solution, and each local solution is a restriction of this solution. Moreover, if at $\lambda = \lambda_0$ Eq.* (3.3) *has a local solution $u_{0\gamma}$ defined on $[a,a+\gamma]\times\Omega$, then for any*

$\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ one can find number I such that for all $i > I$ Eq. (3.3) *has a local solution $u_\gamma = u_\gamma(\lambda_i)$ defined on $[a, a+\gamma] \times \Omega$ and $\|u_\gamma(\lambda_i) - u_{0\gamma}\|_{C([a,a+\gamma],BC(\Omega,R^n))} \to 0$.*

Proof. Choose an arbitrary $b \in (a, \infty)$. In order to use Theorem 2.1, we need to bring the Eq. (3.3) to the form $u(t,\cdot) = (F(u,\lambda))(t)$, $t \in [a,b]$. Using the same technique as in the proof of Theorem 3.1 and the corresponding assumptions $(A_\lambda 1)$ - $(A_\lambda 6)$, we get here

$$F(\cdot,\lambda) : C([a,b], BC(\Omega,R^n)) \to C([a,b], BC(\Omega,R^n)),$$

$$(F(u,\lambda))(t,x) = \varphi_\lambda(a,x) + \int_a^t \int_\Omega W_\lambda(t,s,x,y) f_\lambda\Big((S_{\tau_\lambda}^{\varphi_\lambda} u)(s,x,y)\Big) dy ds,$$

$$t \in [a,b], \ x \in \Omega,$$

$$(S_{\tau_\lambda}^{\varphi_\lambda} u)(t,x,y) = \begin{cases} \varphi_\lambda(t - \tau_\lambda(t,x,y), y), & \text{if} \quad t - \tau_\lambda(t,x,y) < a; \\ u(t - \tau_\lambda(t,x,y), y), & \text{if} \quad t - \tau_\lambda(t,x,y) \geq a \end{cases}$$

for all $\lambda \in \Lambda$.

The condition 1) of this theorem allows us to verify the assumption 1) of Theorem 2.1 for each $\lambda \in U_0$ by the same procedure as we used in the proof of Theorem 2. So, we only need to verify the condition 2) of Theorem 1.

Choose an arbitrary $u \in C([a,b], BC(\Omega,R^n))$. Let $\|u_i - u\|_Y \to 0$, i.e, $\|u_i - u\|_{C([a,b],BC(\Omega,R^n))} \to 0$, $i \to \infty$, and $\lambda \to \lambda_0$.

We have the following estimates:

$$|(S_{\tau_\lambda}^{\varphi_\lambda} u_i)(t,x,y) - (S_{\tau_{\lambda_0}}^{\varphi_{\lambda_0}} u)(t,x,y)| \leq |(S_{\tau_\lambda}^{\varphi_\lambda} u_i)(t,x,y) - (S_{\tau_\lambda}^{\varphi_\lambda} u)(t,x,y)| +$$

$$+ |(S_{\tau_\lambda}^{\varphi_\lambda} u)(t,x,y) - (S_{\tau_{\lambda_0}}^{\varphi_\lambda} u)(t,x,y)| + |(S_{\tau_{\lambda_0}}^{\varphi_\lambda} u)(t,x,y) - (S_{\tau_{\lambda_0}}^{\varphi_{\lambda_0}} u)(t,x,y)|.$$

If $\lambda \to \lambda_0$, then the first term on the right-hand side of this inequality goes to 0 uniformly as $\|u_i - u\|_{C([a,b],BC(\Omega,R^n))} \to 0$. By the virtue of the condition 4), the second term on the right-hand side goes to 0 in measure on $([a,b] \times \Omega)$, uniformly in $x \in \Omega$, as $\lambda \to \lambda_0$. The third term on the right-hand side of the inequality goes to 0 uniformly when $\lambda \to \lambda_0$ as the condition 5) holds true. Thus, we have

$$|(S_{\tau_\lambda}^{\varphi_\lambda} u_i)(\cdot,x,\cdot) - (S_{\tau_{\lambda_0}}^{\varphi_{\lambda_0}} u)(\cdot,x,\cdot)| \to 0$$

in measure, uniformly in $x \in \Omega$, as $\|u_i - u\|_{C([a,b],BC(\Omega,R^n))} \to 0$ and $\lambda \to \lambda_0$.

Using this convergence, we can make the following estimates

$$\sup_{t\in[a,b],\ x\in\Omega} \Big| \int_a^t \int_\Omega W_\lambda(t,s,x,y) f_\lambda\Big((S_{\tau_\lambda}^{\varphi_\lambda} u_i)(s,x,y)\Big) dy ds -$$

$$\int_a^t \int_\Omega W_{\lambda_0}(t,s,x,y) f_{\lambda_0}\Big((S_{\tau_{\lambda_0}}^{\varphi_{\lambda_0}} u)(s,x,y)\Big) dy ds \Big| \leq$$

$$\sup_{t\in[a,b],\ x\in\Omega} \Big| \int_a^t \int_\Omega W_\lambda(t,s,x,y) f_\lambda\Big((S_{\tau_\lambda}^{\varphi_\lambda} u_i)(s,x,y)\Big) dy ds -$$

$$\int_a^t \int_\Omega W_\lambda(t,s,x,y) f_{\lambda_0}\Big((S_{\tau_{\lambda_0}}^{\varphi_{\lambda_0}} u)(s,x,y)\Big) dy ds \Big| +$$

$$+ \sup_{t\in[a,b],\ x\in\Omega} \Big| \int_a^t \int_\Omega W_\lambda(t,s,x,y) f_{\lambda_0}\Big((S_{\tau_{\lambda_0}}^{\varphi_{\lambda_0}} u)(s,x,y)\Big) dy ds -$$

$$\int_a^t \int_\Omega W_{\lambda_0}(t,s,x,y) f_{\lambda_0}\Big((S_{\tau_{\lambda_0}}^{\varphi_{\lambda_0}} u)(s,x,y)\Big) dy ds \Big|.$$

Taking into account the condition 3), we conclude that the first term on the right-hand side of the inequality goes to 0 as $\lambda \to \lambda_0$. The second term on the right-hand side of the inequality goes to 0 by the virtue of the condition 2) as $\lambda \to \lambda_0$.

Thus, the condition 2) of Theorem 2.1 is satisfied and Theorem 3.2 is proved. □

We emphasize here that our aim was to formulate the assumptions on the functions involved in the model (3.3) (see conditions 2) – 5) of Theorem 3.2) as general as it possible. Of course, we can strengthen these assumptions in order to make them more conventional e.g. in the following way.

Remark 3.3. If the estimate in the assumption $(A_\lambda 3)$ holds true uniformly with respect to $\lambda \in \Lambda$, then it is possible to get the conclusion of Theorem 3.2 by claiming that for any $b > a$ the functions

$$W_{(\cdot)} : \Lambda \times [a,b] \times [a,b] \times \Omega \times \Omega \to R^n,$$

$$f_{(\cdot)} : \Lambda \times R^n \to R^n,$$

$$\tau_{(\cdot)} : \Lambda \times [a,b] \times \Omega \times \Omega \to [0,\infty),$$

$$\varphi_{(\cdot)} : \Lambda \times (-\infty, b] \times \Omega \to R^n$$

are continuous instead of claiming the conditions 2) – 5) of Theorem 3.2.

We now consider two important special cases of the model (3.3).

As the neural field theory studies processes in cortical tissue, it is realistic to assume that Ω is bounded (see e.g. [13]). The following remark represents the result, analogous to Theorem 3.2 for this case.

Remark 3.4. If Ω is bounded, we can substitute $(A_\lambda 6)$ by

$(A_\lambda^* 6)$ For any $a^* < a$ and each $\varphi_\lambda \in C([a^*,a], C_0(\Omega, R^n))$, $\lambda \in \Lambda$.

In order to get the conclusion of Theorem 3.2, we need the following conditions instead of the conditions 3), 4), and 5), respectively:

For any $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ it holds true that:

3*) For any $u \in R^n$ we have $|f_{\lambda_i}(u) - f_{\lambda_0}(u)| \to 0$;

4*) For all $x \in \Omega$, $|\tau_{\lambda_i}(\cdot,x,\cdot) - \tau_{\lambda_0}(\cdot,x,\cdot)| \to 0$ in measure on $[a,b]\times\Omega$;

5*) For any $a^* < a$ and all $(t,x) \in [a^*,a]\times\Omega$, $|\varphi_{\lambda_i}(t,x) - \varphi_{\lambda_0}(t,x)| \to 0$.

Proof of the statement in Remark 3.4 is given in Appendix A.

In neural field modeling special attention is paid to spatially localized solutions, so-called "bumps". If Ω is unbounded, but the solution to (3.3) is spatially localized, we can relax Theorem 3.2 conditions in the following way.

Remark 3.5. If we replace $(A_\lambda 6)$ by

$(A'_\lambda 6)$ For each $\lambda \in \Lambda$, the prehistory function $\varphi_\lambda \in C((-\infty,a],C_0(\Omega,R^n))$;
and impose the additional condition, corresponding to localization in the spatial variable,

$(A'_\lambda 7)$ For each $\lambda \in \Lambda$ and any $b > a$, $\lim\limits_{|x|\to\infty} \left|W_\lambda(t,s,x,y)\right| = 0$ for all $(t,s,y) \in [a,b]\times[a,b]\times\Omega$,
then, in order to get the conclusion of Theorem 3.2 holds true for spatially localized solutions, we need the following conditions, instead of 2), 3), 4), and 5)respectively:

For any $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ it holds true that:

2′) For any $b > a$, $r > 0$, and each $t \in [a,b]$, $x \in \Omega$, $|x| \le r$ it holds true that

$$\Big|\int_a^t \int_\Omega \Big(W_{\lambda_i}(t,s,x,y)dy - W_{\lambda_0}(t,s,x,y)\Big)dyds\Big| \to 0;$$

3′) For any $u \in R^n$ we have $|f_{\lambda_i}(u) - f_{\lambda_0}(u)| \to 0$;

4′) For all $x \in \Omega$, $|\tau_{\lambda_i}(\cdot,x,\cdot) - \tau_{\lambda_0}(\cdot,x,\cdot)| \to 0$ in measure on $[a,b]\times\Omega$;

5′) For any $(t,x) \in (-\infty,a]\times\Omega$, $|\varphi_{\lambda_i}(t,x) - \varphi_{\lambda_0}(t,x)| \to 0$.

Proof of the statement in Remark 3.5 is given in Appendix B.

As Theorems 2 and 3 are valid for each $a \in R$ in the model (3.1), it is natural to address the question, what happens in the case when $a = -\infty$ (i.e., when (3.1) becomes (1.7)).

Remark 3.6. Solution to (1.7) is not necessarily unique.

The following example illustrates this fact.

Example 3.1. Consider the equation

$$u(t,x) = \int_{-\infty}^t \int_R \exp(-s)\omega(x)u(s,y)dyds,\ t \in R,\ x \in R$$

with some Gaussian function ω. Define the function $u \in C((-\infty,b],BC(R,R))$ as follows:

$$u(t,x) = v(t)\omega(x),$$

where

$$v(t) = V\exp(-\exp(-t)), V \in R,$$

is a solution to

$$\dot{v}(t) = \exp(-t)v(t),$$

satisfying the property $v(t) \to 0$ as $t \to -\infty$. Thus, for any $V \in R$ we get a solution to (1.7) which belongs to $C((-\infty,b], BC(R,R))$.

Nevertheless, it is possible to find conditions, which guarantee wellposedness of the model (1.7). The last part of the present paper is devoted to this problem. We have the following assumptions on the functions involved:

($\mathcal{A}1$) For any $a,b \in R$, $a < b$, $t \in [a,b]$, $x \in \Omega$, the function $W(t,\cdot,x,\cdot) : [a,b]\times\Omega \to R^n$ is measurable.

($\mathcal{A}2$) For any $a,b \in R$, $a < b$, at almost all $(s,y) \in [a,b]\times\Omega$, the function $W(\cdot,s,\cdot,y) : (-\infty,b]\times \Omega \to R^n$ is uniformly continuous.

($\mathcal{A}3$) For any $b \in R$, $t \in (-\infty,b]$, $\int_\Omega \sup_{x\in\Omega} \left|W(t,s,x,y)\right| dy = G(s)$, where $G \in L^1((-\infty,b],\mu,R^n)$.

Assumptions ($\mathcal{A}4$) and ($\mathcal{A}5$) are the same as the corresponding assumptions ($A4$) and ($A5$).

Now, we need to give the definitions of local, maximally extended and global solutions to Eq. (1.7).

Definition 3.2. We define *a local solution* to Eq. (1.7) on $(-\infty,\gamma]\times\Omega$, $\gamma \in R$, to be a function $u_\gamma \in C((-\infty,\gamma], BC(\Omega,R^n))$ that satisfies the equation (1.7) on $(-\infty,\gamma]\times\Omega$. We define a *maximally extended solution* to Eq. (1.7) on $(-\infty,\zeta)\times\Omega$, $\zeta \in R$ to be a function $u_\zeta : (-\infty,\zeta)\times\Omega \to R^n$, whose restriction u_γ to $(-\infty,\gamma]\times\Omega$ is a local solution to Eq. (1.7) for any $\gamma < \zeta$ and $\lim_{\gamma\to\zeta-0} \|u_\gamma\|_{C((-\infty,\gamma],BC(\Omega,R^n))} = \infty$. We define a *global solution* to Eq. (1.7) to be a function $u : R\times\Omega \to R^n$, whose restriction u_γ to $(-\infty,\gamma]\times\Omega$ is its local solution for any $\gamma \in R$.

Theorem 3.3. *Let the assumptions ($\mathcal{A}4$) – ($\mathcal{A}5$) hold true. If for any $r > 0$ there exists $\widetilde{f_r} \in R$ such that for all $u_1,u_2 \in R^n$, $|u_1| \le r$, $|u_2| \le r$, we have $|f(u_1) - f(u_2)| \le \widetilde{f_r}|u_1 - u_2|$, then Eq. (1.7) has a unique global or maximally extended solution and each local solution is a restriction of this global or maximally extended solution (all types of solutions are meant in the sense of Definition 3.2).*

Proof. First, we prove existence of a unique local solution to (1.7). Choose arbitrary $b \in R$. Using the same estimation technique as in the proof ot Theorem 3.1 and the corresponding assumptions ($\mathcal{A}1$) – ($\mathcal{A}5$), we rewrite Eq. (1.7) as the operator equation $u(t,\cdot) = (Fu)(t)$, and consider it on $(-\infty,b]$, where

$$F : C((-\infty,b], BC(\Omega,R^n)) \to C((-\infty,b], BC(\Omega,R^n)),$$

$$(Fu)(t,x) = \int_{-\infty}^{t} \int_\Omega W(t,s,x,y) f\Big(u(s-\tau(s,x,y),y)\Big) dy ds, \ t \in [a,b], \ x \in \Omega.$$

Choose arbitrary $q_0 < 1$, $r > 0$, $\|u_1\|_{C((-\infty,b],BC(\Omega,R^n))} \le r$ and $\|u_2\|_{C((-\infty,b],BC(\Omega,R^n))} \le r$. In order to prove existence of a unique local solution to (1.7) using the Banach fixed point theorem, we need to find $\delta \in R$ such that

$$\max_{t\in(-\infty,\delta]} \|(Fu_1)(t)-(Fu_2)(t)\|_{BC(\Omega,R^n)} \le q_0 \max_{t\in(-\infty,\delta]} \|(u_1)(t)-(u_2)(t)\|_{BC(\Omega,R^n)}.$$

For any $\delta < b$, we get the estimates

$$\sup_{t\in(-\infty,\delta],x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\Omega} W(t,s,x,y)f\big(u_1(s-\tau(s,x,y),y)\big)dyds-$$

$$-\int_{-\infty}^{t}\int_{\Omega} W(t,s,x,y)f\big(u_2(s-\tau(s,x,y),y)\big)dyds\Big| \le$$

$$\le \sup_{t\in(-\infty,\delta],x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\Omega} W(t,s,x,y)\Big(f\big(u_1(s-\tau(s,x,y),y)\big)-$$

$$-f\big(u_2(s-\tau(s,x,y),y)\big)\Big)dyds\Big| \le$$

$$\le \sup_{t\in(-\infty,\delta],x\in\Omega}\int_{-\infty}^{t}\int_{\Omega}\Big|W(t,s,x,y)\Big|\widetilde{f}_r dyds\|u_1-u_2\|_{BC((-\infty,\delta]\times\Omega,R^n)} \le$$

$$\le q\|u_1-u_2\|_{BC((-\infty,\delta]\times\Omega,R^n)}.$$

Here

$$q = \widetilde{f}_r \sup_{t\in(-\infty,\delta],x\in\Omega}\int_{-\infty}^{t}\int_{\Omega}\Big|W(t,s,x,y)\Big|dyds.$$

Using the assumption ($\mathcal{A}3$), we can find $\delta > 0$ such that $q \le q_0$. Thus, the equation (1.7) has a unique local solution, defined on $(-\infty,\delta]\times\Omega$. Now, regarding this solution as a prehistory function for the model (3.1) and taking $a=\delta$, we use Theorem 3.1 and obtain the conclusion of the theorem. □

In order to approach the problem of wellposedness of (1.7), we consider its parameterized version:

$$u(t,x) = \int_{-\infty}^{t}\int_{\Omega} W_\lambda(t,s,x,y)f_\lambda(u(s-\tau_\lambda(s,x,y),y))dyds,$$

$$t\in R, x\in\Omega, \tag{3.4}$$

with a parameter $\lambda \in \Lambda$.

For each $\lambda\in\Lambda$, the assumptions ($\mathcal{A}_\lambda 1$) – ($\mathcal{A}_\lambda 5$), imposed on the functions involved in the model (3.4) repeat the assumptions ($\mathcal{A}1$) – ($\mathcal{A}5$), respectively.

At each $\lambda\in\Lambda$ we define the types of solutions to (3.4) according to Definition 3.2.

Theorem 3.4. *Let the assumptions ($\mathcal{A}_\lambda 1$) – ($\mathcal{A}_\lambda 5$) hold true. Assume that the following conditions are satisfied:*

1) *There is a neighborhood* U_0 *of* λ_0 *such that for any for any* $r > 0$ *there exists* $\widetilde{f_r} \in R$ *(independent of* $\lambda \in U_0$*), for which* $|f_\lambda(u_1) - f_\lambda(u_2)| \leq \widetilde{f_r}|u_1 - u_2|$ *for all* $u_1, u_2 \in R^n$, $|u_1| \leq r$, $|u_2| \leq r$*;*

For any $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ *it holds true that:*

2) *For any* $b \in R$, $\sup\limits_{(-\infty,b],\ x\in\Omega} \left| \int\limits_{-\infty}^{t} \int\limits_{\Omega} W_{\lambda_i}(t,s,x,y)dy - \int\limits_{-\infty}^{t} \int\limits_{\Omega} W_{\lambda_0}(t,s,x,y)dy \right| \to 0$;

3) *For any* $b \in R$, *if* $|u_i(\cdot,\cdot) - u(\cdot,\cdot)| \to 0$ *in measure on* $(-\infty,b]\times\Omega$ *as* $i \to \infty$, *then* $|f_{\lambda_i}(u_i(\cdot,\cdot)) - f_{\lambda_0}(u(\cdot,\cdot))| \to 0$ *in measure on* $(-\infty,b]\times\Omega$ *as* $i \to \infty$*;*

4) *For any* $b \in R$, $\sup\limits_{x\in\Omega} |\tau_{\lambda_i}(\cdot,x,\cdot) - \tau_{\lambda_0}(\cdot,x,\cdot)| \to 0$ *in measure on* $(-\infty,b]\times\Omega$*;*

Then there is a neighborhood U *of* λ_0*, such that for each* $\lambda \in U$*, Eq.* (3.4) *has a unique global or maximally extended solution, and each local solution is a restriction of this solution. Moreover, if at* $\lambda = \lambda_0$ *Eq.* (3.4) *has a local solution* $u_{0\gamma}$ *defined on* $(-\infty,\gamma]\times\Omega$*, then for any* $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ *one can find number* I *such that for all* $i > I$ *Eq.* (3.4) *has a local solution* $u_\gamma = u_\gamma(\lambda_i)$ *defined on* $(-\infty,\gamma]\times\Omega$ *and* $\|u_\gamma(\lambda) - u_{0\gamma}\|_{C((-\infty,\gamma],BC(\Omega,R^n))} \to 0$ *as* $\lambda \to \lambda_0$*.*

Proof Choose an arbitrary $b \in R$. Consider the following operator equation

$$u(t,\cdot) = (F(u,\lambda))(t),\ t \in (-\infty,b],$$

where at each $\lambda \in \Lambda$, by the virtue of the assumptions $(\mathcal{A}_\lambda 1)$ – $(\mathcal{A}_\lambda 5)$,

$$F(\cdot,\lambda) : C((-\infty,b],BC(\Omega,R^n)) \to C((-\infty,b],BC(\Omega,R^n)),$$

$$(F(u,\lambda))(t,x) = \int\limits_{-\infty}^{t} \int\limits_{\Omega} W_\lambda(t,s,x,y) f_\lambda\Big(u(t-\tau_\lambda(t,x,y),y)\Big)dyds,.$$

$$t \in (-\infty,b],\ x \in \Omega$$

Note that by Theorem 3.3 we have a unique solution to Eq. (3.4) defined on $(-\infty,\delta]\times\Omega$ for each $\lambda \in U_0$. We need to prove continuous dependence of these solutions on λ. First, we prove that the operator F is continuous in (u,λ_0) for any fixed $u \in C((-\infty,b],BC(\Omega,R^n))$.

Choose an arbitrary $u \in C((-\infty,b],BC(\Omega,R^n))$. Let $\|u_i - u\|_{C((-\infty,b],BC(\Omega,R^n))} \to 0$, $i \to \infty$, and $\lambda \to \lambda_0$.

We have the following estimates:

$$|u_i(t-\tau_\lambda(t,x,y),y) - u(t-\tau_{\lambda_0}(t,x,y),y)| \leq$$

$$\leq |u_i(t-\tau_\lambda(t,x,y),y) - u(t-\tau_\lambda(t,x,y),y)| +$$

$$+ |u(t-\tau_\lambda(t,x,y),y) - u(t-\tau_{\lambda_0}(t,x,y),y)|.$$

If $\lambda \to \lambda_0$, then the first term on the right-hand side of this inequality goes to 0 uniformly as $\|u_i - u\|_{C((-\infty,b],BC(\Omega,R^n))} \to 0$. By virtue of the condition 4), the second term on the right-hand side goes to 0 in measure on $((-\infty,b]\times\Omega)$, uniformly in $x \in \Omega$, as $\lambda \to \lambda_0$. So,

$$|u_i(\cdot - \tau_\lambda(\cdot,x,\cdot),\cdot) - u(t-\tau_{\lambda_0}(\cdot,x,\cdot),\cdot)| \to 0$$

in measure, uniformly in $x \in \Omega$, as $\|u_i - u\|_{C((-\infty,b],BC(\Omega,R^n))} \to 0$ and $\lambda \to \lambda_0$.

Using this convergence, we obtain

$$\sup_{t\in(-\infty,b],\ x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\Omega} W_\lambda(t,s,x,y)f_\lambda\Big(u_i(t-\tau_\lambda(s,x,y),y)\Big)dyds-$$

$$\int_{-\infty}^{t}\int_{\Omega} W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(s,x,y),y)\Big)dyds\Big| \leq$$

$$\sup_{t\in(-\infty,b],\ x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\Omega} W_\lambda(t,s,x,y)f_\lambda\Big(u_i(t-\tau_\lambda(s,x,y),y)\Big)dyds-$$

$$\int_{-\infty}^{t}\int_{\Omega} W_{\lambda}(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(s,x,y),y)\Big)dyds\Big|+$$

$$+\sup_{t\in(-\infty,b],\ x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\Omega} W_\lambda(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(s,x,y),y)\Big)dyds-$$

$$\int_{-\infty}^{t}\int_{\Omega} W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(s,x,y),y)\Big)dyds\Big|.$$

Taking into account the condition 3), we conclude that the first term on the right-hand side of the inequality goes to 0 as $\lambda\to\lambda_0$. The second term on the right-hand side of the inequality goes to 0 by the virtue of the condition 2) as $\lambda\to\lambda_0$. Thus, the operator F is continuous in (u,λ_0) for any chosen $u\in C((-\infty,b],BC(\Omega,R^n))$. Using this fact, for any $\varepsilon>0$ we can find such $\varepsilon_1>0$ and neighborhood U_1 of λ_0, that

$$\|F(\mathrm{u}_\delta,\lambda)-F(u_{0\delta},\lambda)\|_{C((-\infty,b],BC(\Omega,R^n))}\leq\varepsilon$$

for all $\lambda\in U_1$ and any $\mathrm{u}_\delta\in C((-\infty,\delta],BC(\Omega,R^n))$, satisfying the estimate

$$\|\mathrm{u}_\delta-u_{0\delta}\|_{C((-\infty,\delta],BC(\Omega,R^n))}\leq\varepsilon_1.$$

As the mapping $F(\cdot,\lambda)$ is contracting with the constant $q_0<1$ (see Theorem 3.3) for any $\lambda\in U_0$, for any $m=1,2,\ldots$ we have

$$\|F^m(u_{0\delta},\lambda)-u_{0\delta}\|_{C((-\infty,\delta],BC(\Omega,R^n))}\leq$$

$$\leq\|F^m(u_{0\delta},\lambda)-F^{m-1}(u_{0\delta},\lambda)\|_{C((-\infty,\delta],BC(\Omega,R^n))}+\ldots$$

$$\ldots+\|F(u_{0\delta},\lambda)-u_{0\delta}\|_{C((-\infty,\delta],BC(\Omega,R^n))}\leq$$

$$\leq(q_0^{m-1}+\ldots+q_0+1)(1-q_0)\varepsilon\leq\varepsilon.$$

Due to the convergence of the approximations $F^m(u_{0\delta},\lambda)$ to the fixed point $u_\delta=u_\delta(\lambda)$ of the operator $F(\cdot,\lambda):C((-\infty,\delta],BC(\Omega,R^n))\to C((-\infty,\delta],BC(\Omega,R^n))$ we get $\|u_\delta(\lambda)-u_{0\delta}\|_{C((-\infty,\delta],BC(\Omega,R^n))}\leq\varepsilon$ for each $\lambda\in U_0\bigcap U_1$ and $\varepsilon\to0$ as $\lambda\to\lambda_0$.

Now, addressing the model (3.3) and Theorem 3.2, and taking $\varphi_\lambda=u_\delta(\lambda)$ and $a=\delta$, we prove this theorem. □

We note here that the remark, analogous to Remark 3.3, is valid for Theorem 3.4 as well.

Remark 3.7. If Ω is bounded, we can get the conclusion of Theorem 3.4 replacing 3) and 4) by the following conditions:

For any $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ it holds true that:

3*) For any $u \in R^n$ we have $|f_{\lambda_i}(u) - f_{\lambda_0}(u)| \to 0$;

4*) For all $x \in \Omega$, $|\tau_{\lambda_i}(\cdot, x, \cdot) - \tau_{\lambda_0}(\cdot, x, \cdot)| \to 0$ in measure on $(-\infty, b] \times \Omega$.

Proof of the statement in Remark 3.7 is given in Appendix C.

In case of spatially localized solutions to the (1.7) and (3.4), we have the following remark to Theorem 3.4.

Remark 3.8. If in (3.4) we add the condition, corresponding to localization in the spatial variable,

($\mathcal{A}'_\lambda 6$) For each $\lambda \in \Lambda$ and any $b \in R$, $\lim\limits_{|x| \to \infty} \left| W_\lambda(t, s, x, y) \right| = 0$ for all $(t, s, y) \in (-\infty, b] \times (-\infty, b] \times \Omega$, then, in order to get the conclusion of Theorem 3.4 for spatially localized solutions, we need the following conditions instead of 2), 3), and 4), respectively:

For any $\{\lambda_i\} \subset \Lambda$, $\lambda_i \to \lambda_0$ it holds true that:

2′) For any $b \in R$, $r > 0$ and each $t \in (-\infty, b]$, $x \in \Omega$, $|x| \le r$ it holds true that

$$\Big| \int_{-\infty}^{t} \int_{\Omega} \Big(W_{\lambda_i}(t, s, x, y) - W_{\lambda_0}(t, s, x, y) \Big) dy ds \Big| \to 0;$$

3′) For any $u \in R^n$ we have $|f_{\lambda_i}(u) - f_{\lambda_0}(u)| \to 0$;

4′) For all $x \in \Omega$, $|\tau_{\lambda_i}(\cdot, x, \cdot) - \tau_{\lambda_0}(\cdot, x, \cdot)| \to 0$ in measure on $(-\infty, b] \times \Omega$.

Proof of the statement in Remark 3.8 is given in Appendix D.

4 Conclusions and Outlook

For the nonlinear Volterra integral equations (1.7) and (3.1), which generalize the commonly used in the neural field theory models (1.1) – (1.6), we have defined the notions of local, global and maximally extended solutions. We have obtained conditions which guarantee existence of a unique global or maximally extended solution and its continuous dependence on the equation parameters. These results can also serve as a starting point for the development of numerical schemes for a broad class of neural field models. A key word in this context is justification of such schemes. We will emphasize that our results shed light on the problem of structural stability in nonlocal field models in, e.g. systems biology.

The question of applicability of our results in case of solutions belonging to $C([a,b], L^2(\Omega, R^n))$ (see Remark 3.2) gives rise to the problem of extension of our results for solutions from an abstract space $C([a,b], \mathfrak{B}(\Omega, R^n))$. The difficulties in this case are caused by intrinsic properties of the inner superposition operator in the abstract functional Banach space $\mathfrak{B}(\Omega, R^n)$.

A further development of the present study consists of studying the solvability and continuous dependence on parameters in the Volterra model with an abstract parameterized measure

$$\begin{gathered} u(t,x,\lambda) = \varphi_\lambda(a,x) + \int_a^t \int_\Omega W_\lambda(t,s,x,y) f_\lambda(u(s-\tau_\lambda(s,x,y),y,\lambda))\nu_\lambda(dy)ds, \\ t \in [a,\infty), x \in \Omega; \\ u(\xi,x,\lambda) = \varphi_\lambda(\xi,x),\ \xi \le a, x \in \Omega. \end{gathered} \tag{4.1}$$

This formulation of the problem will allow us to approach the homogenized Amari model with non-Lebesgue measure, which can be derived from the parameterized Amari model (1.4) in case of non-periodicity of the connectivity kernel in the fine-scale variable (see [19]). Considering the model (4.1) will also allow us to investigate e.g. the delayed Hopfield model model (see e.g. [23])

$$\dot{u}_i(t) = -\alpha u_i(t) + \sum_{j=1}^{N} \omega_{ij} f\big(u_i(t-\tau(t))\big) + I_i(t), t \in [a,\infty), i = 1,\dots,N.$$

Appendix A. Proof of The Statement in Remark 3.4

We refer here to the proof of Theorem 3.2 and note that conditions in Remark 3.4 imply that

$$|(S_{\tau_\lambda}^{\varphi_\lambda} u_i)(\cdot,x,\cdot) - (S_{\tau_{\lambda_0}}^{\varphi_{\lambda_0}} u)(\cdot,x,\cdot)| \to 0$$

uniformly on $(([a,b]\times\Omega)\setminus\Theta_\lambda)\times R^n$ ($\mu(\Theta_\lambda)\to 0$), for each $x\in\Omega$, as $\|u_i - u\|_{C([a,b],C_0(\Omega,R^n))} \to 0$ and $\lambda \to \lambda_0$.

Choose arbitrary $\varepsilon > 0$. For the b chosen in the proof of Theorem 3.2 we find

$$a^* = \min_{t\in[a,b];\ (x,y)\in\Omega^2} (t - \tau_\lambda(t,x,y)).$$

Define the piecewise constant functions $\overline{u} : [a,b]\times R^n \to R^n$ and $\overline{\varphi}_{\lambda_0} : [a^*,a]\times R^n \to R^n$ as $\overline{u}(t,x) \in R^n$ for $t \in [a,b]$, $\xi \in [a^*,a]$, $x \in \Omega$ such that

$$\begin{cases} |\overline{u}(t,x) - u(t,x)| \le \varepsilon/2, \text{ if } |\overline{u}(t,x)| > |u(t,x)|; \\ |\overline{u}(t,x) - u(t,x)| < \varepsilon/2, \text{ if } |\overline{u}(t,x)| < |u(t,x)|; \end{cases}$$

$$\begin{cases} |\overline{\varphi}_{\lambda_0}(\xi,x) - \varphi_{\lambda_0}(\xi,x)| \le \varepsilon/2, \text{ if } |\overline{\varphi}_{\lambda_0}(\xi,x)| > |\varphi_{\lambda_0}(\xi,x)|; \\ |\overline{\varphi}_{\lambda_0}(\xi,x) - \varphi_{\lambda_0}(\xi,x)| < \varepsilon/2, \text{ if } |\overline{\varphi}_{\lambda_0}(\xi,x)| < |\varphi_{\lambda_0}(\xi,x)|. \end{cases}$$

We get the estimate

$$\begin{aligned} &\left| f_\lambda\Big((S_{\tau_\lambda}^{\varphi_\lambda} u_i)(t,x,y)\Big) - f_{\lambda_0}\Big((S_{\tau_{\lambda_0}}^{\varphi_{\lambda_0}} u)(t,x,y)\Big)\right| \le \\ &\le \left| f_\lambda\Big((S_{\tau_\lambda}^{\varphi_\lambda} u_i)(t,x,y)\Big) - f_\lambda\Big((S_{\tau_{\lambda_0}}^{\overline{\varphi}_{\lambda_0}} \overline{u})(t,x,y)\Big)\right| + \\ &+ \left| f_\lambda\Big((S_{\tau_{\lambda_0}}^{\overline{\varphi}_{\lambda_0}} \overline{u})(t,x,y)\Big) - f_{\lambda_0}\Big((S_{\tau_{\lambda_0}}^{\overline{\varphi}_{\lambda_0}} \overline{u})(t,x,y)\Big)\right| + \end{aligned}$$

$$+\left|f_{\lambda_0}\Big((S^{\overline{\varphi}_{\lambda_0}}_{\tau_{\lambda_0}}\overline{u})(t,x,y)\Big)-f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(t,x,y)\Big)\right|.$$

Using the functions $\overline{u}$ and $\overline{\varphi}_{\lambda_0}$, it is easy to conclude that the first and the third terms on the right-hand side of this inequality are less or equal to 2ε and ε, respectively, on $(([a,b]\times\Omega)\setminus\Theta_\lambda)\times\Omega$, where $\mu(\Theta_\lambda)\to 0$ as $\lambda\to\lambda_0$. In addition, the condition 4*) provide convergence to 0 of the second term on the right-hand side of the inequality as $\lambda\to\lambda_0$.

Using the convergence obtained above, we get

$$\max_{t\in[a,b],\ x\in\Omega}\left|\int_a^t\int_\Omega W_\lambda(t,s,x,y)f_\lambda\Big((S^{\varphi_\lambda}_{\tau_\lambda}u_i)(s,x,y)\Big)dyds-\right.$$

$$\left.\int_a^t\int_\Omega W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(s,x,y)\Big)dyds\right|\le$$

$$\max_{t\in[a,b],\ x\in\Omega}\left|\int_a^t\int_\Omega W_\lambda(t,s,x,y)f_\lambda\Big((S^{\varphi_\lambda}_{\tau_\lambda}u_i)(s,x,y)\Big)dyds-\right.$$

$$\left.\int_a^t\int_\Omega W_\lambda(t,s,x,y)f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(s,x,y)\Big)dyds\right|+$$

$$+\max_{t\in[a,b],\ x\in\Omega}\left|\int_a^t\int_\Omega W_\lambda(t,s,x,y)f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(s,x,y)\Big)dyds-\right.$$

$$\left.\int_a^t\int_\Omega W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(s,x,y)\Big)dyds\right|.$$

Taking into account the condition 3*), we have the first term on the right-hand side of this inequality going to 0 as $\lambda\to\lambda_0$. The second term on the right-hand side of the inequality goes to 0 by the virtue of the condition 2) as $\lambda\to\lambda_0$. Thus, the statement in Remark 3.4 is valid.

Appendix B. Proof of The Statement in Remark 3.5

Conditions in Remark 3.5 imply the following changes in the proof of Theorem 3:

$$|(S^{\varphi_\lambda}_{\tau_\lambda}u_i)(\cdot,x,\cdot)-(S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(\cdot,x,\cdot)|\to 0$$

uniformly on $(([a,b]\times\Omega)\setminus\Theta_\lambda)\times R^n$ ($\mu(\Theta_\lambda)\to 0$), for each $x\in\Omega$, as $||u_i-u||_{C([a,b],C_0(\Omega,R^n))}\to 0$ and $\lambda\to\lambda_0$.

Choose arbitrary $\varepsilon>0$. Define the piecewise constant functions $\overline{u}:[a,b]\times R^n\to R^n$ and $\overline{\varphi}_{\lambda_0}:(-\infty,a]\times R^n\to R^n$ as $\overline{u}(t,x)\in R^n$ for $t\in[a,b]$, $\xi\in(-\infty,a]$, $x\in\Omega$ such that

$$\begin{cases}|\overline{u}(t,x)-u(t,x)|\le\varepsilon/2, \text{ if } |\overline{u}(t,x)|>|u(t,x)|;\\ |\overline{u}(t,x)-u(t,x)|<\varepsilon/2, \text{ if } |\overline{u}(t,x)|<|u(t,x)|;\end{cases}$$

$$\begin{cases}|\overline{\varphi}_{\lambda_0}(\xi,x)-\varphi_{\lambda_0}(\xi,x)|\le\varepsilon/2, \text{ if } |\overline{\varphi}_{\lambda_0}(\xi,x)|>|\varphi_{\lambda_0}(\xi,x)|;\\ |\overline{\varphi}_{\lambda_0}(\xi,x)-\varphi_{\lambda_0}(\xi,x)|<\varepsilon/2, \text{ if } |\overline{\varphi}_{\lambda_0}(\xi,x)|<|\varphi_{\lambda_0}(\xi,x)|.\end{cases}$$

We get the estimate

$$\left|f_\lambda\Big((S^{\varphi_\lambda}_{\tau_\lambda}u_i)(t,x,y)\Big)-f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(t,x,y)\Big)\right|\le$$

$$\le\left|f_\lambda\Big((S^{\varphi_\lambda}_{\tau_\lambda}u_i)(t,x,y)\Big)-f_\lambda\Big((S^{\overline{\varphi}_{\lambda_0}}_{\tau_{\lambda_0}}\overline{u})(t,x,y)\Big)\right|+$$

$$+\left|f_\lambda\Big((S^{\overline{\varphi}_{\lambda_0}}_{\tau_{\lambda_0}}\overline{u})(t,x,y)\Big)-f_{\lambda_0}\Big((S^{\overline{\varphi}_{\lambda_0}}_{\tau_{\lambda_0}}\overline{u})(t,x,y)\Big)\right|+$$

$$+\left|f_{\lambda_0}\Big((S^{\overline{\varphi}_{\lambda_0}}_{\tau_{\lambda_0}}\overline{u})(t,x,y)\Big)-f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(t,x,y)\Big)\right|.$$

Using the functions $\overline{u}$ and $\overline{\varphi}_{\lambda_0}$, it is easy to conclude that the first and the third terms on the right-hand side of this inequality are less or equal to 2ε and ε, respectively, on $(([a,b]\times\Omega)\setminus\Theta_\lambda)\times\Omega$, where $\mu(\Theta_\lambda)\to 0$ as $\lambda\to\lambda_0$. In addition to that, the condition 4′) provide convergence to 0 of the second term on the right-hand side of the inequality as $\lambda\to\lambda_0$.

Using the convergence obtained above, $(A'_\lambda 3)$, $(A_\lambda 5)$, and conditions 2′) and 3′), we get

$$\max_{t\in[a,b],\ x\in\Omega}\left|\int_a^t\int_\Omega W_\lambda(t,s,x,y)f_\lambda\Big((S^{\varphi_\lambda}_{\tau_\lambda}u_i)(s,x,y)\Big)dyds-\right.$$

$$\left.\int_a^t\int_\Omega W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(s,x,y)\Big)dyds\right|\le$$

$$\max_{t\in[a,b],\ x\in\Omega}\left|\int_a^t\int_{\{x\in\Omega,|x|\le r'\}} W_\lambda(t,s,x,y)f_\lambda\Big((S^{\varphi_\lambda}_{\tau_\lambda}u_i)(s,x,y)\Big)dyds-\right.$$

$$\left.\int_a^t\int_{\{x\in\Omega,|x|\le r'\}} W_\lambda(t,s,x,y)f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(s,x,y)\Big)dyds\right|+$$

$$+\max_{t\in[a,b],\ x\in\Omega}\left|\int_a^t\int_{\{x\in\Omega,|x|\le r'\}} W_\lambda(t,s,x,y)f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(s,x,y)\Big)dyds-\right.$$

$$\left.\int_a^t\int_{\{x\in\Omega,|x|\le r'\}} W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big((S^{\varphi_{\lambda_0}}_{\tau_{\lambda_0}}u)(s,x,y)\Big)dyds\right|+\epsilon_{r'}(t,x).$$

Here $\epsilon_{r'}(t,x)\to 0$ uniformly as $r'\to\infty$. Taking into account the condition 3′), we have the first term on the right-hand side of this inequality going to 0 as $\lambda\to\lambda_0$. The second term on the right-hand side of the inequality goes to 0 by the virtue of the condition 2′) as $\lambda\to\lambda_0$. Thus, the statement in Remark 3.5 is valid.

Appendix C. Proof of The Statement in Remark 3.7

The following changes in the proof of Theorem 3.4 stem from the conditions of Remark 3.7:

$$|u_i(t-\tau_\lambda(t,x,y),y)-u(t-\tau_{\lambda_0}(t,x,y),y)|\to 0$$

uniformly on $(((-\infty,b]\times\Omega)\setminus\Theta_\lambda)\times R^n$ $(\mu(\Theta_\lambda)\to 0)$ for each $x\in\Omega$, as $\|u_i-u\|_{C((-\infty,b],BC(\Omega,R^n))}\to 0$ and $\lambda\to\lambda_0$.

Choose an arbitrary $\varepsilon>0$. Define the piecewise constant function $\overline{u}:(-\infty,b]\times R^n\to R^n$ as $\overline{u}(t,x)\in R^n$ for $t\in(-\infty,b]$, $x\in\Omega$ such that

$$\begin{cases} |\overline{u}(t,x)-u(t,x)|\le\varepsilon/2, \text{ if } |\overline{u}(t,x)|>|u(t,x)|; \\ |\overline{u}(t,x)-u(t,x)|<\varepsilon/2, \text{ if } |\overline{u}(t,x)|<|u(t,x)|. \end{cases}$$

Using the function introduced above, we get the estimate

$$\Big|f_\lambda\Big(u_i(t-\tau_\lambda(t,x,y),y)\Big)-f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)\Big|\le$$
$$\le\Big|f_\lambda\Big(u_i(t-\tau_\lambda(t,x,y),y)\Big)-f_\lambda\Big(\overline{u}(t-\tau_{\lambda_0}(t,x,y),y)\Big)\Big|+$$
$$+\Big|f_\lambda\Big(\overline{u}(t-\tau_{\lambda_0}(t,x,y),y)\Big)-f_{\lambda_0}\Big(\overline{u}(t-\tau_{\lambda_0}(t,x,y),y)\Big)\Big|+$$
$$+\Big|f_{\lambda_0}\Big(\overline{u}(t-\tau_{\lambda_0}(t,x,y),y)\Big)-f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)\Big|.$$

Here, the first and the third terms on the right-hand side of this inequality are less or equal to 2ε and ε, respectively, on $(((-\infty,b]\times\Omega)\setminus\Theta_\lambda)\times R^n$, where $\mu(\Theta_\lambda)\to 0$ as $\lambda\to\lambda_0$. In addition, the condition 4*) provide convergence to 0 of the second term on the right-hand side of the inequality as $\lambda\to\lambda_0$.

Using the convergence obtained above and $(\mathcal{A}_\lambda 4)$, we get

$$\max_{t\in(-\infty,b],\ x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\Omega}W_\lambda(t,s,x,y)f_\lambda\Big(u_i(t-\tau_\lambda(t,x,y),y)\Big)dyds-$$
$$\int_{-\infty}^{t}\int_{\Omega}W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)dyds\Big|\le$$
$$\max_{t\in(-\infty,b],\ x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\Omega}W_\lambda(t,s,x,y)f_\lambda\Big(u_i(t-\tau_\lambda(t,x,y),y)\Big)dyds-$$
$$\int_{-\infty}^{t}\int_{\Omega}W_\lambda(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)dyds\Big|+$$
$$+\max_{t\in(-\infty,b],\ x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\Omega}W_\lambda(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)dyds-$$
$$\int_{-\infty}^{t}\int_{\Omega}W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)dyds\Big|.$$

Taking into account the condition 3*), we have the first term on the right-hand side of this inequality going to 0 as $\lambda\to\lambda_0$. The second term on the right-hand side of the inequality goes to 0 by the virtue of the conditions 2) as $\lambda\to\lambda_0$. Thus, the statement in Remark 3.7 is valid.

Appendix D. Proof of The Statement in Remark 3.8

Referring to the proof of Theorem 3.4 we get the following changes caused by conditions of Remark 3.8:

$$|u_i(t-\tau_\lambda(t,x,y),y)-u(t-\tau_{\lambda_0}(t,x,y),y)|\to 0$$

uniformly on $(((-\infty,b]\times\Omega)\setminus\Theta_\lambda)\times R^n$ ($\mu(\Theta_\lambda)\to 0$) for each $x\in\Omega$, as $\|u_i-u\|_{C((-\infty,b],BC(\Omega,R^n))}\to 0$ and $\lambda\to\lambda_0$.

Choose an arbitrary $\varepsilon>0$. Define the piecewise constant function $\overline{u}:(-\infty,b]\times R^n\to R^n$ as $\overline{u}(t,x)\in R^n$ for $t\in(-\infty,b]$, $x\in\Omega$ such that

$$\begin{cases} |\overline{u}(t,x)-u(t,x)|\le\varepsilon/2, \text{ if } |\overline{u}(t,x)|>|u(t,x)|;\\ |\overline{u}(t,x)-u(t,x)|<\varepsilon/2, \text{ if } |\overline{u}(t,x)|<|u(t,x)|.\end{cases}$$

Using this function, we get the estimate

$$\Big|f_\lambda\Big(u_i(t-\tau_\lambda(t,x,y),y)\Big)-f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)\Big|\le$$

$$\le\Big|f_\lambda\Big(u_i(t-\tau_\lambda(t,x,y),y)\Big)-f_\lambda\Big(\overline{u}(t-\tau_{\lambda_0}(t,x,y),y)\Big)\Big|+$$

$$+\Big|f_\lambda\Big(\overline{u}(t-\tau_{\lambda_0}(t,x,y),y)\Big)-f_{\lambda_0}\Big(\overline{u}(t-\tau_{\lambda_0}(t,x,y),y)\Big)\Big|+$$

$$+\Big|f_{\lambda_0}\Big(\overline{u}(t-\tau_{\lambda_0}(t,x,y),y)\Big)-f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)\Big|.$$

Using the function $\overline{u}$, it is easy to conclude that the first and the third terms on the right-hand side of this inequality are less or equal to 2ε and ε, respectively, on $(((-\infty,b]\times\Omega)\setminus\Theta_\lambda)\times R^n$, where $\mu(\Theta_\lambda)\to 0$ as $\lambda\to\lambda_0$. In addition, the condition 4′) provide convergence to 0 of the second term on the right-hand side of the inequality as $\lambda\to\lambda_0$.

Using the convergence obtained above, $(\mathcal{A}'_\lambda 3)$, $(\mathcal{A}_\lambda 4)$, and conditions 2′) and 3′), we get

$$\max_{t\in(-\infty,b],\ x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\Omega}W_\lambda(t,s,x,y)f_\lambda\Big(u_i(t-\tau_\lambda(t,x,y),y)\Big)dyds-$$

$$\int_{-\infty}^{t}\int_{\Omega}W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)dyds\Big|\le$$

$$\max_{t\in(-\infty,b],\ x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\{x\in\Omega,|x|\le r'\}}W_\lambda(t,s,x,y)f_\lambda\Big(u_i(t-\tau_\lambda(t,x,y),y)\Big)dyds-$$

$$\int_{-\infty}^{t}\int_{\{x\in\Omega,|x|\le r'\}}W_\lambda(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)dyds\Big|+$$

$$+\max_{t\in(-\infty,b],\ x\in\Omega}\Big|\int_{-\infty}^{t}\int_{\{x\in\Omega,|x|\le r'\}}W_\lambda(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)dyds-$$

$$\int_{-\infty}^{t}\int_{\{x\in\Omega,|x|\le r'\}}W_{\lambda_0}(t,s,x,y)f_{\lambda_0}\Big(u(t-\tau_{\lambda_0}(t,x,y),y)\Big)dyds\Big|+\epsilon_{r'}(t,x).$$

Here $\epsilon_{r'}(t,x) \to 0$ uniformly as $r' \to \infty$. Taking into account the condition 3′), we have the first term on the right-hand side of this inequality going to 0 as $\lambda \to \lambda_0$. The second term on the right-hand side of the inequality goes to 0 by the virtue of the conditions 2′) as $\lambda \to \lambda_0$. Thus, the statement in Remark 3.8 is valid.

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PAPER IV

Memoirs on Differential Equations and Mathematical Physics
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EXISTENCE, UNIQUENESS AND CONTINUOUS DEPENDENCE ON PARAMETERS OF SOLUTIONS TO NEURAL FIELD EQUATIONS

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. We obtain conditions for the existence and uniqueness of solutions to generalized neural field equations involving parameterized measure. We study continuous dependence of these solutions on the spatiotemporal integration kernel, delay effects, firing rate, external input and measure. We also construct the connection between the delayed Amari and Hopfield network models.

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Key words and phrases. Neural field equations, Hopfield networks, well-posedness.

Introduction

The main object of our study is the following parameterized integral equation involving integration with respect to an arbitrary measure:

$$u(t,x,\lambda)$$
$$= \int\limits_{-\infty}^{t} ds \int\limits_{\Omega} W(t,s,x,y,\lambda) f\big(u(s-\tau(s,x,y,\lambda),y,\lambda),\lambda\big)\nu(dy,\lambda)$$
$$+I(t,x,\lambda), \quad t>a, \quad x\in\Omega, \quad \lambda\in\Lambda \tag{1}$$

with the initial (prehistory) condition

$$u(\xi,x,\lambda)=\varphi(\xi,x,\lambda), \quad \xi\le a, \quad x\in\Omega, \quad \lambda\in\Lambda. \tag{2}$$

Here, the function u represents the activity of a neural element at time t and position x. The generalized spatio-temporal connectivity kernel W determines the time-dependent coupling between elements at positions x and y. The non-negative activation function f gives the firing rate of a neuron with activity u. The non-negative function τ represents the time-dependent axonal delay effects in the neural field, which require a prehistory condition given by the function φ. The function $I(t,x)$ represents a variable external input. All the above functions involve a parametrization by the parameter λ which, as well as introducing of an arbitrary parameterized measure $\nu(\,\cdot\,,\lambda)$, gives us some investigation advantages.

The equation (1) covers a wide variety of neural field models:

The most well-known Amari model [1]

$$\partial_t u(t,x) = -u(t,x) + \int\limits_{R} \omega(x-y) f(u(t,y))dy + I(t,x), \quad t\ge 0, \quad x\in R,$$

can be obtained from the equation (1) by taking

$$W(t,s,x,y,\lambda) = \exp\big(-(t-s)\big)\omega(x-y),$$
$$\tau(t,x,y,\lambda) = \varphi(\xi,x,\lambda)\equiv 0.$$

The two-population Amari model (see [2], [16])

$$\begin{pmatrix}\partial_t u_e\\ \alpha\partial_t u_i\end{pmatrix}(t,x) = -\begin{pmatrix}u_e\\ u_i\end{pmatrix}(t,x)$$
$$+\int\limits_{R}\begin{pmatrix}\omega_{ee} & -\omega_{ei}\\ \omega_{ie} & -\omega_{ii}\end{pmatrix}(x-y)\begin{pmatrix}f_e(u_e(t,x))\\ f_i(u_i(t,x))\end{pmatrix}dy$$
$$+\begin{pmatrix}I_e\\ I_i\end{pmatrix}(t,x), \quad t\ge 0, \quad x\in R,$$

can be obtained from the equation (1) by taking

$$W(t,s,x,y,\lambda) = \operatorname{diag}\Big(\exp(-(t-s)), \exp\big(-(t-s)/\alpha\big)/\alpha\Big)\begin{pmatrix}\omega_{ee} & -\omega_{ei}\\ \omega_{ie} & -\omega_{ii}\end{pmatrix}(x-y),$$
$$\tau(t,x,y,\lambda) = \varphi(\xi,x,\lambda) \equiv 0.$$

The delayed Amari model (see e.g. [5])

$$\partial_t u(t,x) = -Lu(t,x) + \int_\Omega \omega(t,x,y) f\big(u(t-\tau(x,y),y)\big)\,dy + I(t,x),$$
$$t \in \Big[-\max_{x,y\in\overline{\Omega}} \tau(x,y), \infty\Big),\quad x\in\Omega\subset B_{R^m}(0,r),\quad L = \operatorname{diag}(l_1,\dots,l_n),\quad l_i > 0$$

with a time-dependent connectivity kernel is also a special case of the model (1) with

$$W(t,s,x,y,\lambda) = \operatorname{diag}\Big(l_1\exp\big(-l_1(t-s)\big),\dots,l_n\exp\big(-l_n(t-s)\big)\Big)\omega(t,x,y),$$
$$\tau(t,x,y,\lambda) = \tau(x,y),\quad \varphi(\xi,x,\lambda)\equiv 0.$$

Another special case of the equation (1) arises in models that take into account the microstructure of the neural field (see [4, 9, 13])

$$\begin{gathered}\partial_t u(t,x) = -u(t,x) + \int_{R^m} \omega^\varepsilon(x-y) f(u(t,y))\,dy,\\ \omega^\varepsilon(x) = \omega(x,x/\varepsilon),\quad 0<\varepsilon\ll 1,\\ t\ge 0,\quad x\in R^m.\end{gathered} \tag{3}$$

If the microstructure is periodic, then, as the heterogeneity parameter $\varepsilon \to 0$, the above model converges (see e.g. [12]) to the homogenized Amari model

$$\partial_t u(t,x_c,x_f) = -u(t,x_c,x_f) + \int_{R^m}\int_{\mathcal{Y}} \omega(x_c-y_c, x_f-y_f) f\big(u(t,y_c,y_f)\big)\,dy_c\,dy_f, \tag{4}$$
$$t>0,\quad x_c\in R^m,\quad x_f\in\mathcal{Y}\subset R^k,$$

where x_c and x_f are the coarse-scale and fine-scale spatial variables, respectively. Taking

$$\Omega = R^m\times\mathcal{Y}\ \ (\mathcal{Y} \text{ is some } k\text{-dimensional torus [15]}),$$
$$x=(x_c,x_f),\quad y=(y_c,y_f),$$
$$W(t,s,x,y,\lambda) = \exp\big(-(t-s)\big)\omega(x_c-y_c, x_f-y_f)$$

in (1) with

$$\tau(t,x,y,\lambda) = \varphi(\xi,x,\lambda)\equiv 0,$$

we get the model (4). It should be pointed out here that the case of non-periodic microstructure in the model (3) that leads (see [12]) to non-Lebesgue measure in (4) is also covered by (1). It is more realistic to assume some small deviations from the periodicity in the neural networks structure reflected in the properties of the connectivity kernel with respect to the second argument. Hence, it is natural to ask whether the solution of the model (3) with a non-periodic perturbation of the periodic connectivity kernel in some sense is "close" to the solution in the non-perturbed case. One possible answer to this question is suggested in Appendix. The answer is based on the main result of the paper which is the existence, uniqueness and continuous dependence of solutions to (1) on the model parameters.

Another application of the main result is the possibility to connect the models in use in the neural field theory to the well-known Hopfield network model [8] utilizing the parameterized measure involved in (1). As the network models of the Hopfield type are used for numerical simulations of the neural fields, our results thus justify implementation of such numerical schemes.

The paper is organized in the following way. In Section 1 a special case (that is relevant in the neural field theory) of the general statement on the solvability and continuous dependence on a parameter of solutions to the Volterra operator equation from the paper [3] is given. Based on this theorem, analogous results are obtained in Section 2 for the generalized neural field model (1). Section 3 is devoted to the connection between the delayed Amari and Hopfield network models. In addition, a mathematical justification of the two known numerical schemes is offered, which illustrates a generality of the methods suggested in the paper. Finally, Appendix contains a short informal description of the homogenization procedure for the neural field equations with non-periodic microstructure based on the convergence of Banach algebras with mean value.

1. Preliminaries

In this section we provide an overview of the notation used in the paper, introduce the main definitions and formulate a fixed point theorem for locally contracting Volterra operators.

Let us introduce the following notations:

- R^m is the m-dimensional real vector space with the norm $|\cdot|$;
- Λ is some metric space;
- $B_\Lambda(\lambda_0, r)$ is the ball in the space Λ of the radius $r > 0$ centered at the point $\lambda_0 \in \Lambda$;
- Ω is a closed subset of R^m;
- $\partial\Omega$ is the boundary of the Ω;
- $\Omega_r = \Omega \cap B_{R^m}(0, r)$;

– $BC(\Omega, R^n)$ is the space of bounded continuous functions $\vartheta : \Omega \to R^n$ with the norm $\|\vartheta\|_{BC(\Omega,R^n)} = \sup_{x\in\Omega} |\vartheta(x)|$;

– $C_{comp}(\Omega, R^n)$ is the locally convex space of continuous functions $\vartheta : \Omega \to R^n$, with a compact support, equipped with the topology of uniform convergence on compact subsets;

– $Y(\mathbb{I}) = C(\mathbb{I}, BC(\Omega, R^n))$ consists of all continuous functions $\upsilon : \mathbb{I} \to BC(\Omega, R^n)$, with the norm $\|\upsilon\|_{Y(\mathbb{I})} = \max_{t\in\mathbb{I}} \|\upsilon(t)\|_{BC(\Omega,R^n)}$ if $\mathbb{I}$ is compact; if $\mathbb{I}$ is not compact, then $Y(\mathbb{I})$ is a locally convex linear space equipped with the topology of uniform convergence on compact subsets of $\mathbb{I}$;

Let $[a, b]$ be a compact subinterval of the real line. In the three forthcoming definitions we use the following notation: $Y = Y([a, b])$, $Y_\xi = Y([a, a + \xi])$ for any $\xi \in (0, b-a)$.

Definition 1. An operator $\Psi : Y \to Y$ is said to be a Volterra operator if for any $\xi \in (0, b-a)$ and any $y_1, y_2 \in Y$ the equality $y_1(t) = y_2(t)$ on $[a, a+\xi]$ implies that $(\Psi y_1)(t) = (\Psi y_2)(t)$ on $[a, a+\xi]$.

Choosing an arbitrary $\xi \in (0, b-a)$, we introduce the following three important operators. Let $E_\xi : Y \to Y_\xi$ be defined as $(E_\xi y)(t) = y_\xi(t)$, $t \in [a, a+\xi]$, where $y_\xi(t)$ is a restriction of the function $y(t)$ to the subinterval $[a, a + \xi]$; conversely, to each $y_\xi \in Y_\xi$ the operator $P_\xi : Y_\xi \to Y$ assigns one of the extensions $y \in Y$ of the element y_ξ (P_ξ may not be uniquely defined); the operator $\Psi_\xi : Y_\xi \to Y_\xi$ is given by $\Psi_\xi y_\xi = E_\xi \Psi P_\xi y_\xi$. Note that for any Volterra operator $\Psi : Y \to Y$ the operator $\Psi_\xi : Y_\xi \to Y_\xi$ is also a Volterra operator and is independent of the choice of P_ξ.

Definition 2. A Volterra operator $\Psi : Y \to Y$ is called locally contracting if there exist $q < 1$, $\theta > 0$, such that for all elements $y_1, y_2 \in Y$ the following two conditions are satisfied:

$\mathfrak{q}_1$) $\|E_\theta \Psi y_1 - E_\theta \Psi y_2\|_{Y_\theta} \le q\|E_\theta y_1 - E_\theta y_2\|_{Y_\theta}$,

$\mathfrak{q}_2$) for any $\gamma \in [0, b-a-\theta]$, the equality $E_\gamma y_1 = E_\gamma y_2$ implies that

$$\big\|E_{\gamma+\theta} \Psi y_1 - E_{\gamma+\theta} \Psi y_2\big\|_{Y_{\gamma+\theta}} \le q\big\|E_{\gamma+\theta} y_1 - E_{\gamma+\theta} y_1\big\|_{Y_{\gamma+\theta}}. \tag{5}$$

Definition 3. If there exists $\gamma \in (0, b-a]$ and a function $y_\gamma \in Y_\gamma$, which satisfies the equation $\Psi_\gamma y_\gamma = y_\gamma$, then we call y_γ a local solution of the Volterra equation

$$y(t) = (\Psi y)(t), \quad t \in [a, b]. \tag{6}$$

In the case if $\gamma = b - a$, we call this solution global (relative to the interval $[a, b]$).

To study continuous dependence on a parameter, we need some more definitions.

Definition 4. Let $F(\cdot, \cdot) : Y \times \Lambda \to Y$ be a family of Volterra operators depending on a parameter $\lambda \in \Lambda$. This family is called uniformly locally contracting if for each $\lambda \in \Lambda$ the operator $F(\cdot, \lambda)$ is locally contracting and the constants $q \geq 0$ and $\theta > 0$ from Definition 3, are independent of $\lambda \in \Lambda$.

The following theorem concerning the well-posedness of the operator equation

$$y(t) = (F(y, \lambda))(t), \quad t \in [a, b], \quad \lambda \in \Lambda, \tag{7}$$

is a special case of Theorem 1 in Burlakov, et al [3]. It represents the main theoretical tool for the problems to be studied in this paper.

Theorem 1. *Assume that for some $\lambda_0 \in \Lambda$ and $r_0 > 0$, the family of Volterra operators $F(\cdot, \lambda) : Y \to Y$ ($\lambda \in B_\Lambda(\lambda_0, r_0)$) is uniformly locally contracting and the mapping $F(\cdot, \cdot) : Y \times \Lambda \to Y$ is continuous at (y, λ_0) for all $y \in Y$.*

Then there exists $r > 0$, such that the equation (7) has a unique global solution $y(t, \lambda)$ for all $\lambda \in B_\Lambda(\lambda_0, r)$, and

$$\|y(\cdot, \lambda) - y(\cdot, \lambda_0)\|_Y \to 0 \quad as \quad \lambda \to \lambda_0.$$

Moreover, for each $\lambda \in B_\Lambda(\lambda_0, r)$, any local solution of the equation (7) is also unique and is a restriction of the corresponding global solution.

2. The Main Result

In this section we justify the property of well-posedness for the generalized neural field equation (1).

The following assumptions will be imposed on the functions involved:

(**A1**) The function $f : R^n \times \Lambda \to R^n$ is continuous, bounded and Lipschitz one in the first variable uniformly with respect to $\lambda \in \Lambda$.

(**A2**) For any $b \in R$ and $r > 0$, the delay function $\tau : (-\infty, b] \times \Omega \times \Omega_r \times \Lambda_c \to [0, \infty)$ is uniformly continuous, where Λ_c is some compact subset of Λ.

(**A3**) The initial (prehistory) function $\varphi : (-\infty, a] \times \Omega \times \Lambda_c \to R^n$ is uniformly continuous.

(**A4**) The external input function $I : [a, \infty) \times \Omega \times \Lambda \to R^n$ generates a continuous mapping $\lambda \mapsto I(\cdot, \cdot, \lambda)$ from Λ to the space $Y[a, \infty)$.

(**A5**) For any $b > a$ and $r > 0$, the kernel function $W : [a, b] \times [-r, r] \times \Omega \times \Omega_r \times \Lambda_c \to R^n$ is uniformly continuous.

(**A6**) The complete σ-additive measures $\nu(\cdot, \lambda)$ ($\lambda \in \Lambda$) are finite on compact subsets of Ω and weakly continuous with respect to $\lambda \in \Lambda$ i.e. the measures can be interpreted as linear functionals on the separable locally convex space $C_{comp}(\Omega, R^n)$.

(**A7**) For any $b > a$,

$$\max_{t\in[a,b]} \Bigg(\int_{-\infty}^{t} ds \sup_{x\in\Omega,\lambda\in\Lambda} \int_{\Omega} \big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda)\Bigg) < \infty.$$

(**A8**) For any $b > a$,

$$\lim_{r\to\infty} \sup_{t\in[a,b],\, x\in\Omega,\, \lambda\in\Lambda} \int_{-\infty}^{t} ds \int_{\Omega-\Omega_r} \big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda) = 0.$$

Definition 5. Let $\lambda \in \Lambda$. We define a local solution to the problem (1), (2) on $[a, a+\gamma]\times R^n$, $\gamma \in (0,\infty)$, to be a function $u_\gamma \in Y([a, a+\gamma])$ that satisfies the equation (1) on $[a, a+\gamma]$ and the prehistory condition (2). We define a global solution to the problem (1), (2) to be a function $u \in Y([a,\infty))$, whose restriction u_γ to $[a, a+\gamma]$ is its local solution for any $\gamma \in (0,\infty)$.

Theorem 2. *Suppose that the assumptions* (**A1**)–(**A8**) *are fulfilled. Then the initial value problem* (1), (2) *has a unique continuous solution* $u(\,\cdot\,,\,\cdot\,,\lambda)\in Y([a,\infty))$ *for any* $\lambda \in \Lambda$*, and the correspondence* $\lambda \mapsto u(\,\cdot\,,\,\cdot\,,\lambda)$ *is a continuous mapping from* Λ *to* $Y([a,\infty))$*. Moreover, for each* $\lambda \in \Lambda$*, any local solution of the problem* (1), (2) *is also unique and it is a restriction of the corresponding global solution.*

Proof. Due to the definition of the topology in $Y([a,\infty))$, it suffices to prove this result for the case of an arbitrary compact interval $[a,b] \subset [a,\infty)$. In what follows we therefore keep fixed an arbitrary $b > a$ and keep the notation Y for the space $Y([a,b])$.

For each $\lambda \in \Lambda$ and $\varphi(\xi, x, \lambda)$ satisfying the assumption (**A3**) we define the following integral operator

$$(F(u,\lambda))(t,x) = I_1(t,x,\lambda) + I_2(t,x,\lambda) + \int_{a}^{t} ds \int_{\Omega} W(t,s,x,y,\lambda) f\big((S(u,\lambda))(t,s,x,y,\lambda),\lambda\big)\nu(dy,\lambda), \quad (8)$$

where

$$(S(u,\lambda))(t,x,y,\lambda) = \begin{cases} \varphi(t-\tau(t,x,y,\lambda),x,\lambda) & \text{if } t-\tau(t,x,y,\lambda) < a, \\ u(t-\tau(t,x,y,\lambda),y,\lambda) & \text{if } t-\tau(t,x,y,\lambda) \ge a, \end{cases} \quad (9)$$

and

$$I_1(t,x,\lambda) = \varphi(a,x,\lambda) + I(t,x,\lambda),$$

$$I_2(t,x,\lambda) = \int_{-\infty}^{a} ds \int_{\Omega} W(t,s,x,y,\lambda) f\big(\varphi(s-\tau(s,x,y,\lambda),x,\lambda),\lambda\big)\nu(dy,\lambda).$$

Below we assume that $|f(u)| \le M$ for all $u \in R^n$.

We have to apply Theorem 1. Towards this end, we need to show that the operator family $F(\,\cdot\,,\lambda)$ $(\lambda \in \Lambda)$ satisfies the assumptions of this theorem.

At the first step of the proof we will show that $F(u,\lambda) \in Y$ for each $u \in Y$, $\lambda \in \Lambda$. Applying the assumption (**A8**) for the given $\varepsilon > 0$, we find $r > 0$ such that

$$\sup_{t\in[a,b],\ x\in\Omega,\ \lambda\in\Lambda} \int\limits_{-\infty}^{t} ds \int\limits_{\Omega-\Omega_r} \big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda) < \frac{\varepsilon}{M}\,. \tag{10}$$

For this r and a fixed $\lambda \in \Lambda$, we find a positive $\delta = \delta(\lambda)$ (u is kept fixed) such that

$$\Big|W(t,s,x,y,\lambda)f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big) - W(t_0,s_0,x_0,y_0,\lambda)f\big((S(u,\lambda))(s_0,x_0,y_0,\lambda),\lambda\big)\Big| < \frac{\varepsilon}{((b-a)\nu(\Omega_r,\lambda))} \tag{11}$$

for all $t, t_0, s, s_0 \in [a,b]$, $x, x_0 \in \Omega$, $y, y_0 \in \Omega_r$, satisfying

$$|t-t_0| < \delta, \quad |s-s_0| < \delta, \quad |x-x_0| < \delta, \quad |y-y_0| < \delta.$$

We show first that $F(\,\cdot\,,\lambda) : Y \to Y$ for each $\lambda \in \Lambda$. In other words, we have to prove that the mapping $t \mapsto (F(u,\lambda))(t,\,\cdot\,)$ is a continuous function from $[a,b]$ to $BC(\Omega, R^n)$.

As the assumptions (**A3**), (**A4**) imply $\varphi(a,\,\cdot\,,\lambda) \in BC(\Omega,R^n)$ and $I(\,\cdot\,,\,\cdot\,,\lambda) \in Y$ $(\lambda \in \Lambda)$, we only need to check that $I_2(\,\cdot\,,\,\cdot\,,\lambda) \in Y$ and $F_0(u,\lambda) \in Y$ for all $u \in Y$ and $\lambda \in \Lambda$, where

$$(F_0(u,\lambda))(t,x) = \int\limits_a^t ds \int\limits_\Omega W(t,s,x,y,\lambda)f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)\nu(dy,\lambda).$$

The proofs are similar, so we concentrate on the more involved case of F_0.

For any $t \in [a,b]$, we have

$$\begin{aligned}\big|(F_0(u,\lambda))(t,x) &- (F_0(u,\lambda))(t,x_0)\big| \\ &\le \int\limits_a^t ds \int\limits_{\Omega_r} \Big|W(t,s,x,y,\lambda)f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big) \\ &\quad - W(t,s,x_0,y,\lambda)f\big((S(u,\lambda))(s,x_0,y,\lambda),\lambda\big)\Big|\nu(dy,\lambda) \\ &\quad + \int\limits_a^b ds \int\limits_{\Omega-\Omega_r} \Big(\big|W(t,s,x,y,\lambda)\big| + \big|W(t,s,x_0,y,\lambda)\big|\Big)\nu(dy,\lambda) < 3\varepsilon\end{aligned}$$

as long as $|x-x_0| < \delta = \delta(\lambda)$ due to the estimates (10) and (11). This proves the continuity of $(F_0(u,\lambda))(t,x)$ in x.

The boundedness of this function for each $t \in [a,b]$ follows from the assumption (**A7**) and boundedness of the function $f : R^n \to R^n$.

Finally, we check that $t \mapsto (F_0(u,\lambda))(t,\,\cdot\,)$ is a continuous mapping from $[a,b]$ to $BC(\Omega, R^n)$ if $u \in Y$:

$$\sup_{x\in\Omega}\big|(F_0(u,\lambda))(t,x)-(F_0(u,\lambda))(t_0,x)\big|$$

$$\le \sup_{x\in\Omega}\Big|\int_a^t ds\int_\Omega W(t,s,x,y,\lambda)f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)$$

$$-\int_a^{t_0} ds\int_\Omega W(t_0,s,x,y,\lambda)f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)\Big|\nu(dy,\lambda)$$

$$\le \int_{t_0}^t ds\sup_{x\in\Omega}\int_\Omega \big|W(t,s,x,y,\lambda)\big|M\nu(dy,\lambda) < \varepsilon$$

as long as $t-t_0<\delta$. (Here we have assumed that $t>t_0$ and again used the assumption (**A7**).) We have therefore proved that $F_0(\,\cdot\,,\lambda), F(\,\cdot\,,\lambda) : Y \to Y$ for each $\lambda \in \Lambda$.

At the second step of the proof we show that the Volterra operator (8) is a local contraction in the first variable, uniformly with respect to the parameter λ.

We choose arbitrary constants $q<1$, $\gamma\in[0,b-a)$ and $\lambda\in\Lambda$. Let $\widetilde{f}$ be the Lipschitz constant for the function f. Since the space Y consists of continuous functions, we can unify the two properties from Definition 2 into a single one and prove that $u_1(t,\,\cdot\,)=u_2(t,\,\cdot\,)$, $t\in[a,a+\gamma)$, where $u_1,u_2\in Y$, implies the inequality (5) for the chosen $q<1$ and some $\theta>0$. Indeed,

$$\big\|F(u_1,\lambda)-F(u_2,\lambda)\big\|_Y$$

$$=\sup_{t\in[a,a+\gamma+\theta],\;x\in\Omega}\Big|\int_a^t ds\int_\Omega W(t,s,x,y,\lambda)f\big((S(u_1,\lambda))(s,x,y,\lambda)\big)\nu(dy,\lambda)$$

$$-\int_a^t ds\int_\Omega W(t,s,x,y,\lambda)f\big((S(u_2,\lambda))(s,x,y,\lambda)\big)\nu(dy,\lambda)\Big|$$

$$\le \sup_{t\in[a+\gamma,a+\gamma+\theta],\;x\in\Omega}\Big|\int_{a+\gamma}^t ds\int_\Omega W(t,s,x,y,\lambda)\Big(f\big((S(u_1,\lambda))(s,x,y,\lambda)\big)$$

$$-f\big((S(u_2,\lambda))(s,x,y,\lambda)\big)\Big)\nu(dy,\lambda)\Big|$$

$$\leq \sup_{t\in[a+\gamma,a+\gamma+\theta],\ x\in\Omega} \int\limits_{a+\gamma}^{t} ds \int\limits_{\Omega} \big|W(t,s,x,y,\lambda)\big|\widetilde{f}\nu(dy,\lambda)\|u_1-u_2\|_Y$$
$$\leq \widetilde{q}\|u_1-u_2\|_Y,$$

where

$$\widetilde{q} = \widetilde{f} \sup_{t\in[a+\gamma,a+\gamma+\theta],\ x\in\Omega} \int\limits_{a+\gamma}^{t} ds \int\limits_{\Omega} \big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda).$$

Using the assumption (**A7**), we can always find a $\theta > 0$ such that $\widetilde{q} \leq q < 1$. This proves the property of local contractivity of the operator $F(\,\cdot\,,\lambda) : Y \to Y$ for any $\lambda \in \Lambda$. Moreover, we easily obtain from $\gamma \in [0, b-a)$ the estimate on $\widetilde{q}$ that this property is uniform with respect to γ and λ, i.e. $\theta > 0$ and $q < 1$ can be chosen to be independent of $\gamma \in [0, b-a)$ and $\lambda \in \Lambda$.

At the third and final step of the proof we show the continuity of the mapping $F : Y \times \Lambda \to Y$. We pick arbitrary $\lambda_0 \in \Lambda$, $u_0 \in Y$, where continuity will be examined, and arbitrary sequences $\lambda_N \to \lambda_0$, $u_N \to u_0$ $(N \to \infty)$.

We start with estimation of the following difference:

$$\big|(S(u_N,\lambda_N))(s,x,y,\lambda_N) - (S(u_0,\lambda_0))(s,x,y,\lambda_0)\big|$$
$$\leq \big|(S(u_N,\lambda_N))(s,x,y,\lambda_N) - (S(u_0,\lambda_N))(s,x,y,\lambda_0)\big|$$
$$+ \big|(S(u_0,\lambda_N))(s,x,y,\lambda_0) - (S(u_0,\lambda_0))(s,x,y,\lambda_0)\big|.$$

The first term on the right-hand side of this inequality is less than $\varepsilon/2$ for all $s \in (-\infty, b]$, $x, y \in \Omega$, $N \geq N_1$ as $u_N \to u_0$ $(N \to \infty)$. By virtue of the assumptions (**A2**) and (**A3**), the second term on the right-hand side is less than $\varepsilon/2$ for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_r$, $N \geq N_2(r)$. Thus, for any $r > 0$, we have

$$\big|(S(u_N,\lambda_N))(s,x,y,\lambda_N) - (S(u_0,\lambda_0))(s,x,y,\lambda_0)\big| \leq \varepsilon \tag{12}$$

for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_r$, $N \geq N_3(r)$.

Then, choosing $\varepsilon > 0$, we find a number $r_0 > 0$ such that the estimate (10) holds true. Increasing, if necessary, the value of r_0, we may, in addition, assume without loss of generality that $\nu(\Omega_{r_0}, \lambda_0) > 0$ and $\nu(\partial\Omega_{r_0}, \lambda_0) = 0$, so that

$$\lim_{N\to\infty} \nu(\Omega_{r_0}, \lambda_N) = \nu(\Omega_{r_0}, \lambda_0)$$

(see e.g. [7, Chapter VI, Theorem 2]).

Using this r_0, we estimate the following difference:

$$\Big|f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big|$$
$$\leq \Big|f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) - f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big)\Big|$$
$$+ \Big|f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_0\big) - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big|.$$

By virtue of the assumption (**A1**), the first term on the right-hand side of the inequality is less than ε for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_4(r_0)$. Using the assumption (**A1**) and the estimate (12), we get that the second term on the right-hand side of the inequality is less than ε for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_3(r_0)$. Thus, taking into account (**A1**) and (**A7**), we obtain the inequality

$$\Big| \int\limits_{-\infty}^{t} ds \int\limits_{\Omega_{r_0}} W(t,s,x,y,\lambda_N)\Big(f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) \\ - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big)\nu(dy,\lambda_N)\Big| < \varepsilon \quad (13)$$

for all $t \in [a,b]$, $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_5(r_0)$.

The assumption (**A5**) yields

$$\big|W(t,s,x,y,\lambda_N) - W(t,s,x,y,\lambda_0)\big| < \frac{\varepsilon}{M((b-a)\nu(\Omega_r,\lambda))} \quad (14)$$

for all $t \in [a,b]$, $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_6(r_0)$.

Using the assumptions (**A3**), (**A4**), and (**A6**), we find a natural number $N_7(r_0)$ such that

$$\begin{gathered} \sup_{t\in[a,b],\ x\in\Omega} \Big| \int\limits_{\Omega_{r_0}} \Phi(t,x,y)\big(\nu(dy,\lambda_N) - \nu(dy,\lambda_0)\big)\Big| < \varepsilon, \\ \nu(\Omega_{r_0},\lambda_N) \leq 2\nu(\Omega_{r_0},\lambda_0), \\ \sup_{x\in\Omega} \big|\varphi(a,x,\lambda_N) - \varphi(a,x,\lambda_0)\big| < \varepsilon, \\ \sup_{t\in[a,b],\ x\in\Omega} \big|I(t,x,\lambda_N) - I(t,x,\lambda_0)\big| < \varepsilon, \ \ |\lambda_N - \lambda_0| < \delta \end{gathered} \quad (15)$$

for all $N \geq N_7(r_0)$. Here, the function

$$\Phi(t,x,y) = \int\limits_{-\infty}^{t} W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\, ds$$

is uniformly continuous on the set $[a,b] \times \Omega \times \Omega_{r_0}$, so that

$$\int\limits_{\Omega_{r_0}} \Phi(t,x,y)\nu(dy,\lambda_N) \longrightarrow \int\limits_{\Omega_{r_0}} \Phi(t,x,y)\nu(dy,\lambda_0)$$

as $n \to \infty$ uniformly with respect to the variables $t \in [a,b]$, $x \in \Omega$.

Next, we estimate

$$\begin{aligned}
&\sup_{t\in[a,b],\ x\in\Omega} \big|I_2(t,x,\lambda_N) - I_2(t,x,\lambda_0)\big| \\
&\le \sup_{t\in[a,b],\ x\in\Omega} \Bigg| \int_{-\infty}^{t} ds \int_{\Omega} W(t,s,x,y,\lambda_N) \\
&\times f\Big(\varphi\big(s-\tau(s,x,y,\lambda_N),x,\lambda_N\big),\lambda_N\Big)\nu(dy,\lambda_N) \\
&- \int_{-\infty}^{t} ds \int_{\Omega} W(t,s,x,y,\lambda_0) f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg| \\
&\le \sup_{t\in[a,b],\ x\in\Omega} \Bigg| \int_{-\infty}^{t} ds \int_{\Omega-\Omega_{r_0}} W(t,s,x,y,\lambda_N) \\
&\times f\big(\varphi(s-\tau(s,x,y,\lambda_N),x,\lambda_N),\lambda_N\big)\nu(dy,\lambda_N) \\
&- \int_{-\infty}^{t} ds \int_{\Omega-\Omega_{r_0}} W(t,s,x,y,\lambda_0) f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg| \\
&+ \sup_{t\in[a,b],\ x\in\Omega} \Bigg| \int_{-\infty}^{t} ds \int_{\Omega_{r_0}} W(t,s,x,y,\lambda_N) \Big(f\big(\varphi(s-\tau(s,x,y,\lambda_N),x,\lambda_N),\lambda_N\big) \\
&- f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\Big)\nu(dy,\lambda_N)\Bigg| \\
&+ \sup_{t\in[a,b],\ x\in\Omega} \Bigg| \int_{-\infty}^{t} ds \int_{\Omega_{r_0}} \big(W(t,s,x,y,\lambda_N) - W(t,s,x,y,\lambda_0)\big) \\
&\times f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N)\Bigg| \\
&+ \sup_{t\in[a,b],\ x\in\Omega} \Bigg| \int_{-\infty}^{t} ds \int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) \\
&\times f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N) \\
&- \int_{-\infty}^{t} ds \int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg|.
\end{aligned}$$

The first term on the right-hand side of the inequality is less than 2ε as the estimate (10) and the assumption (**A1**) hold true. Each of the second and the third terms on the right-hand side of the inequality is less than

ε due to (13) and (**A1**), (**A7**), (14), respectively, for all $N > N_8(r_0) = \max\{N_5(r_0), N_6(r_0)\}$. The estimate (15) yields the last term on the right-hand side of the inequality is less than ε for all $N > N_7(r_0)$.

Thus, we get that

$$\sup_{t\in[a,b],\ x\in\Omega} \left|I_2(t,x,\lambda_N) - I_2(t,x,\lambda_0)\right| < 5\varepsilon \tag{16}$$

for all $N \geq N_9(r_0) = \max\{N_7(r_0), N_8(r_0)\}$.

Finally, taking into account the estimates (10), (11), (13)–(16) and the assumption (**A7**), we obtain

$$\begin{aligned}
&\left\|F(u_N,\lambda_N) - F(u_0,\lambda_0)\right\|_Y \leq \sup_{x\in\Omega}\left|\varphi(a,x,\lambda_N) - \varphi(a,x,\lambda_0)\right| \\
&\quad + \sup_{t\in[a,b],\ x\in\Omega}\left|I(t,x,\lambda_N) - I(t,x,\lambda_0)\right| \\
&\qquad\qquad + \sup_{t\in[a,b],\ x\in\Omega}\left|I_2(t,x,\lambda_N) - I_2(t,x,\lambda_0)\right| \\
&\quad + \sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int_a^t ds\int_\Omega W(t,s,x,y,\lambda_N) \\
&\qquad\qquad \times f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big)\nu(dy,\lambda_N) \\
&\quad - \int_a^t ds\int_\Omega W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg| \\
&\leq 7\varepsilon + \sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_N) \\
&\qquad\qquad \times f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big)\nu(dy,\lambda_N) \\
&\quad - \int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg| + 2\varepsilon \\
&\leq 9\varepsilon + \sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_N) \\
&\quad \times \Big(f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) \\
&\qquad\qquad - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big)\nu(dy,\lambda_N)\Bigg| \\
&+ \sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int_a^t ds\int_{\Omega_{r_0}} \big(W(t,s,x,y,\lambda_N) - W(t,s,x,y,\lambda_0)\big)
\end{aligned}$$

$$\times f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N)\Bigg|$$

$$+\sup_{t\in[a,b],x\in\Omega}\Bigg|\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0)$$

$$\times f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N)$$

$$-\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0)f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg|$$

$$\le 10\varepsilon+(b-a)\nu(\Omega_{r_0},\lambda_N)\,\frac{\varepsilon}{((b-a)\nu(\Omega_{r_0},\lambda_0))}$$

$$+\sup_{t\in[a,b],\;x\in\Omega}\Bigg|\int_{\Omega_{r_0}}\Phi(t,x,y)(\nu(dy,\lambda_N)-\nu(dy,\lambda_0))\Bigg|<13\varepsilon$$

for all $N\ge N_9(r_0)$.

The proof is complete. □

Remark 1. If Ω is compact, then the assumption (**A8**) is fulfilled automatically and can therefore be omitted, while the assumptions (**A2**)–(**A5**) only require continuity of the corresponding functions instead of their uniform continuity in the variable x.

3. The Hopfield Model with Delay

In this section we prove convergence of the generalized Hopfield network to the Amari neural field equation.

Consider the following delayed Hopfield network model (see e.g. [14])

$$\dot z_i(t,N)=-\alpha z_i(t,N)+\sum_{j=1}^{N}\omega_{ij}(N)f\big(z_j(t-\tau_{ij}(t,N),N)\big)+\mathrm{J}_i(t,N),\quad t>a,\quad i=1,\dots,N,\tag{17}$$

parameterized by a natural parameter N. Here at each natural N, $z_i(\,\cdot\,,N)$ are n-dimensional vector functions, $\omega_{ij}(N)$ are real $n\times n$-matrices (connectivities), $\tau_{ij}(\,\cdot\,,N)$ are nonnegative real-valued continuous functions (axonal delays), $f:R^n\to R^n$ are firing rate functions which are Lipschitz and bounded and $\mathrm{J}_i(\,\cdot\,,N)$ are continuous external input n-dimensional vector functions.

The initial conditions for (17) are given as

$$z_i(\xi,N)=\varphi_i(\xi,N),\quad \xi\le a,\quad i=1,\dots,N.\tag{18}$$

We use the general well-posedness result from the previous section to justify the convergence of a sequence of the delayed Hopfield equations (17) (with the initial conditions (18)) to the Amari equation involving a spatio-temporal delay

$$\partial_t u(t,x) = -\alpha u(t,x) + \int_\Omega \omega(x,y) f(u(s-\tau(t,x,y),y))\nu(dy) + J(t,x), \quad t>a, \quad x\in\Omega, \tag{19}$$

with the initial (prehistory) condition

$$u(\xi,x) = \varphi(\xi,x), \quad \xi \le a, \quad x\in\Omega. \tag{20}$$

On the above functions we impose the following assumptions:

(**B1**) The function $f : R^n \to R^n$ is continuous, bounded and Lipschitz one.

(**B2**) The spatio-temporal delay $\tau : R\times\Omega\times\Omega \to [0,\infty)$ is continuous.

(**B3**) The initial (prehistory) function $\varphi : (-\infty, a]\times\Omega \to R^n$ is continuous.

(**B4**) For any $b > a$, the external input function $J : [a,b]\times\Omega \to R^n$ is uniformly continuous and bounded with respect to the second variable.

(**B5**) The kernel function $\omega : \Omega\times\Omega \to R^n$ is continuous.

(**B6**) $\nu(\,\cdot\,)$ is the Lebesgue measure on Ω.

(**B7**) For any $b > a$,

$$\sup_{x\in\Omega} \int_\Omega |\omega(x,y)|\nu(dy) < \infty.$$

(**B8**) For any $b > a$,

$$\lim_{r\to\infty} \sup_{x\in\Omega} \int_{\Omega-\Omega_r} |\omega(x,y)|\nu(dy) = 0.$$

The following theorem represents the main result of this section.

Theorem 3. *For each natural number N let $\{\Delta_i(N),\ i=1,\dots,N\}$ be a finite family of open subsets of Ω satisfying the conditions*

$$\bigcup_{i=1}^{N} \overline{\Delta}_i(N) = \Omega_N \quad \textit{and} \quad \lim_{N\to\infty} \textit{mesh}\,\big\{\Delta_i(N),\ i=1,\dots,N\big\} = 0.$$

Let $y_i(N)$ $(i=1,\dots,N)$ be arbitrary points in $\Delta_i(N)$. Finally, let the assumptions (**B1**)*–*(**B8**) *be fulfilled. Then the sequence of the solutions $z_i(t,N)$ $(t\in R)$ of the initial value problem* (17), (18)*, where the coefficients are defined by*

$$\begin{gathered}\omega_{ij}(N) = \beta_i(N)\omega(y_i(N), y_j(N)), \quad \textit{where } \beta_i(N) = \nu(\Delta_i(N)),\\ \tau_{ij}(t,N) = \tau(t, y_i(N), y_j(N)), \quad \mathrm{J}_i(t,N) = J(t, y_i(N)),\end{gathered} \tag{21}$$

converges for any $b > a$ to the solution $u(t,x)$ $(t \in R,\ x \in \Omega)$ of the initial value problem (19), (20) *as $N \to \infty$, in the following sense:*

$$\lim_{N\to\infty} \sup_{t\in[a,b]} \Big(\sup_{1\le i\le N} \Big(\sup_{x\in\Delta_i(N)} |u(t,x) - z_i(t,N)| \Big)\Big) = 0. \tag{22}$$

In order to prove this theorem, we will need to use the following statement.

Lemma 1. *Assume that for each natural number N we have a finite family of open subsets $\{\Delta_i(N),\ i = 1,\dots,N\}$ of Ω satisfying the conditions*

$$\bigcup_{i=1}^{N} \overline{\Delta}_i(N) = \Omega_N \quad \textit{and} \quad \lim_{N\to\infty} \textit{mesh}\big\{\Delta_i(N),\ i = 1,\dots,N\big\} = 0.$$

Let $y_i(N)$ $(i = 1,\dots,N)$ be arbitrary points in $\Delta_i(N)$, $\mathfrak{D}_i(N)$ be the Dirac measures at $y_i(N)$ and $\beta_i(N) = \nu(\Delta_i(N))$. Then the sequence of the discrete weighted measures

$$\nu_N = \sum_{i=1}^{N} \beta_i(N)\mathfrak{D}_i(N) \tag{23}$$

weakly converges (in the sense of the weak topology on the dual space to $C_{comp}(\Omega)$) to the Lebesgue measure on Ω.

Proof. We simply observe that for any continuous and compactly supported function $\Phi(x)$, $x \in \Omega$, we get

$$\int_\Omega \Phi(x)\nu_N(dx) = \sum_{i=1}^{N} \Phi\big(y_i(N)\big)\beta_i(N)$$
$$= \sum_{i=1}^{N} \Phi\big(y_i(N)\big)\nu\big(\Delta_i(N)\big) \longrightarrow \int_\Omega \Phi(x)\nu(dx), \tag{24}$$

as $N \to \infty$, due to the properties of the Riemann–Stiltjes integrals (see e.g. Chapter 2 in [11]). □

Proof of the Theorem 3. In order to apply Theorem 2, we first of all define the metric space $\Lambda = \{\lambda_N,\ N = 0,1,2,\dots\}$, where $\lambda_0 = \infty$, $\lambda_N = N$ for natural numbers N, and the distance is given by $d(\lambda_N,\lambda_M) = |1/N - 1/M|$ $(N, M \ne 0)$ and $d(\lambda_N,\lambda_0) = 1/N$ $(N \ne 0)$, so that $\lambda_N \to \lambda_0$ simply means that $N \to \infty$. Multiplication by the function $\eta(t-s)$, where $\eta(\sigma) = \exp(-\alpha\sigma)$, followed by integration, converts the equation (19) into the equation (1), where f, τ,

$$W(t,s,x,y) = \exp(-\alpha(t-s))\omega(x,y),$$
$$I(t,x) = \int_a^t \exp(-\alpha(t-s))J(s,x)\,ds$$

are all independent of λ, and the measures are defined as $\nu(\,\cdot\,,\lambda_N) = \nu_N$ (see (23)) and $\nu(\,\cdot\,,\lambda_0) = \nu$, respectively.

The assumptions (**A1**)–(**A5**) of Theorem 2 are trivial, the assumption (**A6**) is fulfilled due to Lemma 1 and the above definition of convergence in Λ.

Taking into account that

$$\max_{t\in[a,b]} \int_{-\infty}^{t} \exp(-\alpha(t-s))\,ds = \frac{1}{\alpha},$$

it is straightforward to check the assumptions (**A7**) and (**A8**).

From Theorem 2 it now follows that the solutions $u(t,x,N)$ of the initial boundary value problems

$$\partial_t u(t,x,N) = -\alpha u(t,x,N)$$
$$+ \int_{\Omega} \omega(x,y) f\big(u(s-\tau(t,x,y),y,N)\big)\nu_N(dy) + J(t,x), \;\; t > a, \;\; x \in \Omega, \quad (25)$$

with the initial (prehistory) condition

$$u(\xi,x,N) = \varphi(\xi,x), \;\; \xi \le a, \;\; x \in \Omega, \quad (26)$$

converge to the solution $u(t,x)$ ($t \in R$, $x \in \Omega$) of the initial value problem (19), (20), as $N \to \infty$, uniformly on $[a,b]\times\Omega$ for any $b > a$. Evidently, replacing x by $y_i(N)$ in the equation (25) and in the initial condition (26) yields the initial value problem (17), (18). It remains therefore to notice that the set $z_i(t,N) = u(t,y_i(N),N)$ ($i = 1,\ldots,N$) is a (unique) solution of the latter problem. □

The theoretical results of this section can be applied to justify numerical integration schemes. For example, Faye et al [5] considered discretization of the following delayed Amari model

$$\partial_t u(t,x) = -\alpha u(t,x) + \int_{\Omega} \omega\big(|x-y|\big) f\Big(u\Big(t - \frac{|x-y|}{v}, y\Big)\Big)\,dy \quad (27)$$

in the cases

I. $u(t,x) \in R$, $\Omega = [-L,L]$,

II. $u(t,x) \in R^2$, $\Omega = [-L,L]$,

III. $u(t,x) \in R$, $\Omega = [-L,L]^2$.

Faye et al have justified their numerical schemes using convergence of the trapezoidal integration rule and the rectangular method to the corresponding integrals. We will show how our results can be applied for the more involved case III:

$$\partial_t u_{ij}(t) = -\alpha u_{ij}(t) + \sum_{k=1}^{M}\sum_{l=1}^{M} \omega\big(|(x_i^1, x_j^2) - (x_k^1, x_l^2)|\big) \times f\Big(u_{kl}\Big(t - \frac{|(x_i^1, x_j^2) - (x_k^1, x_l^2)|}{v}\Big)\Big)\, dy. \quad (28)$$

Here,

$$x = (x^1, x^2), \quad u_{ij}(t) = u\big(t, (x_i^1, x_j^2)\big), \quad i, j = 1, \ldots, M.$$

Denoting

$$z_i(t) = u_{ij}(t), \quad \omega_{ij} = \omega\big(|(x_i^1, x_j^2) - (x_k^1, x_l^2)|\big),$$
$$\tau_{ij}(t) = \frac{|(x_i^1, x_j^2) - (x_k^1, x_l^2)|}{v},$$
$$i = iM + j, \quad j = kM + l, \quad N = M^2,$$

in (28), we get the Hopfield network model (17). Applying Theorem 3, we prove convergence of the numerical scheme (28) to the equation (27).

Rankin et al [10] discretize the Amari model (27) for

$$u(t, x) \in R, \quad \Omega = [-L, L]^2, \quad v = \infty,$$

also by substituting Ω with the grid $\{(x_i^1, x_j^2),\, i, j = 1, \ldots, M\}$ and then use a combination of the Fourier transform and the inverse Fourier transform to obtain the solution numerically. Discretization of the Amari model on a hyperbolic disc $\Omega = \{x = (r, \theta),\ r \in [0, r_0],\ r_0 \in R,\ \theta \in [0, 2\pi)\}$ using the rectangular rule for the quadrature $\{(r_i, \theta_j),\ i = 1, \ldots, M,\ j = i = 1, \ldots, N\}$ was implemented in [6] to study of the localized solutions. As it easy to conclude from Theorem 3, the solutions obtained in both these cases converge to the corresponding analytical solutions as $M \to \infty$ and $N \to \infty$.

We emphasize here that Theorem 3 also allows one to justify discretization schemes on unbounded domains for equations involving spatio-temporal-dependent delay as well.

Appendix

In this section we consider the following neural field model with a general (i.e. non-periodic) microstructure:

$$\begin{aligned} \partial_t u(t, x) &= -u(t, x) + \int_{R^m} \omega_i^\varepsilon(x - y) f(u(t, y))\, dy, \\ \omega_i^\varepsilon(x) &= \omega_i(x, x/\varepsilon), \quad 0 < \varepsilon \ll 1, \\ & t \geq 0, \quad x \in R^m. \end{aligned} \quad (29)$$

which is a parametrized version of (3).

Question: What can we say about behavior of the solutions u_n to the equation (29) as $\omega_i^\varepsilon \to \omega_0^\varepsilon$ uniformly ($i \to \infty$), where ω_0^ε is periodic with respect to the second argument?

Following the idea of homogenization of the equation (3) (see [12])), we first look at the family of homogenized problems

$$\partial_t u(t,x_c,x_f) = -u(t,x_c,x_f)$$
$$+\int\limits_{R^m}\int\limits_{K_n} \omega_i(x_c-y_c,x_f-y_f)f(u(t,y_c,y_f))\,dy_c\,\nu_n(dy_f), \tag{30}$$
$$t>0,\ \ x_c\in R^m,\ \ x_f\in K_i\subset R^k$$

and the corresponding limit problem as $i\to\infty$

$$\partial_t u(t,x_c,x_f) = -u(t,x_c,x_f)$$
$$+\int\limits_{R^m}\int\limits_{K_0} \omega_0(x_c-y_c,x_f-y_f)f(u(t,y_c,y_f))\,dy_c\,\nu_0(dy_f), \tag{31}$$
$$t>0,\ \ x_c\in R^m,\ \ x_f\in K_0\subset R^k.$$

As in [12], we assume that for each $i=0,1,2,\dots$, the connectivity kernel $\omega_i(x,\cdot)$ $(x\in R^m)$ belongs to A_i, where $A_i=C(K_i)$ are some Banach algebras of continuous functions defined on the compact sets $K_i\subset R^k$ and equipped with the mean values $\mathfrak{M}_i$ (which give rise to the finite measure ν_i defined on K_i). Further, we assume that there is a compact $\overline{K}$ such that $\bigcup\limits_{i=0}^{\infty} K_i\subseteq\overline{K}$, so we can extend the measures ν_i corresponding to the mean values $\mathfrak{M}_i$ $(i=0,1,2,\dots)$, to the compact $\overline{K}$ by putting $\nu_i(\overline{K}\setminus K_i)=0$. Finally, we assume that convergence of the connectivity kernels is a consequence of a convergence of the associated Banach algebras with mean. More precisely, we suppose that:

1) the compacts K_i converge to the compact K_0 in the Hausdorff metric;
2) $\mathfrak{M}_n(\chi\big|_{K_n})\to\mathfrak{M}_0(\chi\big|_{K_0})$ for any function $\chi\in C(\overline{K})$ (here $\chi\big|_{K_i}$ denotes the restriction of the function $\chi\in C(\overline{K})$ to the set K_i).

Thus, we get

$$\int\limits_{K_n}\chi(x)\nu_n(dx)\longrightarrow\int\limits_{K_0}\chi(x)\nu_0(dx)$$

for any $\chi\in C(\overline{K})$, which means that the sequence of measures ν_n weakly converges to the measure ν_0. Hence, we can apply Theorem 2 to the problems (30) and (31) and get uniform convergence of the corresponding solutions. This approach can serve as a possible answer to the above-formulated question.

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