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Water waves radiated from an oscillating line source in shear flow with a free surface and non-zero surface velocity.

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Lektorutdanning i Realfag

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1. Sammendrag

Denne oppgaven beskriver et todimensjonalt linearisert vannbølge problem, der en oscillerende kilde er plassert i vannet. Vi antar at vannet er verken viskøst eller kompressibelt, og at trykket på overflaten er homogent og atmosfærisk. Vannet der kilden er plassert har en overflatehastighet. Vi kaller den retningen denne overflatehastigheten går for medstrøms, og den andre retningen motstrøms. Den neddykkede kilden er plassert i vann med skjærstrøm, dette betyr at hastigheten på vannet i horisontal retning avhenger lineært med dybden. Denne skjærstrømmen kan både gå i samme og motsatt retning av overflatehastigheten. I utgangspunktet ligger den frie overflaten helt i ro, og all bølgeaktivitet i vannet kommer fra den neddykkede kilden. Vi tar utgangspunkt i Euler's bevegelseslikninger for fluider. Det er tidligere blitt jobbet med samme problem med utgangspunkt i Laplace's likning. Dette ga derimot ikke en helt rett tolkning fordi Laplace's likning ikke er oppfylt i hele vannet, og dermed ble hele eksistensen av den kritiske bølgen ikke oppdaget. Vi jobber en stund med endelig konstant dybde på havet, men hoveddelen av oppgaven handler om uendelig dybde. Kilden ligger hele tiden på en bestemt endelig dybde.

I denne oppgaven løses problemet analytisk for amplituden til bølgene i fjernfeltet. Vi oppdager at det totalt kan eksistere fire dispersive bølger, i tillegg til en femte kritisk bølge som ikke er dispersiv. Alle disse fem bølgene skapes av den neddykkede kilden. Til sammen lager disse fem bølgene hele det synlige bølgesystemet. Vi finner for uendelig dybde at de fire dispersive bølgene har forskjellige områder av eksistens. Eksistensen av bølgene diskuteres i ett dimensjonsløst skjærstrøm og frekvens plan. Det er kun den første bølgen, som går i samme retning som overflatehastigheten, som alltid vil eksistere. Det er også en andre bølge som går i positiv retning, men med et noe mindre område av eksistens. De to bølgene som går i motsatt retning av skjærstrømmen, henholdsvis bølge tre og fire, har enda mindre område der de eksisterer. Den fjerde bølgen har bare et lite område der den eksisterer. Den kritiske bølga vil eksistere så lenge strømmen i vannet ved kildens dybde er null. Ved alle andre hastigheter vil bølgen eksistere. Den kritiske bølgen vil alltid ha samme fasehastighet som strømmens hastighet ved kildens dybde. Dette betyr at den kritiske bølgen kan gå både oppstrøms og motstrøms. Dette er fordi den ikke er avhengig av overflatehastigheten men av hastigheten på kildens dyp. Dette viser at den kritiske bølgen bare er en manifestasjon til overflaten av det som skjer på kildens dyp.

Vi finner at den kritiske bølgen kan resonnere med alle de fire dispersive bølgene. Dette gis ved grafer i ett dimensjonsløst skjærstrøm, frekvens plan. Den fjerde bølgen krever en større dybde i forhold til overflatestrømmen enn de andre bølgene. For tre av disse bølgene, henholdsvis bølge en, to og fire, skaper dette en samlet bølge som har en faseforskyvning. I tillegg inneholder dette ett ledd

som inneholder avstanden til kilden. Dette betyr at i lengden går denne amplituden til uendelig. For den tredje bølgen bryter derimot teorien litt sammen og gir bare en amplitude som er uendelig. Vi finner i tillegg at de to bølgene som går i motsatt retning av overflatehastigheten kan resonnere med hverandre og skape Doppler resonans. Denne blir her kalt for Dopplerbølgen. Også for denne bølgen bryter teorien sammen og gir en amplitude som er uendelig. Den kritiske bølgen vil kun være i resonans med en bølge av gangen, med ett eneste unntak. Den kritiske bølgen kan nemlig også resonnere med Dopplerbølgen. Dette krever en enda større minste dybde i forhold til overflatehastigheten for å kunne skje. Allikevel så er det maks ved to punkter i dimensjonsløst skjærstrøm, frekvens plan at resonans mellom alle tre motstrømsbølgene er mulig. Lineær teori vil gi uendelig amplitude også for denne bølgen. Lineær teori gir uendelig amplitude ved resonans fordi vi tvinger til bølgen å være periodisk. Det er derimot mulig med lineær resonans, noe vi har ved resonans mellom de dispersive bølgene og den kritiske bølgen. Da får vi en amplitude som vokser med avstanden fra kilden.

2. Introduction

We will first introduce water wave theory, through potential theory and potential flow. This theory requires that the fluid is irrotational. The Helmholtz theorem of fluid dynamics states that fluid particles will remain with the same vorticity. This means that if the fluid starts without rotation, it will follow potential flow. We will explain potential theory now in a three-dimensional (x,y,z) coordinate frame. Potential theory introduces the velocity potential, ϕ. Where the fluid velocity vector, $\boldsymbol{V}$, is given by the gradient of the velocity potential, $\boldsymbol{V} = \nabla\phi$. We apply boundary conditions to the velocity potential to find the wave function. One of these boundary conditions is the kinematic boundary conditions, which states that the fluid particles on the surface will follow the surface direction. Newman (1977) gives the kinematic boundary condition for three dimensions as $0 = \frac{D}{Dt}(y-\zeta) = \frac{\partial\phi}{dy} - \frac{\partial\zeta}{dt} - \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} - \frac{\partial\phi}{\partial z}\frac{\Phi\zeta}{\partial z}$. The last two terms here are of second order. In linearized theory we only keep the first order contributions. The kinematic boundary condition in linearized theory becomes: $\frac{\partial\phi}{dy} = \frac{\partial\zeta}{dt}$. The second boundary condition is the dynamic boundary condition. The dynamic boundary condition states that the pressure at the surface of the water has to be the same as the pressure of the atmosphere. Newman (1977) shows the dynamic boundary condition as $\zeta = -\frac{1}{g}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\nabla\phi\right)$. The second term in the parenthesis is clearly of the second order, and we linearize the dynamic boundary condition to become $\zeta = -\frac{1}{g}\frac{\partial\Phi}{\partial t}$. These boundary conditions should be used on the exact free surface where $y = \zeta$ but when we linearize this we use it on the undisturbed free surface $y = 0$. Newman (1977) showed that by combining the kinematic and dynamic boundary conditions we find the combined boundary condition $\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0.$

We now turn to the simplest form of two-dimensional waves. These are called plane waves and are given by $\zeta(x, t) = Acos(kx - \omega t)$. The amount of waves per unit length is called the wave number, k, and equals $= \frac{2\pi}{\lambda}$, where λ is the wavelength and ω is the radian frequency of the wave. It is usual to work with complex numbers, rewriting so the surface elevation equation becomes $\zeta(x, t) = Ae^{i(-kx+\omega t)}$. Working with potential theory, Laplace's equation $\nabla^2\phi = 0$ is satisfied stating that the curl of the velocity gradient equals zero. This is a known fact from vector calculus. Seeking a solution on the form $\phi = Re(Y(y)e^{-ikx+i\omega t}$. Solving for this we find that $Y(y) = Ce^{ky} + De^{-ky}$. With infinite bottom, we must avoid unbounded velocity at the bottom of the water at infinite. This means that D must be zero, and $Y(y) = Ce^{ky}$. Comparing this with the equation for the surface elevation we find that the velocity potential must be: $\phi = \frac{gA}{\omega}e^{ky}sin(kx - \omega t)$. This solution is too general, and we must impose the combined boundary condition, which gives an additional requirement. This requirement is the dispersion relation, relating the wave number and frequency, $k = \frac{\omega^2}{g}$. If we work with finite bottom we must apply another boundary condition, $\frac{\partial\phi}{\partial y}|_{y=-H} = 0$. This states that there can be no fluid velocity through the bottom. We now find that $\phi = \frac{gAk}{\omega}\frac{cosh(k(y+H))}{cosh(kh)}sin(kx - \omega t)$ and the dispersion relation becomes $k\tanh(kh) = \frac{\omega^2}{g}$. Where we see that including bottom conditions makes both the velocity potential and the dispersion relation more complicated.

We will now turn our attention towards the problem we will focus on. This is the waves made from a submerged oscillating line source in a shear flow in two dimensions. The first mathematical solutions to submerged sources were given by Kochin(1939, 1940), which can be seen in the review article of Wehausen & Laitone (1960). However, the theory has not been expanded to account for shear flows. Tyvand and Lepperød (2014) found solutions to the submerged line oscillating line source in a shear flow with zero surface velocity based on Laplace's equation and potential theory, finding the two dispersive waves for infinite bottom. Tyvand and Lepperød (2015) extended this to also include non-zero velocities. They found that including surface velocity increases the amount of dispersive waves to four, as well as making Doppler effects possible in certain situations where two of the waves flow together. Tyvand and Lepperød (2014 and 2015) assumed that the flow perturbation obeys Laplace's Equation, even with the existence of vorticity. Their argument was based on Lord Kelvin's Circulation theorem, stating that the vorticity is conserved in two dimensions. The submerged source however violates Laplace's equation in one point, rendering this argument uncertain. Because of this, Tyvand and Ellingsen (2016) saw it fruitful to solve the problem from a fundamental approach. Instead of using Laplace's equation, they based their research on the more fundamental Euler's equation of motion. Ellingsen and Tyvand (2016) did this for non-zero surface velocity, giving results that did not agree with the analysis of Tyvand and Lepperød (2014). This proved that the latter model is physically inconsistent. Because the model given by Tyvand and Ellingsen (2016) starts from first principles for inviscid incompressible flow, this model is superior. Even though the amplitudes predicted by Tyvand and Lepperød (2014 and 2015) are wrong; their predictions give a useful background for discussion of the dispersive waves. Especially relating to the dispersion relation, which remains the same for the dispersive waves. We now generalize the work of Ellingsen and Tyvand (2016) to also include non-zero surface velocities. This creates the opportunity for Doppler Effect resonance between the two downstream waves, as well as increasing the number of waves as shown by Tyvand and Lepperød (2015). It will now be especially interesting to see how the dispersive waves respond to the existence of the critical wave.

That potential theory cannot be used for three-dimensional flow with a shear flow was known from before. This is because a varying perturbation vorticity must exist in three dimensions. Now Ellingsen and Tyvand (2016) have also shown that potential theory has its shortcomings for two-dimensional flows with a shear current if there are singularities in the fluid domain. We find that solving the problem from Euler's equation of motion that we linearize with respect to perturbative quantities, we find that the wave pattern is different from Tyvand and Lepperød(2015). Especially the existence of a non-dispersive critical layer created by the singular oscillating source manifesting to the surface. This wave takes some of the energy and mass flow from the dispersive waves and will because of this change the values of the amplitudes for the dispersive waves, compared to doing this from Laplace's equation. Ellingsen and Tyvand (2016) found that the critical wave would be directed in the negative direction when the surface velocity is zero. They showed that the critical wave would have the speed and direction of the flow at the depth of the source. This shows that the critical wave is only a manifestation of what happens at the singular source. When we solve this accepting any surface velocities, we will find that the critical wave can go in both directions. This depends only on the value of the velocity at the depth of the source.

We will solve to find the amplitudes in the far field, ignoring any near-field contributions. The far-field elevation takes a few wavelengths to build up. The critical wave in this problem must not be confused with other forms of critical waves. Several papers have been written where the critical waves are due to interference from waves coming into the area from outside. All waves in this problem has its origin in the singular source.

3. Mathematical model

The fluid in this problem is inviscid and incompressible. The fluid has a steady shear flow varying linearly with depth along the horizontal x-axis. The free surface of the fluid has constant atmospheric pressure. The fluid depth is constant, H. We use the Cartesian coordinates x, z, where the z-axis is directed upwards in the gravity field and the x-axis along the undisturbed free surface. We denote the constant density of the fluid by ρ and g is the gravitational constant. We denote the elevation of the surface by ζ(x, z). The velocity perturbation vector is $(\hat{u}, \hat{w})$ where we use ^ to mark that this is before Fourier transform. An oscillatory point source located at (0, -D) is the driving source to the wave motion. We will linearize the water wave problem with respect to the surface elevation, as well as the velocity and pressure perturbations. A sketch of the problem is shown in figure 3.1.

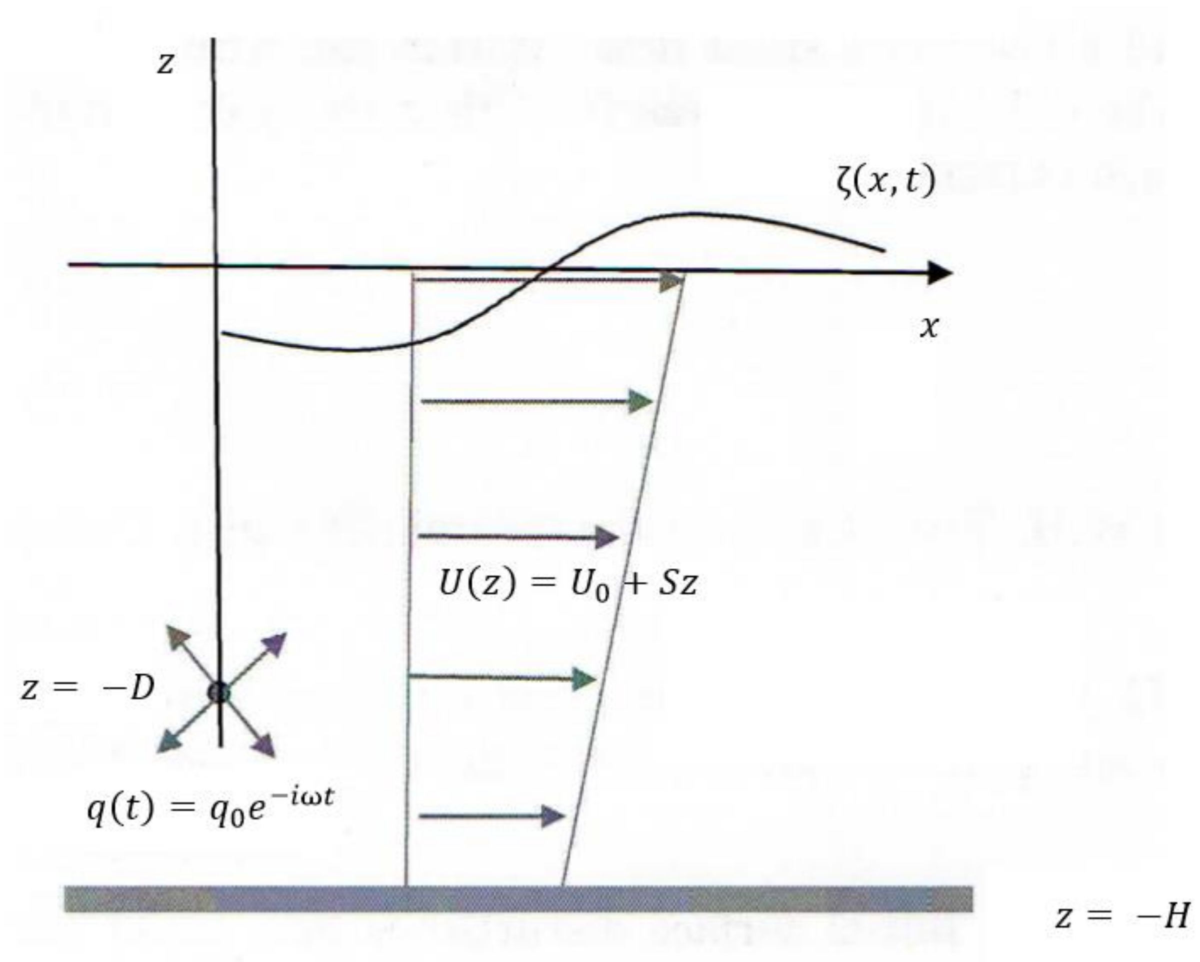

Figure 3.1: Sketch of the overall problem, with the source at depth D. Source Tyvand (2016), private communication

The singular source at any depth D oscillates with

$$q(t) = q_0 e^{-i\omega t}. \tag{3.1}$$

Where the physical is the real part. The shear flow, *U(Z)*, in the x-direction is given by

$$U(Z) = U_0 + Sz, \;\; z \leq 0. \tag{3.2}$$

Where U_0 is the surface velocity, which is uniform. The shear flow constant S is uniform. We restrict the surface velocity U_0 to be positive, stating that the direction of U_0 decides what we choose as positive direction. Thus, the shear flow constant S can take any real value.

The linearized kinematic free surface condition is that the velocity of the particles on the surface has to follow the wave profile

$$\hat{w} = \zeta_t + U_0 \zeta_x \,, \; z = 0. \tag{3.3}$$

Where the subscripts denote partial derivatives. This condition states that fluid particles at the surface follows the motions of the wave profile on the surface.

Euler's equation of motion can be written as

$$\boldsymbol{a} = -\frac{1}{\rho} \nabla P - g\boldsymbol{e_z}.$$

Where $\boldsymbol{a}$ is the acceleration vector and $\boldsymbol{e_z}$ is the vertical unit vector. P is the total pressure and the equation for the pressure is $P = -\rho g z + \hat{p}$, where $\hat{p}$ is the small perturbation.

We neglect surface tension and the dynamic boundary condition is given by the continuity of the tangential component of the Euler equation along the free surface. Mathematically this is written in the following way

$$\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{n})\boldsymbol{n} = -g\boldsymbol{e_z} + g(\boldsymbol{e_z} \cdot \boldsymbol{n})\boldsymbol{n} \,, \; z = \zeta(x,y,t). \tag{3.4}$$

Where $\boldsymbol{n}$ is the surface normal vector. This equation has been taken from Ellingsen and Tyvand(2016) and says that any fluid particles that exist on the surface, will always stay on the surface. Gravity will accelerate the fluid particles downward, but the pressure gradient normal to the surface keeps it on the surface. Thus the fluid particles will flow along the surface. From linear theory, we find that the surface normal is given by $\boldsymbol{n} = \boldsymbol{e_z} - \nabla\zeta$. Linearizing this and taking the x-component of equation 3.4 we find the dynamic boundary condition

$$\hat{u}_t + U_0 \hat{u}_x + S\hat{w} = -g\zeta_x, z = 0. \tag{3.5}$$

The continuity equation is given by

$$\nabla \cdot \hat{\boldsymbol{v}} = \hat{u}_x + \hat{w}_z = q_0 e^{-i\omega t} \delta(x)\delta(z + D). \tag{3.6}$$

Where δ is the Dirac's delta function. The continuity equation in general states that the mass is conserved in the system. In our problem, we have a mass contribution added to the system from the singular source. The singularity however is located within the body of the fluid, and not at the surface. This means that for z = 0 the right side of the equality becomes zero, and we find that the continuity equation for the surface is

$$\hat{u}_x = -\widehat{w}_z \text{ , z = 0.} \tag{3.7}$$

We now want to eliminate the horizontal velocity in the dynamic boundary condition (3.5) by the use of the continuity equation (3.7) at the surface to get the dynamic free-surface condition

$$\widehat{w}_{zt} + U_0\widehat{w}_{zx} - S\widehat{w}_x = g\zeta_{xx}, \ z = 0. \tag{3.8}$$

Where we had to differentiate the dynamic boundary condition once in the z-direction. Ellingsen and Tyvand (2015) did this with $U_0 = 0$ in their simplified case. The difference that we will see several times in the mathematical model is that when we include surface velocity different from zero the operator $\frac{\partial}{\partial t}$ changes to $\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}$.

We now make use of the linearized kinematic free surface condition (3.3) to eliminate ζ from the equation. Doing this we get

$$\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)(\widehat{w}_{zt} + U_0\widehat{w}_{zx} - S\widehat{w}_x) = \left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)g\zeta_{xx}, \ z = 0. \tag{3.9}$$

And the final result of this operation is

$$\widehat{w}_{ztt} + 2U_0\widehat{w}_{ztx} + {U_0}^2\widehat{w}_{zxx} = S\widehat{w}_{xt} + (U_0S + g)\widehat{w}_{xx}, \text{ z = 0.} \tag{3.10}$$

We also have the bottom condition

$$\widehat{w} = 0, \ z = -H. \tag{3.11}$$

Stating that the fluid velocities through the bottom of the water will be zero.

To analyze this problem we will have to Fourier transform the variables. The variables will be Fourier transformed in the following way

$$(\hat{u}, \widehat{w}, \hat{p}) = q_0\frac{1}{2\pi}\int_{-\infty}^{\infty}\big(u(z), w(z), p(z)\big)e^{ikx - i\omega t}dk. \tag{3.12}$$

The integral consists of waves travelling in both the positive x-direction with k>0, as well as waves travelling in the negative x-direction with k<0.

We now want to use the Euler equation given by

$$\widehat{\boldsymbol{v}}_t + U(z)\boldsymbol{i} + \widehat{\boldsymbol{v}} \cdot \nabla(U(z)\boldsymbol{i} + \widehat{\boldsymbol{v}}) = -\frac{\nabla P}{\rho}. \quad (3.13)$$

Which rewritten in our 2D situation is

$$\widehat{\boldsymbol{v}}_t + U(z)\widehat{\boldsymbol{v}}_x + \widehat{\boldsymbol{v}}\boldsymbol{k}\frac{\partial}{\partial z}(Sz)\boldsymbol{i} = -\frac{\nabla P}{\rho}. \quad (3.14)$$

Taking the x and z component of this equation respectively we get the following equations

$$\hat{u}_t + U(z)\hat{u}_x + S\widehat{w} = -\frac{P_x}{\rho} \quad (3.15a)$$

$$\widehat{w}_t + U\widehat{w}_x = -\frac{P_z}{\rho} \quad (3.15b)$$

We will now Fourier transform equations (3.15a and b). We Fourier transform the vertical velocity w and the pressure p. From the Fourier transformation, we will generally get the following changes: $\frac{\partial}{dx} \to ik \text{ and } \frac{\partial}{dt} \to -i\omega$

This gives us the following two equations

$$-i(\omega - kU)u + Sw = -ik\frac{p}{\rho}, \quad (3.16a)$$

$$-i(\omega - kU)w = -\frac{p_z}{\rho}. \quad (3.16b)$$

We also Fourier transform the continuity equation (3.6) yielding

$$iku + w_z = \delta(z + D). \quad (3.17)$$

We now want to solve these three equations for w eliminating u and p on the way to get an equation for w alone.

We first solve 3.16a with respect to u getting

$$u = \frac{\frac{kp}{\rho} + \frac{Sw}{i}}{\omega - kU}. \quad (3.18a)$$

We now use this in equation 3.17 getting

$$ik^2\frac{p}{\rho} + Skw + w_z(\omega - kU) = \delta(z + D)(\omega - kU). \quad (3.18b)$$

We now differentiate once in the z-direction remembering that $U(Z) = U_0 + Sz$ getting

$$ik^2\frac{p_z}{\rho} + Skw_z + w_{zz}(\omega - kU) - Skw_z = \delta_z(z + D)(\omega - kU) - Sk\delta(z + D). \quad (3.18c)$$

We are now in position to use 3.16b to replace $\frac{p_z}{\rho}$ and noticing the two cancelling terms we get

$$w_{zz} - k^2 w = \delta_z(z+D) - \frac{kS}{\omega - kU}\delta(z+D). \quad (3.18d)$$

In the second term on the right side, we have a Dirac's delta function; this will effectively remove everything that is not at exactly z= –D. To avoid the apparent greater z-dependency than we really have, as well as simplify further mathematics, the equation will be written as this

$$w_{zz} - k^2 w = \delta_z(z+D) - \frac{kS}{\omega - kU_0 + kSD}\delta(z+D). \quad (3.19)$$

Where we have written $U(z) = U_0 - SD$ fully, and inserted –D for z.

We also want to Fourier transform the free surface (ζ). This we do in the following way

$$\zeta(x,t) = q_0 \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega,k) e^{ikx - i\omega t} dk. \quad (3.20)$$

4. Finding the dispersion relation:

The homogenous solution of (3.18d) satisfying the bottom condition (3.11) is

$$w_h(z) = A(\omega,k)\sinh(k(z+H)). \quad (4.1)$$

In addition, we have two particular solutions, these we will find using the variation of parameters technique. From the first inhomogeneous terms in equation (3.18d) we get the first particular solution using the variation of parameter technique. The variation of parameters technique is well explained by the University of Utah (no year).

$$w_{p1} = \frac{1}{2}(e^{k(z+H)}\int_{-D}^{0} \frac{\delta'(z+D)}{ke^{k(z+H)}} dz - e^{-k(z+H)}\int_{-D}^{0} \frac{\delta\prime(z+D)}{ke^{-k(z+H)}} dz \quad (4.2a)$$

Where ʻ means $\frac{d}{dz}$. We use integration by parts on both integrals giving us

$$W_{p1} = \frac{1}{2k} e^{k(z+H)}(\delta(z+D)e^{-k(z+H)} + k\int_{-D}^{0} e^{-k(z+H)}\delta(z+D)dz - \frac{1}{2k} e^{-k(z+H)}(\delta(z+D)e^{k(z+H)}$$
$$- k\int_{-D}^{0} ke^{k(z+H)}\delta(z+D)dz. \quad (4.2b)$$

First, we notice that the two terms without integrals cancel each other, and we are left only with the integrals. We now use that: $\int_{-D}^{0} f(z)\,\delta(z+D)dz = f(-D)\Theta(z+D)$ where we do not put restrictions on the integral for positive z, as these are irrelevant and above the surface at z = 0. Giving us

$$W_{p1} = \frac{1}{2}(e^{k(z+h)}e^{k(D-H)} - e^{-k(z+H)}e^{-k(D-H)})\,\Theta(z+D). \quad (4.2c)$$

We recognize the first parenthesis as $\cosh(\mathrm{k}(\mathrm{z}+\mathrm{D}))$ and write our first particular solution as

$$w_{p1} = \cosh\big(k(z+D)\big)\,\Theta(z+D). \quad (4.2d)$$

We now look at the second term, again using variation of parameters technique giving us the following equation

$$w_{p2} = e^{k(z+H)} \int_{-D}^{0} \frac{\frac{kS}{\omega - U_0 k + kSD}\delta(z+D)}{-2ke^{k(z+H)}} dz + e^{-k(z+H)} \int_{-D}^{0} \frac{-\frac{kS}{\omega - U_0 k + kSD}\delta(z+D)}{-2ke^{-k(z+H)}} dz. \quad (4.3a)$$

We again make use of: $\int_{-D}^{0} f(z)\,\delta(z+D)dz = f(-D)\Theta(z+D)$ just as above, we now get the particular solution

$$w_{p2} = -\frac{S}{\omega - U_0 k + kSD}\frac{1}{2}\left(e^{k(z+H)}e^{k(D-H)} - e^{-k(z+H)}e^{-k(D-H)}\right)\Theta(z+D). \quad (4.3b)$$

Where we notice that the parenthesis becomes sinh(k(z+D) giving us:

$$w_{p2} = -\frac{S}{\omega - U_0 k + kSD}\sinh(k(z+D))\Theta(z+D) \quad (4.3c)$$

Where the Heaviside Theta function (Θ(z+D)) has been introduced.

Adding the homogenous and the particular solutions together we find the full solution for the vertical velocity.

$$w = A(\omega,k)\sinh\big(k(z+H)\big) + \cosh\big(k(z+D)\big)\,\Theta(z+D) - \frac{S}{\omega - U_0 k + kSD}\sinh(k(z+D))\Theta(z+D) \quad (4.3d)$$

We will now Fourier transform the kinematic boundary condition (3.3). Doing this we find the relationship:

$$Asinh(kH) = -\cosh(kD) + \frac{S}{\omega - U_0 k + kSD}\sinh(kD) - i(\omega - U_0 k)B \quad (4.4)$$

We now also transform the dynamic free-surface condition (3.8). Then we find the relationship.

$$\big((\omega - U_0 k)\coth(KH) + S\big)Asinh(kH) + ((\omega - U_0 k)\sinh(kH) + Scosh(kD) - \frac{(\omega - U_0 k)Scosh(kD) + S^2\sinh(kD)}{\omega - U_0 k + kSD} = -igkB \quad (4.5)$$

We are interested in the free surface elevation *B(ω,k)*, not *A* so we now insert the right hand side of (4.4) for $Asinh(kH)$ in equation (4.5). After reorganizing we find the following relationship

$$\frac{i}{(\omega - U_0 k)}(gk - (\omega - U_0 k)[(\omega - U_0 k)\coth(kH) + S])B = \frac{\sinh(k(H-D)}{\left(\frac{\omega}{S} - \frac{U_0 k}{s} + kD\right)\sinh(kH)} + \frac{\cosh(k(H-D)}{\sinh(kH)}. \quad (4.6)$$

Here we have B multiplied by a factor, which is zero when k has one of the values corresponding to the dispersion relation

$$gk = S(\omega - U_0 k) + (\omega - U_0 k)^2 \coth(kH). \tag{4.7}$$

5. Far field solution for finite depth:

We have now found the equation for B. We now want to use this to find the equation for the free surface ζ. The only problem is that the integral (320) is not well defined, as we have poles sitting directly on the axis of integration. To solve this in a way that is both physically and mathematically acceptable we say that the source has been slowly increasing in strength since t = - ∞.

Mathematically this replaces equation 3.1 with

$$q(t) = q_0 e^{-i\omega t + \varepsilon \omega t}. \tag{5.1}$$

Where we will make $\varepsilon \rightarrow 0^+$ eventually. Doing this will make the following change in our equations due to the Fourier transform

$$\omega \rightarrow \omega(1 + i\varepsilon). \tag{5.2}$$

We first want to find the equation for the wave number of the critical wave, k_c. We find this when the first factor in the denominator of the first term on the right side of equation (4.6) is zero. Without the transformation made by (5.2) we find the factor is

$$\frac{\omega}{S} - \frac{U_0 k}{S} + kD = 0. \tag{5.3a}$$

Solving for k, and renaming this the wave number for the critical wave k_c, we find

$$k_c = \frac{\omega}{U_0 - SD}. \tag{5.3b}$$

This gives a mathematical dilemma of infinite $|k_c|$ when $U_0 = SD$. This is no problem physically, because when the shear flow at the depth of the source equals the surface velocity there will be no velocity at the oscillatory source, and we will not get any critical layer whatsoever.

The critical wave number is negative when $U_0 < SD$ and positive when $U_0 > SD$. This means that the critical wave will travel in the positive x-direction if $U_0 > SD$ and in the negative x-direction if $U_0 < SD$. When Ellingsen and Tyvand (2016) worked with the case where $U_0 = 0$, they found the possibility for resonance with the negative wave number. This was because the critical wave number would always be negative. When we include surface velocities, we find that the critical wave number can take both positive and negative values. This is because the flow at the depth of the source can be both positive and negative. This means that there is a possibility that this critical wave might be in resonance with any of the four dispersive waves found when $\Gamma(k) = 0$.

We now want to rewrite the factor in the denominator of the first term in equation (4.6) using k_c.

$$\frac{\omega}{S}-\frac{U_0k}{S}+kD=(\frac{\omega}{SD}-k\left(\frac{U_0}{SD}-1\right)D=\frac{1}{S}\left(\omega-k(U_0-SD)\right)=\frac{U_0-SD}{S}\left(\frac{\omega}{U_0-SD}-k\right)$$
$$=-\frac{U_0-SD}{S}(k-k_c) \tag{5.4}$$

We now rewrite the denominator using the critical wave number, as well as introducing the effects of (5.2). The effect of (5.2) will be that the poles are moved slightly off the real axis, making the integral (3.20) well defined. Rewriting (3.20) using the B from (4.6) and introducing a two new functions as well as the critical wave number we get

$$\frac{g}{q_0}\zeta=-i\lim_{\varepsilon\to0^+}\int_{-\infty}^{\infty}(\omega-U_0k)\left[\frac{\sinh(k(H-D))}{-\frac{U_0-SD}{S}(k-k_c(1+i\varepsilon))}+\cosh(k(H-D))\right]\times\frac{e^{ikx-i\omega t}}{(\Gamma(k)-i\varepsilon\Phi(k))\sinh(kH)}\frac{dk}{2\pi}. \tag{5.5}$$

Where the following has been defined

$$k_c=\frac{\omega}{U_0-SD} \tag{5.6a}$$

$$\Gamma(k)=k-\frac{(\omega-U_0k)^2}{g}\coth(kH)-\frac{(\omega-U_0k)S}{g} \tag{5.6b}$$

$$\Phi(k)=\frac{2\omega(\omega-U_0k)}{g}\coth(kH)+\frac{\omega S}{g} \tag{5.6c}$$

Where we see that both the phase velocity and the group velocity of the critical wave will be U_0-SD, thus the critical wave is not dispersive.

For simplicity, we will now define $f(k)$ in the following way

$$\frac{g}{(\omega-U_0k)q_0}\zeta=\lim_{\varepsilon\to0^+}\int_{-\infty}^{\infty}f(k)dk. \tag{5.7a}$$

This means that

$$f(k)=-i(\omega-U_0k)\left[\frac{\sinh(k(H-D))}{-\frac{U_0-SD}{S}(k-k_c(1+i\varepsilon))}+\cosh(k(H-D))\right]\times\frac{e^{ikx-i\omega t}}{2\pi(\Gamma(k)-i\varepsilon\Phi(k))\sinh(kH)}. \tag{5.7b}$$

We now want compute this using residue integration. Residue integration is done on a closed path, if there are no singularities inside the path the integration will yield zero. If there are singularities inside the integration these will have to be multiplied by $\pm2\pi i$. The sign will be + if the integration is done in a counter-clockwise direction, and – if the path is in a clockwise direction. In our equations, there are singularities when the denominator is zero, and we will first find the poles in these situations. We ignore any near-field poles that stems from the closing integral not becoming exactly zero, but terms that will vanish in the far-field. We only give the far-field solutions. There is one pole for the critical wave number, which is when $\left(k - k_c(1 + i\varepsilon)\right) = 0$. This pole stems from the critical layer, a phenomenon that potential theory is unable to describe. Because Tyvand and Lepperød (2015) looked at this problem using potential theory, they did not discover this wave. There are also poles for the four wave numbers when the dispersion relation, $\Gamma(k)$, is zero.

We First look at the pole for the critical wave. As we said in the previous section this pole is located at $k = k_c(1 + i\varepsilon)$. This pole has been moved slightly below the k-axis when $U_0 < SD$ and slightly above the k-axis when $U_0 > SD$ due to the value of k_c. Ellingsen and Tyvand (2016) worked with this when $U_0 = 0$. In their situation, the pole was always below the k-axis. This is because they had chosen that $S > 0$ and let this decide which direction is positive. Then $0 < SD$ and the critical wave number is always negaitve. When we include surface velocities, it is better to let the surface velocity decide the direction we see as the positive direction.

We now look at how we close the integration using contour paths. There are two possibilities, one closing through a semicircle through positive imaginary k-values, the other closing through negative k-values.

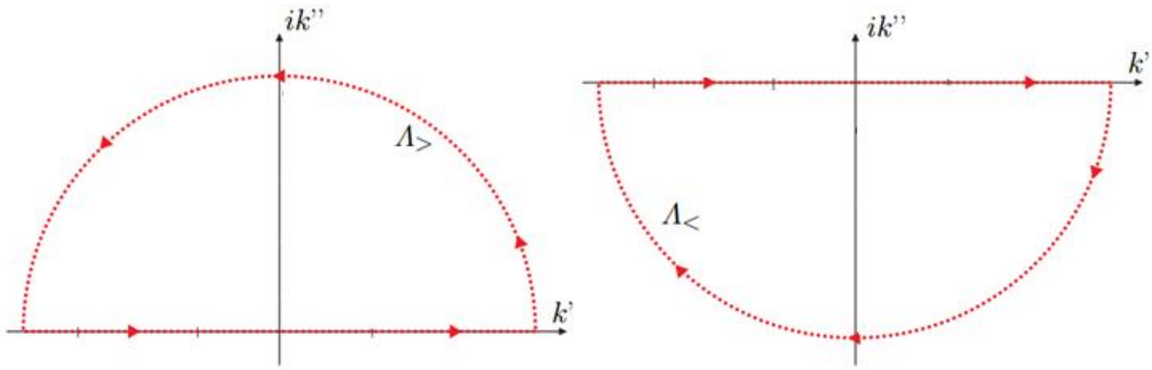

Figure 5.1: The two ways to close the contour integral. $\wedge_{-}$(right) closes with a negative semicircle, and encloses the poles with a negative imaginary part, while $\wedge_{+}$ (left) closes the integration with a positive semicircle and encloses the poles with a positive imaginary part.

We define $\wedge_{-}$ as the path closing the integration through a semicircle below the k-axis. Similarly $\wedge_{+}$ is the path closing the integration through a semicircle above the k-axis. These two ways to close the integration can be seen in figure 5.1. By making these changes that we have done here, we ensure ourselves that the only contribution to the integral will be from the poles within the integration path.

The pole for the critical wave is a simple pole. For a function f(x) with a simple pole at $x = x_{pole}$ we have the formula: $\operatorname*{Res}_{x\to x_{pole}} f(x) = \lim_{x\to x_{pole}} \left(x - x_{pole}\right) f(x)$. We find the residue from this simple pole by

$$Res_{k=kc} f(k) = \lim_{k\to k_c} (k - k_c) f(k) = i\,(\omega - U_0 k_c)\,\frac{e^{ik_c x - i\omega t}\sinh(k_c(H-D))}{\frac{U_0 - SD}{S}\,2\pi\,\Gamma(k_c)\sinh(k_c H)}. \tag{5.8}$$

This means that the contribution from the critical wave is given by:

$$2\pi i\, Res_{k=kc}\, f(k) = -(\omega - U_0 k_c)\,\frac{e^{ik_c x - i\omega t}\sinh(k_c(H-D))}{\frac{U_0 - SD}{S}\Gamma(k_c)\sinh(k_c H)} \tag{5.9a}$$

$$-2\pi i\, Res_{k=kc}\, f(k) = (\omega - U_0 k_c)\,\frac{e^{ik_c x - i\omega t}\sinh(k_c(H-D))}{\frac{U_0 - SD}{S}\Gamma(k_c)\sinh(k_c H)} \tag{5.9b}$$

Where we notice that the only difference is the first sign, which is positive when $U_0 > SD$ and negative when $U_0 < SD$. Due to the $\frac{U_0 - SD}{S}$ in the denominator of (5.9b) will be negative, we can write it more compactly using absolute value around $U_0 - SD$:

$$\pm 2\pi i\, Res_{k=kc}\, f(k) = -(\omega - U_0 k_c)\,\frac{e^{ik_c x - i\omega t}\sinh(k_c(H-D))}{\frac{|U_0 - SD|}{S}\Gamma(k_c)\sinh(k_c H)} \tag{5.9c}$$

This means that the surface elevation of the critical wave alone is given by:

$$\frac{g}{q_0}\zeta = -(\omega - U_0 k_c)\,\frac{e^{ik_c x - i\omega t}\sinh(k_c(H-D))}{\frac{|U_0 - SD|}{S}\Gamma(k_c)\sinh(k_c H)} \tag{5.10}$$

In equation (5.10) we see that the dispersive wave numbers are not a part of the equation. This means that the dispersive waves do not affect the critical wave in any matter. Another thing we see is that in the denominator we have an absolute mark around $|U_0 - SD|$ which will be the absolute value of the flow speed at the depth of the source. Showing that this part of the critical layer amplitude is only due to the speed through the source.

That the dispersive wave number does not affect the critical wave at all made it possible for Tyvand (2016) to work with the critical wave alone, without taking into account the effect of the dispersive waves. Tyvand (2016) looked at the formation of the critical layer at the source, which we will look at in subchapter 12, and its manifestation to the surface. He based this research on two potential flow fields, one above and one below the critical layer. That the flow above and below the critical layer is potential was shown by Ellingsen and Tyvand (2016), because only the source produces additional perturbation vorticity. We will look further at the manifestation of this critical layer now.

We will now look more into the ratio between the surface wave and the oscillation of the critical layer created within the fluid at the depth of the source as shown by Tyvand(2016). Tyvand (2016) found the ratio between the amplitude of the critical wave internally at the depth of the source compared to the amplitude of the manifestation on the surface. He showed that the relation between these two are given by:

$$\frac{\zeta_c}{\zeta_0} = \frac{\sinh\big(k(H-D)\big)}{\sinh(kH)+\left(\frac{B}{A}\right)\sinh(kD)} \tag{5.11}$$

Where ζ_c is the deformation amplitude of the critical layer and ζ_0 is the amplitude of the critical wave at the surface. Tyvand (2016) also showed that the relation between B and A in equation 5.11 is given by

$$\frac{B}{A} = \frac{kDcosh(kH)-(1+\frac{g}{S^2D}))\sinh(kH)}{\left(1+\frac{g}{S^2D}\right)\sinh(kD)-kDcosh(kD)}. \tag{5.12}$$

Using the dimensionless wave number of the critical wave $K = \frac{\omega D}{U_0 - SD} = k_c D$ and the Richardson number $Ri = \frac{g}{S^2 D}$ Tyvand (2016) found that the relation between the deflection amplitude and the surface amplitude could be written as

$$\frac{\zeta_c}{\zeta_0} = \cosh(K) - (1 + Ri)\,\frac{\sinh(K)}{K}. \tag{5.13}$$

One of the most interesting parts of equation (5.13) is that the ratio has no dependence on the depth of the water. This means that given a specific deflection amplitude, the surface manifestation is not dependent on the depth of the water. We see from the limit $Ri \to \infty$ when gravitation is dominant that the surface wave flattens. We can also see that the critical layer will never hide in the middle of the fluid, because when $K \to 0, \frac{\zeta_c}{\zeta_0} = -Ri$ and we will always get a critical wave amplitude. However, there is a possibility to get a ration of zero. This happens when $Ri = Kcoth(K-1)$. This means that the internal oscillations in the fluid will vanish, and simply be a flat vortex street. There will still be a critical wave on the surface. We can also see that the ration can have a negative value. This requires that $Ri > Kcoth(K-1)$ and means that the internal and surface wave oscillates in opposite phase. If gravity is weaker and $Ri < Kcoth(K-1)$ the internal and surface wave will flow with the same phase. Even though the ration between the deflection amplitude and the surface manifestation does not depend on the depth of the water, the depth of the water will affect the deflection amplitude. Tyvand(2016) found the deflection amplitude with dimensionless numbers to be

$$\frac{|U_0-SD|(U_0-SD)}{SDq_0}\ \zeta_c = -\frac{\frac{(1+Ri)\sinh(K)}{K}-\cosh(K)}{K\cosh(Kh)-(1+Ri)\sinh(Kh)}\sinh(Kh-1). \qquad (5.14)$$

Where the dimensionless fluid depth $h = \frac{H}{D}$ has been introduced.

An overall sketch of the problem including the oscillating layer at the depth of the source can be seen in figure 5.2.

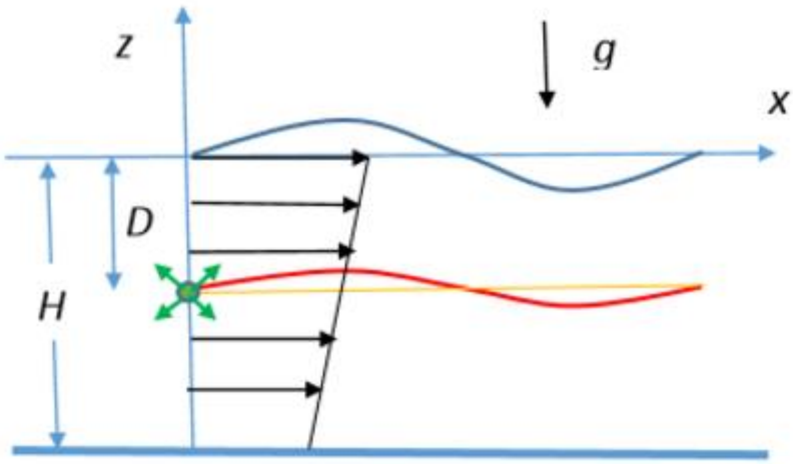

Figure 5.2: Sketch of the problem including the oscillating critical layer. Source Tyvand(2016), private communication

We know from Tyvand and Lepperød (2015) that there can be up to four dispersive waves with infinite bottom given with explicit solutions. For finite bottom we also get up to four waves, the difference is that the dispersion relation in this case will be implicit. The poles due to the dispersive waves were originally where the dispersion relation is zero, $\Gamma(k) = 0$. With the introduction of (5.2), these poles are now moved slightly away from $k_{1,2,3,4}$ and are now found at

$$k = k_{1,2,3,4} + i\varepsilon\frac{\Phi(k_{1,2,3,4})}{\Gamma\prime(k_{1,2,3,4})}. \qquad (5.15)$$

Where $\Gamma'\left(k_{1,2,3,4}\right) = \frac{d\Gamma\left(k_{1,2,3,4,}\right)}{dk} = 1 + \frac{2U_0(\omega-U_0k)}{g}\coth(kH) - \frac{U_0S}{g} + \frac{(\omega-U_0k)^2H}{g\sinh^2(kH)}$

We will now need to find the sign of the imaginary part for the different wave numbers to decide how we are going to close the integration.

Due to the difficulties of understanding the implicit dispersion relation, we will from now change to infinite bottom. This will make the dispersion relation explicit, and much easier to understand, as well as building a foundation to continue solving for finite bottom later. This will be a good first look at this situation, and can be used as a special solution for deep water. The residue is though rather easy to find and can be found without knowing the sign of the imaginary part. It is given by

$$\underset{k=k_{1,2,3,4}}{Res} f(k) = \lim_{k\to k_{1,2,3,4}} \left(k - k_{1,2,3,4}\right) f(k) = -i\left(\omega - U_0 k_{1,2,3,4}\right) \frac{e^{ik_{1,2,3,4}x-i\omega t}}{2\pi\Gamma'(k_{1,2,3,4})\sinh(k_{1,2,3,4}H)} \left[\frac{\sinh\left(k_{1,2,3,4}(H-D)\right)}{-\frac{U_0-SD}{S}(k_{1,2,3,4}-k_c)} + \cosh(k_{1,2,3,4}(H-D))\right]. \tag{5.16}$$

Where we see in the first part of the second parenthesis that the critical wave number affects the dispersive waves. This is a contrast to what we found above, that the dispersive waves do not affect the critical wave.

6. Translating to infinite depth

From here on, we will change our focus to the case of infinite depth. This will give information for deep ocean. Tyvand and Lepperød (2015) worked with this using potential theory, where they accepted only positive wave numbers. We continue working with positive and negative wave numbers, where the sign of the wave number still tell us the direction of the wave. Except for this, we will find that our dispersion relation is the same as Tyvand and Lepperød (2015). Only the amplitudes have been changed due to the critical wave that they do not discover by using potential theory. Because we use Fourier-Transform we will have to work with both positive and negative wave numbers, this also means that we will have to work with absolute sign in the solution of the differential equation (3.19).

Removing the bottom condition (3.11) the homogenous solution (4.1) will now become

$$w_h = A(\omega, k)e^{|k|z}. \tag{6.1}$$

Where we have to use the absolute mark to avoid unbounded movement deep under the surface.

First, we have to write the new equation for w

$$w = Ae^{|k|z} + \cosh\left(k(z+D)\right)\Theta(z+D) - \frac{1}{\frac{\omega-U_0k}{S}+kD}\sinh(k(z+D))\Theta(z+D). \tag{6.2}$$

Using equation 3.3, we get:

$$A = -\cosh(kD) + \frac{1}{\frac{\omega-U_0k}{S}+kD}\sinh(kD) - i(\omega - U_0k)B \tag{6.3}$$

We now use equation 3.8 finding for positive x

$$(\omega - U_0 k)\left(A + \sinh(kD) - \frac{1}{\frac{\omega - U_0 k}{S} + kD}\cosh(kD) \right) + S(A + \cosh(kD)$$

$$- \frac{1}{\frac{\omega - U_0 k}{S} + kD}\sinh(kD) = -igkB, x > 0 \quad (6.4a)$$

For negative x we find

$$(\omega - U_0 k)\left(-A + \sinh(kD) - \frac{1}{\frac{\omega - U_0 k}{S} + kD}\cosh(kD) \right) + S(A + \cosh(kD)$$

$$- \frac{1}{\frac{\omega - U_0 k}{S} + kD}\sinh(kD)) = -igkB, < 0 \quad (6.4b)$$

Eliminating A and solving for B, we get the following two equations that replaces equation 4.6 for infinite bottom.

$$i\frac{g}{\omega - U_0 k}\left(k - \frac{(\omega - U_0 k)(\omega - U_0 k + S)}{g}\right)B \ = e^{-kD}\left(1 + \frac{1}{\frac{\omega - U_0 k}{S} + kD}\right), k > 0 \quad (6.5a)$$

$$i\frac{g}{\omega - U_0 k}\left(-k - \frac{(\omega - U_0 k)(\omega - U_0 k - S)}{g}\right)B = e^{kD}\left(1 - \frac{1}{\frac{\omega - U_0 k}{S} + kD}\right), k < 0 \quad (6.5b)$$

We call the parenthesis on the left side of (6.5a) $\Gamma_+(k) = k - \frac{(\omega - U_0 k)(\omega - U_0 k + S)}{g} = 0$ and on the left side of (6.5b) $\Gamma_+(k) = -k - \frac{(\omega - U_0 k)(\omega - U_0 k - S)}{g} = 0$. We recognize in equations (6.5a and be) critical wave number in the denominator on the right side. Unchanged by the change of bottom to infinite bottom the critical wave number is still

$$k_c = \frac{\omega}{U_0 - SD}. \quad (6.6)$$

While the parenthesis on the left side give us the dispersion relation

$$k_1 = \frac{2\omega U_0 + U_0 S + g + \sqrt{(U_0 S + g)^2 + 4\omega U_0 g}}{2U_0^2} \quad (6.7a)$$

$$k_2 = \frac{2\omega U_0 + U_0 S + g - \sqrt{(U_0 S + g)^2 + 4\omega U_0 g}}{2U_0^2} \quad (6.7b)$$

$$k_3 = -\frac{-2\omega U_0 + U_0 S + g + \sqrt{(U_0 S + g)^2 - 4\omega U_0 g}}{2U_0^2} \quad (6.7c)$$

$$k_4 = -\frac{-2\omega U_0 + U_0 S + g - \sqrt{(U_0 S + g)^2 - 4\omega U_0 g}}{2U_0^2} \quad (6.7d)$$

Where the first and second wave has a positive wave number. While the third and fourth wave has a negative wave number.

7. Existence of the dispersive waves for infinite bottom

Tyvand and Lepperød (2015) has already looked at the existence of these waves. In their work, they only worked with positive wave numbers, therefore $k_3$and k_4 have a different sign in their work than here. Except for this we find the exact same dispersive wave numbers that they found. Given by the dimensionless numbers, $k^*_{1,2,3,4} = \frac{U_0^2}{g} k_{1,2,3,4}, \omega^* = \frac{U_0}{g}\omega$ and $S^* = \frac{U_0}{g} S$,all four waves have different areas of existence. We have numbered the waves to refer to the same waves here, as in the work of Tyvand and Lepperød (2015). Wave number one and two are found by solving for k when

$$\Gamma_+(k) = k - \frac{(\omega - U_0 k)(\omega - U_0 k + S)}{g} = 0 \qquad (7.1a)$$

Similarly, the wave number for dispersive wave three and four are found when

$$\Gamma_+(k) = -k - \frac{(\omega - U_0 k)(\omega - U_0 k - S)}{g} = 0 \qquad (7.1b)$$

Solving these equations, we find the wave numbers given by

$$\frac{2U_0^2}{g} k_1 = 1 + \frac{U_0}{g} S + \frac{2U_0}{g}\omega + \sqrt{\left(1 + \frac{U_0}{g} S\right)^2 + \frac{4U_0}{g}\omega}, \frac{2U_0^2}{g} k_1 > 0 \qquad (7.2a)$$

$$\frac{2U_0^2}{g} k_2 = 1 + \frac{U_0}{g} S + \frac{2U_0}{g}\omega - \sqrt{\left(1 + \frac{U_0}{g} S\right)^2 + \frac{4U_0}{g}\omega}, \frac{2U_0^2}{g} k_2 > 0 \qquad (7.2b)$$

$$\frac{2U_0^2}{g} k_3 = -1 - \frac{U_0}{g} S + \frac{2U_0}{g}\omega - \sqrt{\left(1 + \frac{U_0}{g} S\right)^2 - \frac{4U_0}{g}\omega}, \frac{2U_0^2}{g} k_3 < 0 \qquad (7.2c)$$

$$\frac{2U_0^2}{g} k_4 = -1 - \frac{U_0}{g} S + \frac{2U_0}{g}\omega + \sqrt{\left(1 + \frac{U_0}{g} S\right)^2 - \frac{4U_0}{g}\omega}, \frac{2U_0^2}{g} k_4 < 0 \qquad (7.2d)$$

Which is exactly the same wave numbers that was found by Tyvand and Lepperød(2015).

Written with dimensionless symbols this is

$$2k_1^* = 1 + S^* + 2\omega^* + \sqrt{(1 + S^*)^2 + 4\omega^*}, k_1^* > 0 \qquad (7.3a)$$

$$2k_2^* = 1 + S^* + 2\omega^* - \sqrt{(1 + S^*)^2 + 4\omega^*}, k_2^* > 0 \qquad (7.3b)$$

$$2k_3^* = -1 - S^* + 2\omega^* - \sqrt{(1 + S^*)^2 - 4\omega^*}, k_3^* < 0 \qquad (7.3c)$$

$$2k_4^* = -1 - S^* + 2\omega^* + \sqrt{(1+S^*)^2 - 4\omega^*}, k_4^* < 0 \quad (7.3d)$$

We will now first look at the different areas in the ω^*, S^* where these waves exist. Most of this discussion will be taken directly from Tyvand and Lepperød(2015). The restrictions are that wave number one and two must be positive, while the wave numbers for wave three and four must be negative. In addition, the square root has to be real.

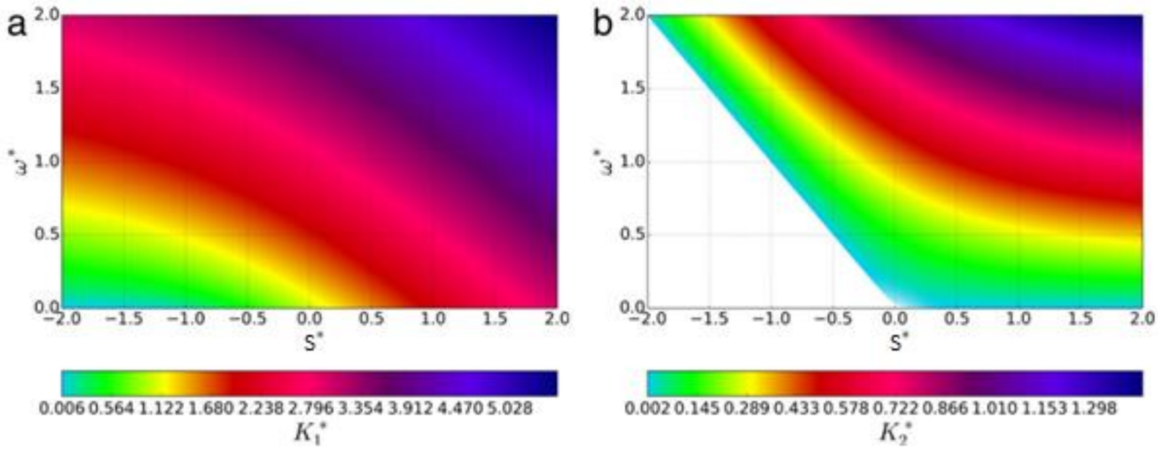

Figure 7.1: Existence of the dipsersive wave 1(a) and 2(b) with the value of their wave number. Source: Tyvand and Lepperød (2015)

In figure 7.1, which was created by Tyvand and Lepperød (2015), we can see that the first wave simply exists everywhere. The second wave can exist for all S and all ω but is limited to $\omega > |S|$ when S is negative as was pointed out by Tyvand and Lepperød(2015).

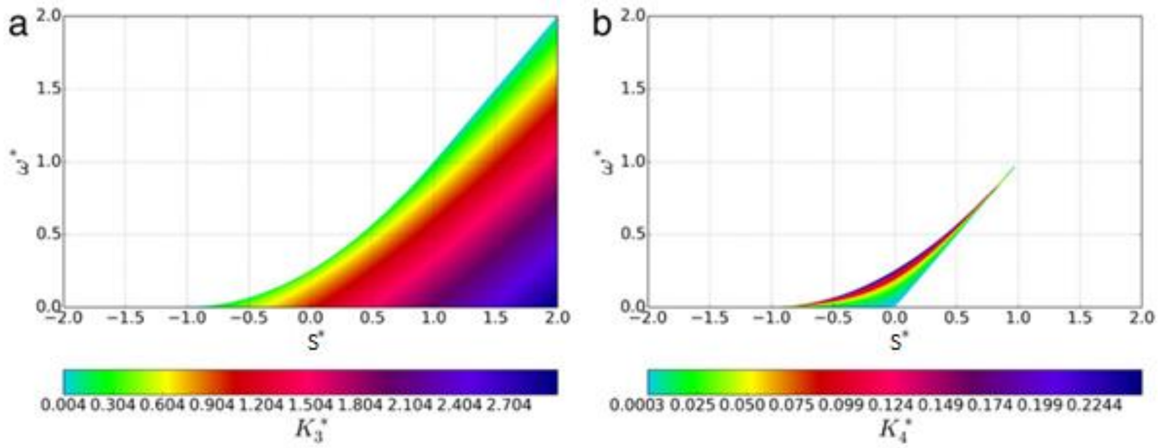

Figure 7.2: Existence of wave three (a) and four (b). Values of the wave number given in absolute value. Source: Tyvand and Lepperød (2015)

In figure 7.2, which was made by Tyvand and Lepperød (2015), we see that the third and the fourth wave has even smaller regions of existence. The third wave is limited by

$$0 < \frac{U_0}{g}\omega < \left(\frac{1+U_0 S/g}{2}\right)^2 \text{ when } -1 < \frac{SU_0}{g} < 1 \quad (7.4a)$$

$$\text{And } 0 < \frac{\omega U_0}{g} < \frac{SU_0}{g} \text{ when } \frac{SU_0}{g} > 1 \quad (7.4b)$$

The fourth wave has an even narrower region of existence given by

$$0 < \frac{\omega U_0}{g} < \left(\frac{1+SU_0/g}{2}\right)^2 \text{ when } -1 < \frac{SU_0}{g} < 0 \quad (7.5a)$$

$$\text{And } \frac{SU_0}{g} < \frac{\omega U_0}{g} < \left(\frac{1+\frac{SU_0}{g}}{2}\right)^2 \text{ when } 0 < \frac{SU_0}{g} < 1 \quad (7.5b)$$

And only exists for S^* in the interval (-1,1)

This is shown graphically with the associated values for the wave number in figure 7.2.

In shallow-water theory, they always work with three waves. This is because when you use this theory the fourth wave will never exist. We can however see that while the area of existence with infinite bottom is little, it does exist under certain circumstances. Especially when the shear flow and wave frequencies are small.

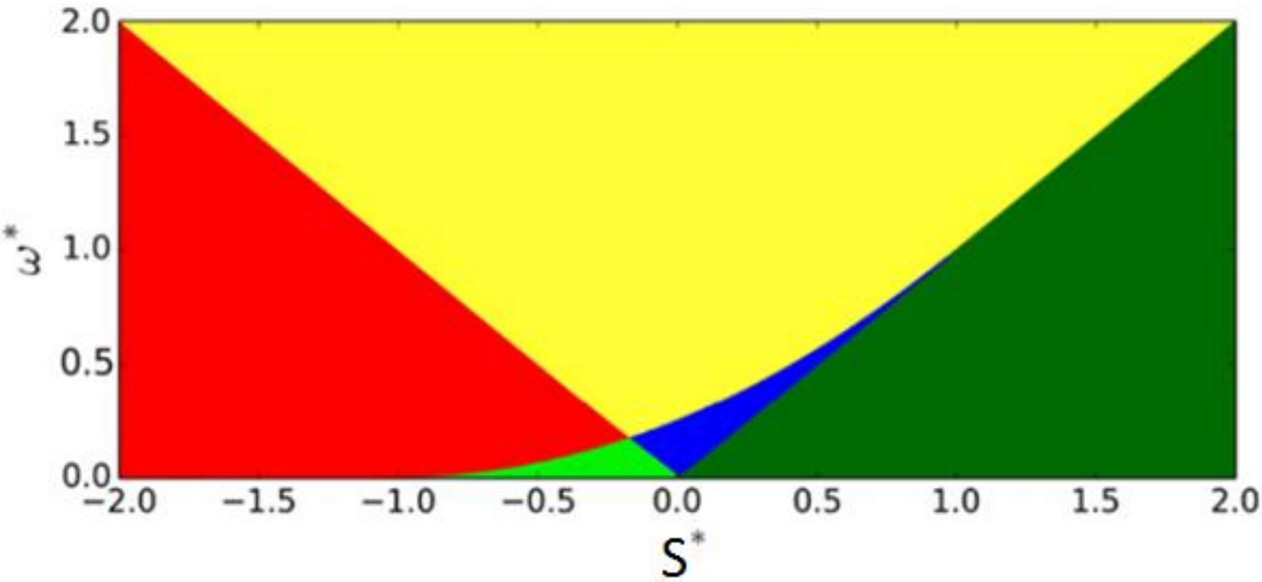

Figure 7.3: Number of existing dispersive waves. Red: One dispersive wave, yellow: two dispersive waves, green: three dispersive waves and blue: four dispersive waves. Source: Tyvand and Lepperød (2015)

Tyvand and Lepperød made a very good figure that shows graphically how many of the dispersive waves will exist in the ω^*, S^* plane. This figure has been repeated here as figure 7.3. Red color means that under these circumstances will only dispersive wave one exist. The yellow area marks where the first and second dispersive wave exist. While the light green area marks where dispersive wave one, three and four exist. The first, second and the third wave exists in the dark green area. Only in the little blue region will all four waves exist.

8. Far-field solution for infinite bottom

We now write the current equation for the free surface

$$\frac{g}{q_0}\zeta(x,t) = -i\int_{-\infty}^{\infty}\frac{(\omega-U_0k)e^{-kD}}{k-\frac{(\omega-U_0k)(\omega-U_0k+S)}{g}}\left(1+\frac{1}{\frac{\omega-U_0k}{S}+kD}\right)e^{ikx-i\omega t}\frac{dk}{2\pi}, k>0 \quad (8.1a)$$

$$\frac{g}{q_0}\zeta(x,t) = -i\int_{-\infty}^{\infty}\frac{(\omega-U_0k)e^{kD}}{-k-\frac{(\omega-U_0k)(\omega-U_0k-S)}{g}}\left(1-\frac{1}{\frac{\omega-U_0k}{S}+kD}\right)e^{ikx-i\omega t}\frac{dk}{2\pi}, k<0 \quad (8.1b)$$

We will now again slightly move the poles off the k-axis by introducing (3.1) with the effects of (3.2)

$$\frac{g}{q_0}\zeta(x,t) = -i\int_{-\infty}^{\infty}\frac{(\omega-U_0k)e^{-kD}}{\Gamma_+(k)-i\varepsilon\Phi_+(k)}\left(1+\frac{1}{-\frac{U_0-SD}{S}(k-k_c(1+i\varepsilon))}\right)e^{ikx-i\omega t}\frac{dk}{2\pi}, k>0 \quad (8.2a)$$

Where $\Gamma_+(k) = k-\frac{(\omega-U_0k)(\omega-U_0k+S)}{g}$, $\phi_+(k) = \frac{2\omega}{g}\left(\omega-U_0k+\frac{S}{2}\right)$ and $k_c = \frac{\omega}{U_0-SD}$

$$\frac{g}{q_0}\zeta(x,t) = -i\int_{-\infty}^{\infty}\frac{(\omega-U_0k)e^{kD}}{\Gamma_-(k)-i\varepsilon\Phi_-(k)}\left(1-\frac{1}{-\frac{U_0-SD}{S}(k-k_c(1+i\varepsilon))}\right)e^{ikx-i\omega t}\frac{dk}{2\pi}, k<0 \quad (8.2b)$$

Where $\Gamma_-(k) = -k-\frac{(\omega-U_0k)(\omega-U_0k-S)}{g}$, $\Phi_-(k) = \frac{2\omega}{g}\left(\omega-U_0k-\frac{S}{2}\right)$ and $k_c = \frac{\omega}{U_0-SD}$

We will now for simpler writing later rename the right side of the equation without the integral sign

$$f_+(k) = -i\,\frac{(\omega-U_0k)e^{-kD}}{\Gamma_+(k)-i\varepsilon\Phi_+(k)}\left(1+\frac{1}{-\frac{U_0-SD}{S}(k-k_c(1+i\varepsilon))}\right)\frac{e^{ikx-i\omega t}}{2\pi} \quad (8.3a)$$

$$f_-(k) = -i\,\frac{(\omega-U_0k)e^{kD}}{\Gamma_-(k)-i\varepsilon\Phi_-(k)}\left(1-\frac{1}{-\frac{U_0-SD}{S}(k-k_c(1+i\varepsilon))}\right)\frac{e^{ikx-i\omega t}}{2\pi} \quad (8.3b)$$

As before we have a simple pole when $k = k_c(1+i\varepsilon)$, we are interested in the real part of the residue, which is given by

$$\operatorname*{Res}_{k-k_c} f_+(k) = \lim_{k\to k_c}(k-k_c)f_+(k) = -\frac{i}{2\pi}\frac{(\omega-U_0k)e^{-k_cD}}{-\frac{U_0-SD}{S}\Gamma_+(k_c)}e^{ik_cx-i\omega t}, U_0>SD \quad (8.4a)$$

$$\operatorname*{Res}_{k\to k_c} f_-(k) = \lim_{k\to k_c}(k-k_c)f_-(k) = \frac{i}{2\pi}\frac{(\omega-U_0k_c)e^{k_cD}}{-\frac{U_0-SD}{S}\Gamma_-(k_c)}e^{ik_cx-i\omega t}, U_0<SD \quad (8.4b)$$

As before the imaginary part of the pole decides which way we have to close the integration and give us the sign in front of the residual. When $U_0 > SD$, the imaginary part of the pole will be positive, and we will get a contribution in the positive direction where we close the integration with a positive semicircle. The critical wave number will in this case be is positive, resulting in a wave in the positive direction. When $U_0 < SD$, we get the opposite, the imaginary part of the pole will be negative and we close the integration with a negative semicircle. The value of the critical wave will in this situation be negative, giving a wave in the negative direction.

Therefore, the contribution to the far field wave system from the critical wave will be

$$2\pi i \operatorname*{Res}_{k\to k_c} f_+(k) = \frac{(\omega-U_0k_c)e^{-k_cD}}{-\frac{U_0-SD}{S}\Gamma_+(k_c)}e^{ik_cx-i\omega t}, k_c > 0 \tag{8.5a}$$

$$-2\pi i \operatorname*{Res}_{k\to k_c} f_-(k) = \frac{(\omega-U_0k_c)e^{k_cD}}{-\frac{U_0-SD}{S}\Gamma_-(k_c)}e^{ik_cx-i\omega t}, k_c < 0 \tag{8.5b}$$

We now need to start looking at the dispersive waves. The poles related to the dispersive waves have been moved slightly off the k-axis and are now located at:

$$k_{1,2} = k_{1,2} + i\varepsilon\frac{\Phi_+(k_{1,2})}{\Gamma_+'(k_{1,2})} \tag{8.6a}$$

$$k_{3,4} = k_{3,4} + i\varepsilon\frac{\Phi_-(k_{3,4})}{\Gamma_-'(k_{3,4})} \tag{8.6b}$$

Where $\Gamma_\pm{}'(k) = \frac{d\Gamma_\pm(k)}{dk}$.

For the positive waves, we have

$$\Phi_+(k_{1,2}) = 2\omega(\omega - U_0k_{1,2} + \frac{S}{2})$$

And $\Gamma_+'(k_{1,2}) = 1 + \frac{2U_0}{g}(\omega - U_0k_{1,2} + \frac{S}{2})$

Where we notice that the parenthesis are identical, yielding that

$$\frac{\Phi_+(k_{1,2})}{\Gamma_+'(k_{1,2})} = \frac{2\omega(\omega-U_0k_{1,2}+\frac{S}{2})}{1+\frac{2U_0}{g}(\omega-U_0k_{1,2}+\frac{S}{2})} \tag{8.7a}$$

We now multiply this with positive number $(\frac{U_0}{g})^2$ and rewrite the equation with the dimensionless numbers we defined when we discussed the existence of the four dispersive waves in subchapter 7. We repeat the dimensionless numbers here: $\omega^* = \frac{U_0}{g}\omega, S^* = \frac{U_0}{g}S$ and $k_{1,2,3,4}^* = \frac{U_0^2}{g}k$. We rewrite the equation into dimensionless numbers to make it easier to discuss and plot the results. We can rewrite our equation to

$$(\frac{U_0}{g})^2 \frac{\Phi_+(k_{1,2})}{\Gamma_+'(k_{1,2})} = \frac{\omega^*(2\omega^* - 2k_{1,2}^* + S^*)}{1 + 2\omega^* - 2k_{1,2}^* + S^*} \tag{8.7b}$$

For the negative waves, we have

$$\Phi_-(k_{3,4}) = 2\omega(\omega - U_0 k_{3,4} - \frac{S}{2})$$

And $\Gamma_-'(k) = -1 + \frac{2U_0}{g}(\omega - U_0 k_{3,4} - \frac{S}{2})$

Where we again notice that the parenthesis are identical. This gives

$$\frac{\Phi_-(k_{3,4})}{\Gamma_-'(k_{3,4})} = \frac{2\omega\left(\omega - U_0 k_{3,4} - \frac{S}{2}\right)}{-1 + \frac{2U_0}{g}(\omega - U_0 k_{3,4} - \frac{S}{2})} \tag{8.8a}$$

We also make this equation dimensionless using the same dimensionless numbers as we did in equation (8.7):

$$\left(\frac{U_0}{g}\right)^2 \frac{\Phi_-(k_{3,4})}{\Gamma_-'(k_{3,4})} = \frac{\omega^*(2\omega^* - 2k_{3,4}^* - S^*)}{-1 + 2\omega^* - 2k_{3,4}^* - S^*} \tag{8.8b}$$

We now aim to substitute the wave numbers to have an equation with only two variables. The dispersive wave numbers are given dimensionless as

$$2k_1^* = 1 + S^* + 2\omega^* + \sqrt{(1 + S^*)^2 + 4\omega^*}, k_1^* > 0$$

$$2k_2^* = 1 + S^* + 2\omega^* - \sqrt{(1 + S^*)^2 + 4\omega^*}, k_2^* > 0$$

$$2k_3^* = -1 - S^* + 2\omega^* - \sqrt{(1 + S^*)^2 - 4\omega^*}, k_3^* < 0$$

$$2k_4^* = -1 - S^* + 2\omega^* + \sqrt{(1 + S^*)^2 - 4\omega^*}, k_4^* < 0$$

We found these wave numbers in subchapter 7.

Where we have named the variables so the same waves get the same number here as in Tyvand and Lepperød's (2015) work. We can now substitute this in the equations above and removing cancelling terms finding the value of the imaginary part of $f_\pm(k)$

$$(\frac{U_0}{g})^2 \frac{\Phi_+(k_1)}{\Gamma_+'(k_1)} = \frac{\omega^*\left(1 + \sqrt{(1 + S^*)^2 + 4\omega^*)}\right)}{\sqrt{(1 + S^*)^2 + 4\omega^*)}}, \tag{8.9a}$$

$$(\frac{U_0}{g})^2 \frac{\Phi_+(k_2)}{\Gamma_+'(k_2)} = -\frac{\omega^*\left(1 - \sqrt{(1 + S^*)^2 + 4\omega^*)}\right)}{\sqrt{(1 + S^*)^2 + 4\omega^*)}}, \tag{8.9b}$$

$$\left(\frac{U_0}{g}\right)^2 \frac{\Phi_-(k_3)}{\Gamma_-'(k_3)} = \frac{\omega^*(1+\sqrt{(1+S^*)^2-4\omega^*})}{\sqrt{(1+S^*)^2-4\omega^*)}}, \quad (8.9c)$$

$$\left(\frac{U_0}{g}\right)^2 \frac{\Phi_-(k_4)}{\Gamma_-'(k_4)} = -\frac{\omega^*(1-\sqrt{(1+S^*)^2-4\omega^*)}}{\sqrt{(1+S^*)^2-4\omega^*)}} \; . \quad (8.9d)$$

We assume here that the denominator for the third and fourth wave is not zero. This situation cause zero group velocity for both wave three and four, and results in Doppler effects that will be discussed in a later chapter.

The square roots in equations (8.9a-d) has to be positive, in addition the dimensionless frequency ω^* also has to be positive. The first and second wave number must be positive, while the third and fourth wave number must be negative. That the square roots and the dimensionless frequency is positive means that equations (8.9a) and (8.9c) are positive for all values of dimensionless shear flow S^* and dimensionless frequency ω^*. Equation 8.9b and 8.9d can apparently take both positive and negative value. However, there are no solutions of k_2 where $1 > \sqrt{(1+S^*)^2+4\omega^*}$ making equation 8.9b positive for all values of dimensionless shear flow S^* and dimensionless frequency ω^* the second wave exists. Similarly, all solution of k_4 requires that $1 > \sqrt{(1+S^*)^2-4\omega^*}$ making equation 8.9d negative for all values of dimensionless shear flow S^* and dimensionless frequency ω^* the fourth wave exists.

This means we close the integration with a positive semicircle for the first three dispersive waves, and a negative semicircle for the fourth dispersive wave. This coincides with the group velocities that Tyvand and Lepperød (2015) found. This also means that the third wave has phase and group velocities in opposite directions as was pointed out by Tyvand and Lepperød(2015).

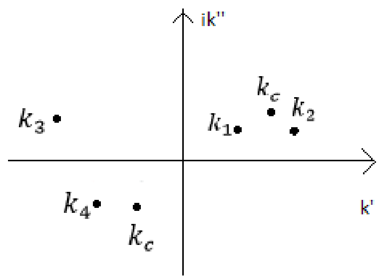

Figure 8.5: Location of the poles in k = k' + ik'' plane. The critical wave exists only in one location, but has been put in twice here due to that it can exist for both positive and negative wave numbers.

The poles have been plotted in a *k = k' + ik''* plane in figure 8.5. In this figure, the critical wave has been plotted twice because it can exist for both positive and negative numbers. We see that k_3 is a bit special, as it exists for negative real value, but positive imaginary value. This is because the group velocity of this wave is positive, while the phase velocity is negative.

The residuals are found using the rules for simple poles. If a function $f(x)$ can be written as $\frac{h(x)}{g(x)}$, the residual will be found at: $\frac{h(x)}{g'(x)}$. This requires that $g(x) = 0$ and $g'(x) \neq 0$ as well as that both *h(x)* and *g(x)* are holomorphic. This means that only the denominator is differentiated once. Doing this for the poles of the dispersion relation gives us the following residuals

$$\operatorname*{Res}_{k\to k_{1,2}} f_+(k) = -\frac{i}{2\pi}\frac{(\omega - U_0 k_{1,2})e^{-k_{1,2}D}}{\Gamma_+'(k_{1,2})}\left(1 + \frac{1}{-\frac{U_0 - SD}{S}(k_{1,2} - k_c)}\right)e^{ik_{1,2}x - i\omega t} \tag{8.10}$$

$$\operatorname*{Res}_{k\to k_{3,4}} f_-(k) = -\frac{i}{2\pi}\frac{(\omega - U_0 k_{3,4})e^{k_{3,4}D}}{\Gamma_-'(k_{3,4})}\left(1 - \frac{1}{-\frac{U_0 - SD}{S}(k_{3,4} - k_c)}\right)e^{ik_{3,4}x - i\omega t} \tag{8.11}$$

We now want to complete the integral and find the wave equations in the positive and negative direction. We first find the wave equations for the positive waves.

$$\begin{aligned}\frac{g}{q_0}\zeta = \int_{\wedge_+} f_+(k)dk = 2\pi i \operatorname*{Res}_{k\to k_1} f_+(k) + 2\pi i \operatorname*{Res}_{k\to k_1} f_+(k) + 2\pi i \operatorname*{Res}_{k\to k_c} f_+(k)\, \Theta(U_0 - SD) \\ = \frac{(\omega - U_0 k_1)e^{-k_1 D}}{\Gamma_+'(k_1)}\left(1 + \frac{1}{-\frac{U_0 - SD}{S}(k_1 - k_c)}\right)e^{ik_1 x - i\omega t} \\ + \frac{(\omega - U_0 k_2)e^{-k_2 D}}{\Gamma_+'(k_2)}\left(1 + \frac{1}{-\frac{U_0 - SD}{S}(k_2 - k_c)}\right)e^{ik_2 x - i\omega t} \\ + \frac{(\omega - U_0 k)e^{-k_c D}}{-\frac{U_0 - SD}{S}\Gamma_+(k)}e^{ik_c x - i\omega t}\, \Theta(U_0 - SD)\end{aligned} \tag{8.12}$$

Where $\Gamma_+(k) = k - \frac{(\omega - U_0 k)(\omega - U_0 k + S)}{g}$ and $\Gamma_+'(k_{1,2}) = 1 + \frac{2U_0}{g}(\omega - U_0 k_{1,2} + \frac{S}{2})$.

In addition to this there are near-field poles that we will not discuss here, meaning that this is only valid in the far-field, meaning that equation 8.12 is only valid in the far-field when $x \to \infty$

Which is the full wave field for positive k's. Remember that the last term only exists when $U_0 > SD$ and represent the critical wave in the positive direction.

The term in the equation that accounts for the second wave assumes that the second wave exists. If this wave does not exist, this term is zero. The areas in the ω^*, S^* plane that the second dispersive wave exists is shown in figure 7.1.

For the negative direction, we will have to do the integral both for positive semicircle, and the negative semicircle because the poles have been moved both below and above the imaginary axis. First we close with a positive semicircle with only the third wave number will have a pole.

$$\frac{g}{q_0}\zeta = \int_{\wedge_+} f_-(k)dk = 2\pi i \operatorname*{Res}_{k\to k_3} f_-(k)$$

$$= \frac{(\omega - U_0 k_3)e^{k_3 D}}{\Gamma_-'(k_3)}\left(1 - \frac{1}{-\frac{U_0 - SD}{S}(k_3 - k_c)}\right)e^{ik_3 x - i\omega t} \tag{8.13a}$$

In addition we will have to close the integration with a negative semicircle to include the poles with negative imaginary value of both the fourth and critical wave.

$$\frac{g}{q_0}\zeta = \int_{\wedge_-} f_-(k)dk = -2\pi i \operatorname*{Res}_{k\to k_4} f_-(k) - 2\pi i \operatorname*{Res}_{k\to k_c} f_-(k)$$

$$= -\frac{(\omega - U_0 k_4)e^{k_4 D}}{\Gamma_-'(k_4)}\left(1 - \frac{1}{-\frac{U_0 - SD}{S}(k_4 - k_c)}\right)e^{ik_4 x - i\omega t}$$

$$+ \frac{(\omega - U_0 k_c)e^{k_c D}}{-\frac{U_0 - SD}{S}\Gamma_-(k_c)} e^{ik_c x - i\omega t}\Theta(SD - U_0) \tag{8.13b}$$

In addition to this there are near-field poles that we will not discuss here, meaning that this is only valid in the far-field.

Even though we have done the integral twice here, the solution will be the sum of these two. This is because the integral originally had all three poles on the axis. We used this way to go around the poles to find the sign of the residual. Tyvand and Lepperød (2015) found this using the radiation condition.

$$\frac{g}{q_0}\zeta = \frac{(\omega - U_0 k_3)e^{k_3 D}}{\Gamma_-'(k_3)}\left(1 - \frac{1}{-\frac{U_0 - SD}{S}(k_3 - k_c)}\right)e^{ik_3 x - i\omega t}$$

$$- \frac{(\omega - U_0 k_4)e^{k_4 D}}{\Gamma_-'(k_4)}\left(1 - \frac{1}{-\frac{U_0 - SD}{S}(k_4 - k_c)}\right)e^{ik_4 x - i\omega t}$$

$$+ \frac{(\omega - U_0 k_c)e^{k_c D}}{-\frac{U_0 - SD}{S}\Gamma_-(k_c)} e^{ik_c x - i\omega t}\Theta(SD - U_0) \tag{8.14}$$

Valid when $x \to -\infty$

Where Where $\Gamma_{-}(k) = -k - \frac{(\omega - U_0 k)(\omega - U_0 k - S)}{g}$ and $\Gamma_{-}'(k) = -1 + \frac{2U_0}{g}(\omega - U_0 k_{3,4} - \frac{S}{2})$

The terms in the equation that accounts for the third and fourth wave, assumes that these waves exists. If this wave does not exist this term will be zero. The domain in the ω^*, S^* plane that the third and fourth dispersive waves exists are shown in figure 7.2. The critical wave however will always exist as long as $U_0 > SD$ which is the same as with dimensionless numbers $S^* > \frac{1}{D^*}$.

In the equations above, we have assumed that the dispersive wave numbers do not equal the critical wave number. In the case where the critical wave number equals the dispersive wave number, we get a double pole in $f_{+}(k)$ or $f_{-}(k)$

It is also possible that the wave number of the third and fourth dispersive wave is equal. This is the situation where their group velocities are both zero. This situation will have a different equation for the waves, which will be subchapter 11.

9. The wave fields in dimensionless form

For better understanding of thes wave fields in subchapter 8 we will now write these equations in a dimensionless form and plot these. We will use the same dimensionless numbers as we have used in previous subchapters, namely: $\omega^* = \frac{U_0}{g}\omega, S^* = \frac{U_0}{g}S, k^*_{1,2,3,4} = \frac{U_0^2}{g}k_{1,2,3,4}$ and $D^* = \frac{g}{U_0^2}D$. In addition we introduce dimensionless room variable $\tilde{x} = \frac{g}{U_0^2}x$ and dimensionless time variable $\tilde{t} = \frac{g}{U_0}t$.

We find the following dimensionless forms of equation 8.12

$$\begin{aligned}\frac{U_0}{q_0}\zeta = &\frac{(\omega^* - k_1^*)e^{-k_1^* D^*}}{\Gamma_{+}'(k_1)}\left(1 + \frac{1}{(D^* - \frac{1}{S^*})(k_1^* - k_c^*)}\right)e^{ik_1^*\tilde{x} - i\omega^*\tilde{t}} \\ &+ \frac{(\omega^* - k_2^*)e^{-k_2^* D^*}}{\Gamma_{+}'(k_2)}\left(1 + \frac{1}{(D^* - \frac{1}{S^*})(k_2^* - k_c^*)}\right)e^{ik_2^*\tilde{x} - i\omega^*\tilde{t}} \\ &+ \frac{(\omega^* - k_c^*)e^{-k_c^* D^*}}{(D^* - \frac{1}{S^*})\Gamma_{+}^*(k_c)}e^{ik_c^*\tilde{x} - i\omega^*\tilde{t}} \qquad (9.1)\end{aligned}$$

Where $\Gamma_{+}'(k) = 1 + 2\omega^* - 2k^* + S)$ and $\Gamma_{+}^*(k) = k^* - (\omega^* - k^*)(\omega^* - k^* + S^*)$ and valid in the far-field when $\tilde{x} \to \infty$.

We repeat the existence of the waves. Wave one exists for all values of ω^* and S^* while the second wave requires that $\omega^* > |S^*|$ for negative S^*.

Similarily we find the dimensionless fo rm of equation 8.14

$$\frac{U_0}{q_0}\zeta = \frac{(\omega^* - k_3^*)e^{k_3^* D^*}}{\Gamma_-'(k_3)}\left(1 + \frac{1}{(D^* - \frac{1}{S^*})(k_3^* - k_c^*)}\right)e^{ik_3^*\tilde{x} - i\omega^*\tilde{t}}$$
$$- \frac{(\omega^* - k_4^*)e^{k_4^* D^*}}{\Gamma_-'(k_4)}\left(1 + \frac{1}{(D^* - \frac{1}{S^*})(k_4^* - k_c^*)}\right)e^{ik_4^*\tilde{x} - i\omega^*\tilde{t}}$$
$$+ \frac{(\omega^* - k_c^*)e^{k_c^* D^*}}{(D^* - \frac{1}{S^*})\Gamma_-^*(k_c)}e^{ik_c^*\tilde{x} - i\omega^*\tilde{t}} \tag{9.2}$$

Where $\Gamma_-'(k) = -1 + 2\omega^* - 2k^* - S)$ and $\Gamma_-^*(k) = -k^* - (\omega^* - k^*)(\omega^* - k^* - S^*)$ and valid in the far-field when $\tilde{x} \to -\infty$.

Both the third and the fourth wave number requires that $0 < \omega^* < \left(\frac{1+S^*}{2}\right)^2$ when $-1 < S^* < 1$. The third wave is further required that $0 < \omega^* < S^*$when $S^* > 1$. The fourth wave exists only for $-1 < S^* < 1$ and has the additional requirement that $S^* < \omega^*$ for $0 < S^* < 1$.

The dimensionless wave numbers are given by:

$$2k_1^* = 1 + S^* + 2\omega^* + \sqrt{(1 + S^*)^2 + 4\omega^*}, k_1^* > 0$$

$$2k_2^* = 1 + S^* + 2\omega^* - \sqrt{(1 + S^*)^2 + 4\omega^*}, k_2^* > 0$$

$$2k_3^* = -1 - S^* + 2\omega^* - \sqrt{(1 + S^*)^2 - 4\omega^*}, k_3^* < 0$$

$$2k_4^* = -1 - S^* + 2\omega^* + \sqrt{(1 + S^*)^2 - 4\omega^*}, k_4^* < 0$$

$$k_c^* = \frac{\omega^*}{1 - S^* D^*}$$

We keep the dimensionless depth at $D^* = 2$ for all our plots. We also only plot for $\tilde{t} = 0$.

The first situation we look at is for $\omega^* = 0.5, S^* = -1$. In this situation only the first of the dispersive waves will exist. In addition, the critical wave number will be positive because $S^* < 1/D^*$. Thus we get no downstream waves at all.

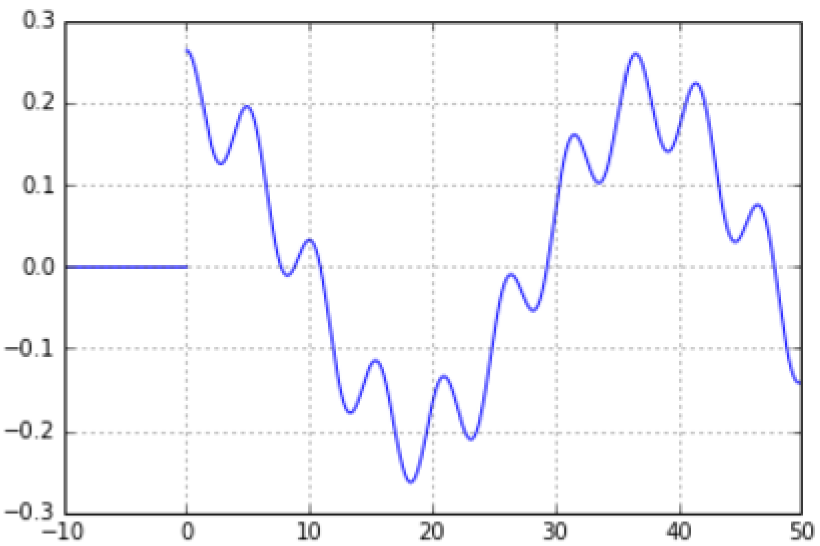

Figure 9.1: Plot of wave amplitudes when $\omega^* = 0.5$, $S^* = -1$ and $D^* = 2$. Only the first dispersive and the critical wave in the positive direction exists in this case. No downstream waves

The second situation we look at is for $\omega^* = 1$, $S^* = 0.2$. In this situation the two dispersive waves in the positive direction will exist. In addition, the critical wave number will be positive because $S^* < 1/D^*$. Thus we get no downstream waves at all.

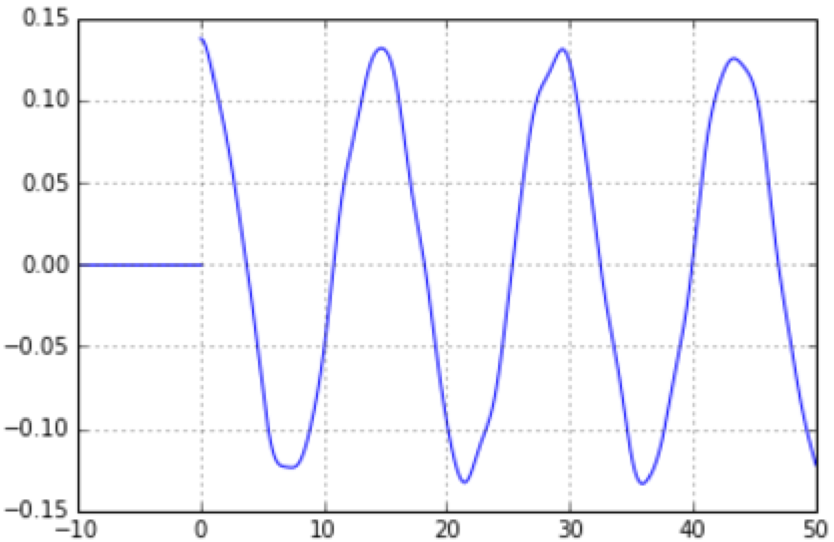

Figure 9.2: Plot of wave amplitudes when $\omega^* = 1$, $S^* = 0.2$ and $D^* = 2$. The first and second dispersive wave aswell as the critical wave in the positive direction exists in this situation. No downstream waves

The third situation we look at is for $\omega^* = 0.5$, $S^* = 1$. In this situation the two dispersive waves in the positive direction will exist, as well as the third dispersive wave in the negative direction. In addition, the critical wave number will be negative because $S^* > 1/D^*$. Thus we get no downstream waves at all.

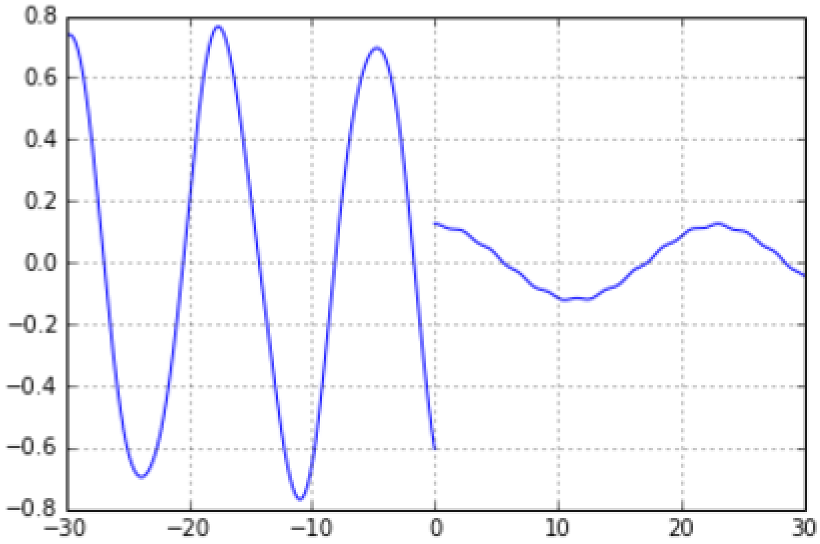

Figure 9.3: Plot of wave amplitudes when $\omega^* = 0.5$, $S^* = 1$ and $D^* = 2$. The first and second dispersive wave exists in the positive direction. In the negative direction we have the third dispersive wave as well as the critical wave.

In the last situation we will look at where all the waves exist. This is possible for $\omega^* = 0.45$, $S^* = 0.4$. For these values the critical wave will be positive, because $S^* > \frac{1}{D^*}$.

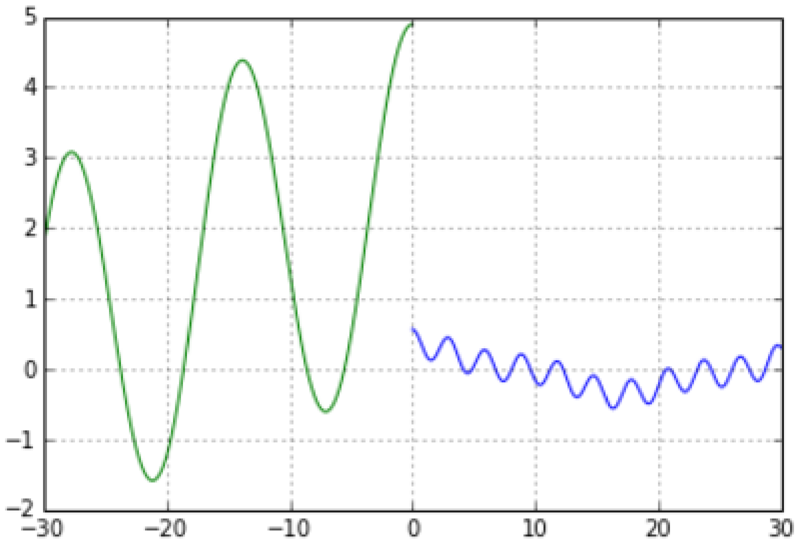

Figure 9.4: Plot of wave amplitudes when $\omega^* = 0.45$, $S^* = 0.4$ and $D^* = 2$. The first and second dispersive wave as well as the critical wave in the positive direction. In the negative direction both the third and fourth dispersive wave exists..

That these plots are discontinuous at $\tilde{x} = 0$ is of little note, as we have omitted near-field poles in our solution.

10. Resonance between the dispersive waves and the critical wave

There are possibilities for resonance between the dispersal waves and the critical wave. This requires that the phase velocities are equal between the respective dispersive waves and the critical wave. Phase velocities are given by: $v_f = \frac{\omega}{k}$. Since ω is a given number, this means that equal wave numbers results in equal phase velocities. We make the critical wave number dimensionless with the same dimensionless numbers as used the previous subchapters.

$$k_c^* = \frac{U_0^2}{g} k_c, S^* = \frac{U_0}{g} S, \omega^* = \frac{U_0}{g} \omega, D^* = \frac{Dg}{U_0^2} \tag{10.1}$$

Which makes the critical wave dimensionless in the following way.

$$k_c^* = \frac{\omega^*}{1 - S^* D^*} \tag{10.2}$$

Earlier we required that $U_0 > SD$ to get a positive direction of flow for the critical wave, while $U_0 < SD$ would result in a negative flow. With this dimensionless equation we see that for a positive direction of flow we now require $S < D$, while $S > D$ would result in a negative direction of flow. Earlier we said that if $U_0 = SD$ we will get no critical wave, this has been translated into that for $S = D$ we get no critical wave. In the three following plots in figure E, F and G, we will see that resonance can only occur on the left side of the line $S = \frac{1}{D}$ for the first and second dispersive wave, while the third dispersive wave will only see resonance on the right side of this line. This is of course because the critical wave number is positive on the left side of this line, while it is negative on the right side of this line

Because the critical wave depends on the depth of the oscillated source, which the dispersive waves do not, the resonance between the waves will depend on a third dimensionless variable, namely the dimensionless depth. We will now first find the curves in S^*, ω^* plane where there will be resonance between the different dispersive waves and the critical wave

$$2k_1^* = 2k_c^*, U_0 > SD$$

$$1 + S^* + 2\omega^* + \sqrt{(1 + S^*)^2 + 4\omega^*} = \frac{2\omega^*}{1 - S^* D^*}, U_0 > SD \quad (10.3)$$

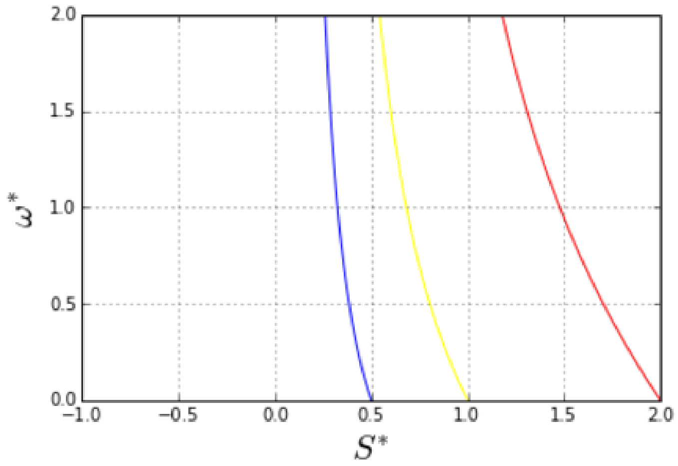

Figure 10.1: Plot of the resonance between the first dispersive wave and the critical wave. Red: D=0.5, yellow: D*=1 and blue D*=2*

In figure 10.1 we see where in ω^*, S^* plane the wave numbers of the dispersive wave one and the critical wave are equal. The first we see is that resonance only occurs for positive dimensionless shear flow. We can also see that as the dimensionless wave frequency approaches zero, the resonance curve approaches the value $S^* = \frac{1}{D^*}$, which equals that $S = \frac{U_0}{D}$. As the dimensionless shear flow approaches zero, the dimensionless wave frequency approaches infinity. The resonance occurs at a lower dimensionless shear frequency as dimensionless shear frequency increases.

For resonance between the critical wave and the second dispersive wave, the wave number has to be

$$k_2^* = k_c^*, U_0 > SD$$

$$1 + S^* + 2\omega^* - \sqrt{(1 + S^*)^2 + 4\omega^*} = \frac{2\omega^*}{1 - S^* D^*}, U_0 > SD \quad (10.4)$$

In the region of the S^*, ω^* plane that the second wave exists.

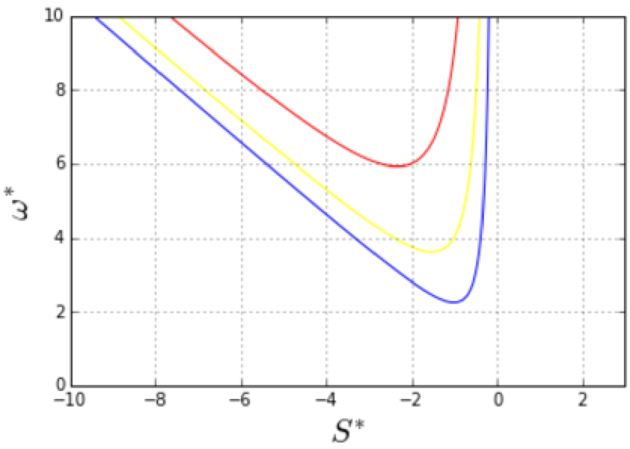

Figure 10.2: Plot of the resonance between the second dispersive wave and the critical wave. Red: D=0.5, yellow: D*=1 and blue D*=2*

In figure 10.2 we see where in ω^*, S^* plane the wave numbers of the dispersive wave two and the critical wave are equal In this figure, we see that resonance will only occur for negative dimensionless shear flow. If we look at the existence of the second wave, we see that all these lines are inside the existence of the second dispersive wave. We see that as dimensionless depth increases, it gets closer to the limit of the existence of the dispersive wave, but always existing. The resonance occurs at lower dimensionless wave frequency as the dimensionless shear flow increases until it reaches a minimum that depends on the depth. The minimum will be at a larger dimensionless shear flow as dimensionless depth increases. After the minimum, the slope increases quickly and as the dimensionless shear flow approaches zero, the dimensionless wave frequency required for resonance approaches infinite.

For resonance between the critical wave and the third dispersive wave, we have

$$k_3^* = k_c^*, U_0 < SD$$

$$-1 - S^* + 2\omega^* - \sqrt{(1+S^*)^2 - 4\omega^*} = \frac{2\omega^*}{1 - S^* D^*}, U_0 < SD \qquad (10.5)$$

In the region of the S^*, ω^* plane that the third wave exists as given in figure Y.

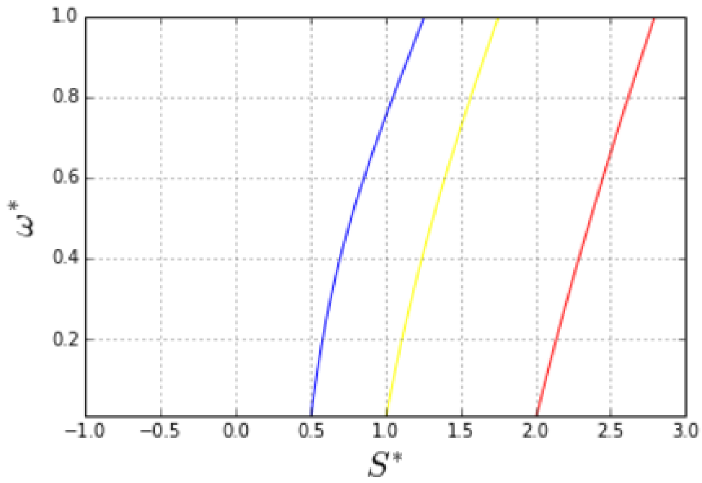

Figure 10.3: Plot of the resonance between third dispersive wave and the critical wave. Red: D=0.5, yellow: D*=1 and blue D*=2*

In figure 10.3, we see where in ω^*, S^* plane the wave numbers of the dispersive wave three and the critical wave are equal. In this figure, we can see that the resonance occurs at a lower dimensionless shear flow as the dimensionless depth increases. We can also see that as the dimensionless wave frequency approaches zero, the resonance curve approaches the value $S^* = \frac{1}{D^*}$, which equals that $S = \frac{U_0}{D}$. This means that we approaches the area where $U_0 - SD = 0$ and the critical wave will not exist. Resonance can only occur for high enough dimensional shear flow to keep the critical wave going in the negative direction. The resonance occurs at a larger dimensionless wave frequency as the dimensionless shear flow increases.

For resonance between the critical wave and the fourth dispersive wave, we have

$2k_4^* = 2k_c^*, U_0 < SD$

$$-1 - S^* + 2\omega^* + \sqrt{(1+S^*)^2 - 4\omega^*} = \frac{2\omega^*}{1 - S^* D^*}, U_0 < SD \quad (10.6)$$

In the region of the ω^*, S^* plane that the fourth wave exists.

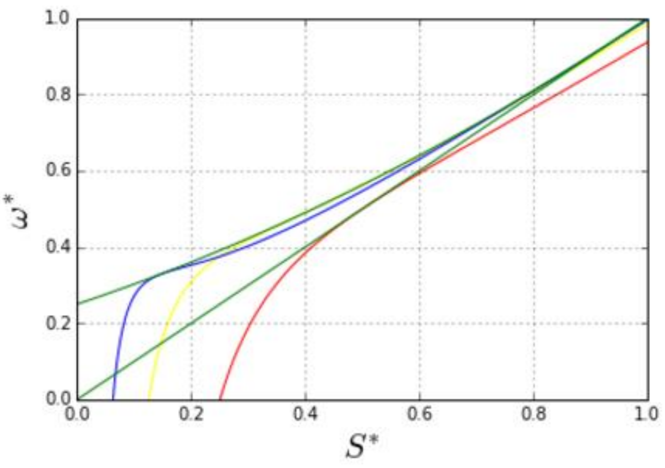

Figure 10.4: Plot of resonance between the fourth dispersive wave and the critical wave. Red: D=2, yellow: D*=4 and blue D*=8. The green line marks the upper and lower limit of existence of the fourth wave.*

We see in figure 10.4, where we have included green lines for the upper and lower limit of existence of the fourth wave, the required S^* and ω^* for resonance. We see that the lowest value of the dimensionless depth required for resonance between the critical wave and the fourth dispersive wave is two(red line). This is higher than what is necessary for all the other dispersive waves. We can also see that first for a dimensionless depth of eight (blue line) the line will touch the upper existence of the fourth wave. Tyvand and Lepperød (2015) pointed out that it is at the upper limit of existence for both third and fourth wave number that Doppler effects are possible, which we will discuss in the next subchapter, 10.

From figures 10.1 to 10.4, as well as figures 7.1 and 7.2, which describes the existence of these waves, we see that the critical wave can actually resonance with all four dispersive waves. The fourth wave however requires a larger dimensional depth before resonance is possible due to the existence of this wave. Remember that the dimensionless depth is given as: $D^* = \frac{g}{U_0^2}D$. This means that the wave requires a larger dimensionless depth of the submerged oscillatory source as U_0 increases. The denominator in the dimensionless depth is the surface velocity squared, showing that the depth has to increase more than the surface velocity for resonance to be possible. The lowest number the dimensionless depth can have for resonance is two, where there will be resonance at only one single point in the ω^*, S^* plane. This can be seen in figure 9.4 where the red line, which is the line for dimensionless depth equal to two, barely touches the lower limit.

When the critical wave number equals one of the dispersive wave numbers we will have double poles for $f_+(k)$ or $f_-(k)$ and the equations for the waves will become quite different than those we found in subchapter 8.

To find the solution of the contour integrals, we need to find the residue. When we have resonance we have a second order pole, because we have two factors that both are zero. When we have a function f(x) with a double pole, the Residue becomes: $\underset{x \to x_{pole}}{Res} f(x) = \lim_{x \to x_{pole}} \left(x - x_{pole}\right)^2 f(x)$. We will now first do this for the positive resonance

$$2\pi i \underset{k=k_c=k_1 \cup k_2}{\operatorname{Res}} f_+(k) = \underset{k=k_c}{\operatorname{Res}} \frac{\gamma_+(k)}{\Gamma_+(k)(k-k_c)} = \lim_{k \to k_c} \frac{d}{dk} \frac{(k-k_c)\gamma_+(k)}{\Gamma_+(k)}$$

$$= \lim_{k \to k_c} \frac{\gamma(k)\Gamma_+(k) + (k-k_c)\gamma'(k)\Gamma_+(k) - \Gamma_+'(k)\gamma(k)}{\Gamma_+(k)^2} \qquad (10.7a)$$

Where $\gamma_+(k) = (k-k_c)(\omega - U_0 k)e^{-kD} e^{ikx - i\omega t} - \frac{(\omega - U_0 k)e^{-kD}}{\frac{U_0 - SD}{S}} e^{ikx - i\omega t}$ which is the nonsingular part of the integrand $f_+(k)$.

We need to use L'Hôpital's rule to find the limit, resulting in

$$2\pi i \underset{k=k_c=k_1 \cup k_2}{\operatorname{Res}} f_+(k) = \lim_{k \to k_c} \frac{2\gamma_+{}'(k)\Gamma_+(k) + (k-k_c)(\gamma_+''(k)\Gamma_+(k) - \Gamma_+''(k)\gamma_+(k)}{2\Gamma_+'(k)\Gamma(k)} \qquad (10.7b)$$

Where we need to use L'Hôpital's rule a second time to find the limit, resulting in

$$2\pi i \underset{k=k_c=k_1 \cup k_2}{\operatorname{Res}} f_+(k) = \frac{\gamma_+'(k)\Gamma_+'(k) - \frac{1}{2}\Gamma_+''^{(k)}\gamma_+(k)}{\Gamma_+{}'(k)^2} \qquad (10.7c)$$

We now need to find the value of the different functions that remain in this solution. Starting with the non-singular function $\gamma_+(k_c)$ and its derivative.

$$\gamma_+(k_c) = -\frac{(\omega - U_0 k_c)e^{-k_c D}}{\frac{U_0 - SD}{S}} e^{ik_c x - i\omega t}$$

$$\gamma_+'(k_c) = \frac{e^{-k_c D}}{\frac{U_0 - SD}{S}} e^{ik_c x - i\omega t}\left(D(\omega - U_0 k_c) + U_0 - ix(\omega - U_0 k_c)\right)$$

We also need the first and second order derivative of the dispersion relation for the positive wave numbers. We already know the first derivative from subchapter 8, and it is quite simple to find the second order derivative.

$$\Gamma_+'(k_c) = 1 + \frac{2U_0}{g}\left(\omega - U_0 k_c + \frac{S}{2}\right)$$

$$\Gamma_+''(k_c) = -\frac{2U_0^2}{g}$$

We also need to do the same for negative wave numbers

$$2\pi i \underset{k=k_c=k_3\cup k_4}{\text{Res}} f_-(k) = \underset{k=k_c}{\text{Res}} \frac{\gamma_-(k)}{\Gamma_-(k)(k-k_c)} = \lim_{k\to k_c} \frac{d}{dk}\frac{(k-k_c)\gamma_-(k)}{\Gamma_-(k)}$$

Doing the exact same thing as with the positive resonance residual, we get the result

$$2\pi i \underset{k=k_c=k_3\cup k_4}{\text{Res}} f_-(k) = \frac{\gamma_-'(k)\Gamma_-'(k)-\frac{1}{2}\Gamma_-''(k)\gamma_-(k)}{\Gamma_-'(k)^2} \tag{10.8}$$

Where $\gamma_-(k) = (k-k_c)(\omega - U_0 k)e^{kD}e^{ikx-i\omega t} + \frac{(\omega-U_0k)}{\frac{U_0-SD}{S}} e^{kD}e^{ikx-i\omega t}$

Is the nonsingular part of the integrand of $f_-(k)$

We now need to find the value of the different functions that remain in this solution. Starting with the non-singular function $\gamma_-(k_c)$ and its derivative.

$$\gamma_-(k_c) = \frac{(\omega - U_0 k_c)}{\frac{U_0 - SD}{S}} e^{k_c D} e^{ik_c x - i\omega t}$$

$$\gamma_-'(k_c) = \frac{e^{k_c D}}{\frac{U_0 - SD}{S}} e^{ik_c x - i\omega t} \left(D(\omega - U_0 k_c) - U_0 + ix(\omega - U_0 k_c)\right)$$

We also need the first and second order derivative of the dispersion relation for the negative wave numbers. We already know the first derivative from subchapter 8, and it is quite simple to find the second order derivative.

$$\Gamma_-'(k_c) = -1 + \frac{2U_0}{g}\left(\omega - U_0 k_c - \frac{S}{2}\right)$$

$$\Gamma_-''(k_c) = -\frac{2U_0^2}{g}$$

We notice that $\Gamma_-''(k_c) = \Gamma_+''(k_c)$ and that the difference between $\gamma_-'(k_c)$ and $\gamma_+'(k_c)$ is that the sign is switched for U_0 and the imaginary part of the parenthesis.

The first and second wave has poles that has a positive imaginary part, and so does the critical wave. This means that we will close the integration still with a positive semicircle when these two waves are in resonance with the critical wave.

For resonance with dispersive wave one, we get the following equation in the positive direction

$$\frac{g}{q_0}\zeta = \int_0^\infty f_+(k)dk = 2\pi i \operatorname*{Res}_{k\to k_c = k_1} f_+(k) + 2\pi i \operatorname*{Res}_{k\to k_2} f_+(k)$$

$$= \frac{\left(D(\omega - U_0 k_c) + U_0 - ix(\omega - U_0 k_c)\right)\left(1 + \frac{2U_0}{g}\left(\omega - U_0 k_c + \frac{S}{2}\right)\right) - \frac{U_0^2}{g}(\omega - U_0 k_c)}{\left(1 + \frac{2U_0}{g}\left(\omega - U_0 k_c + \frac{S}{2}\right)\right)^2} \frac{e^{-k_c D}}{\frac{U_0 - SD}{S}} e^{ik_c x - i\omega t}$$

$$+ \frac{(\omega - U_0 k_2)e^{-k_2 D}}{\Gamma_+{}'(k_2)}\left(1 + \frac{1}{-\frac{U_0 - SD}{S}(k_2 - k_c)}\right) e^{ik_2 x - i\omega t} \qquad (10.9)$$

Where $\Gamma_+'(k_2) = 1 + \frac{2U_0}{g}(\omega - U_0 k_2 + \frac{S}{2})$.

For resonance with dispersive wave two, we get the following equation in the positive direction

$$\frac{g}{q_0}\zeta = \int_0^\infty f_+(k)dk = 2\pi i \operatorname*{Res}_{k\to k_c = k_2} f_+(k) + 2\pi i \operatorname*{Res}_{k\to k_1} f_+(k)$$

$$= \frac{\left(D(\omega - U_0 k_c) + U_0 - ix(\omega - U_0 k_c)\right)\left(1 + \frac{2U_0}{g}\left(\omega - U_0 k_c + \frac{S}{2}\right)\right) - \frac{U_0^2}{g}(\omega - U_0 k_c)}{\left(1 + \frac{2U_0}{g}\left(\omega - U_0 k_c + \frac{S}{2}\right)\right)^2} \frac{e^{-k_c D}}{\frac{U_0 - SD}{S}} e^{ik_c x - i\omega t}$$

$$+ \frac{(\omega - U_0 k_1)e^{-k_1 D}}{\Gamma_+{}'(k_1)}\left(1 + \frac{1}{-\frac{U_0 - SD}{S}(k_1 - k_c)}\right) e^{ik_1 x - i\omega t} \qquad (10.10)$$

Where $\Gamma_+'(k_1) = 1 + \frac{2U_0}{g}(\omega - U_0 k_1 + \frac{S}{2})$.

We see from in both equation 10.9 and 10.10 that the combined waves includes a wave shift as it includes a part that is imaginary. Combined with $e^{ik_c x - i\omega t} = \cos(k_c x - \omega t) + i\sin(k_c x - \omega t)$ this will make the sine function the real part, thus creating the phase shift. The interesting thing that we see for resonance in these waves is that the theory does not give an amplitude that is infinite, as would be expected when we are using linear theory. However, it has an amplitude that increases as the distance from the source increases. In the limit where $x \to \infty$, or very far away from the source, the amplitudes will go towards infinity. This is expected due to the resonance.

The third wave can also be in resonance with the critical wave. In this situation, the group velocity of the two waves will be in opposite direction, as well as the imaginary part will be of opposite signs. Because of this linear theory will give infinite amplitude for the combined wave.

The fourth wave can also be in resonance with the critical wave. When there is resonance, both waves have a negative group velocity and their imaginary part will be negative. We get the following solution for the far-field wave system in the negative x-direction.

$$\frac{g}{q_0}\zeta = \int_{-\infty}^{0} f_-(k)dk = -2\pi i \operatorname*{Res}_{k\to k_c=k_4} f_-(k) + 2\pi i \operatorname*{Res}_{k\to k_c=k_3} f_-(k)$$

$$= -\frac{\left(-1+\frac{2U_0}{g}\left(\omega - U_0k_c - \frac{S}{2}\right)\right)\left(D(\omega - U_0k_c) - U_0 + ix(\omega - U_0k_c)\right) + \frac{U_0^2}{g}(\omega - U_0k_c)}{\left(-1+\frac{2U_0}{g}\left(\omega - U_0k_c - \frac{S}{2}\right)\right)^2}\frac{e^{k_cD}}{\frac{U_0 - SD}{S}}e^{ik_cx - i\omega t}$$

$$+\frac{(\omega - U_0k_3)e^{k_3D}}{\Gamma_-'(k_3)}\left(1 - \frac{1}{-\frac{U_0 - SD}{S}(k_3 - k_c)}\right)e^{ik_3x - i\omega t} \qquad (10.11)$$

Where $\Gamma_-'(k) = -1 + \frac{2U_0}{g}(\omega - U_0k_{3,4} - \frac{S}{2})$

Where we again see that the combined wave in this situation includes a phase shift and an amplitude that depends on x.

11. Doppler-effect resonance

Tyvand and Lepperød (2015) looked at the possibilities for resonance between the dispersive waves. They found that because the group velocity of the first and second wave are always positive and larger than zero, there can never be any resonance between these waves. This equals the fact that the square root in equation 7.3a and b will always be larger than zero. However, between the third and fourth dispersive wave there can be resonance. This is because the square root in equations 7.3c and d can be zero resulting in equal wave numbers. The third wave has a positive group velocity, while the fourth has a negative group velocity. The third wave is a little special, as it has a positive group velocity, but a negative phase velocity. This means that the group velocity of the third wave will always be in the opposite direction of the propagation of this wave. The fourth wave will always have both group and phase velocities in the negative direction. However, when the two wave numbers are equal and resonance is achieved the group velocities will be zero for both the third and the fourth wave. We know from the existence of the third and fourth wave number that the value of S^* has to be in the interval from negative one to positive one. Tyvand and Lepperød (2015) found that the resonance wave number is

$$k_3 = k_4 = \frac{\frac{1}{4}g}{U_0^2}\left(\frac{S^2U_0^2}{g^2} - 1\right) \qquad (11.1)$$

Which written dimensionless, using the same dimensionless variables as we have used before, will be

$$k_3^* = k_4^* = k_d = \frac{1}{4}((S^*)^2 - 1) \qquad (11.2)$$

Where we have defined the dimensionless Doppler wave number. We have assumed that we are in the in the region where the dispersive wave three and four exist as given in figure Y. Because S^* has to be in the interval between -1 and 1 we can quickly see that the wave number will be zero at both ends of the interval. The function is a parabola with lowest value at $S^* = 0$ where the wave number is $k_d = -0.25$. All Doppler wave numbers will be in the interval between these two intervals.

In addition, the wave frequency in this case will be

$$\omega = \frac{(g+U_0 S)^2}{4gU_0} \tag{11.3}$$

This is the wave frequency required to make the square root, which is equal for both wave number three and four in equation 7.3 c and d is zero. Thus making the wave numbers equal. Using this wave number and setting the square root equal to zero we can verify the wave numbers found by Tyvand and Ellingsen (2015)

Which dimensionless becomes

$$\omega^* = \frac{1}{4}(1+S^*)^2 \tag{11.4}$$

With the restrictions on S^*, we see that the dimensionless wave frequency is at its lowest value at the far left end of the interval. At this value, it is zero, while it rises to exactly one at the far right end. This means that Doppler Effects is only possible for dimensionless frequencies between zero and positive one. This is not surprising as the existence of the fourth dispersive wave has exactly the same requirements.

When the conditions we get from equations (11.1) to (11.4) are met, the two waves will flow together as one single wave.

It might be interesting to look at the phase velocity of this combined wave, we know from the existence of the third and fourth wave number that the value of S^* has to be in the interval from negative one to positive one. The equation for the phase velocity of the combined wave is

$$v_f^* = \frac{\omega^*}{k^*} = \frac{(1-S^*)^2}{((S^*)^2-1)} \tag{11.5}$$

Moreover, the group velocity is

$$v_g^* = \frac{d\omega^*}{dk^*} = 0 \tag{11.6}$$

Therefore, we see that this Doppler resonance occurs when the two group velocities are equal and zero as was expected.

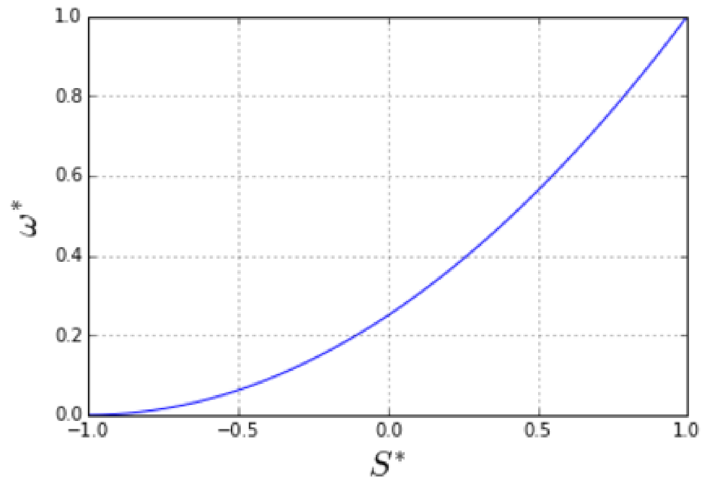

Figure 11.1: The location of the Doppler Resonance.

Shown graphically in figure 11.2 is the location of the Doppler resonance. Compared to the existence of both wave three and four this is at the upper limit of ω^* for existence for any given S^*.

When Doppler effects are present, the two waves flow together with an amplitude that goes to infinity according to linear theory.

We now want to see if there can be resonance between the Doppler wave and the critical wave.

$$k_D = k_c^* \tag{11.7a}$$

$$\frac{1}{4}\left(S^{*2} - 1\right) = \frac{\omega^*}{1 - S^* D^*} \tag{11.7b}$$

We know what the value of ω^* has to be from equation 10.4

$$\frac{1}{4}\left(S^{*2} - 1\right) = \frac{\frac{1}{4}(S+1)^2}{1 - S^* D^*} \tag{11.7c}$$

We multiply this equation with the denominator on the right side, and simplifying. Due to the multiplication, we might construct additional solutions; therefore, we must remember that $S^* > \frac{1}{D^*}$ is a requirement for the critical wave to go in the correct direction

$$S^{*3} D^* + 2S^* - S^* D^* + 2 = 0, \qquad (1 - S^* D^*) \neq 0 \tag{11.8}$$

We factorize equation 10.8, finding

$$(S^* + 1)\left(S^{*^2} D - SD + 2\right) = 0 \tag{11.9}$$

The solutions are now whenever any of these two parenthesis is zero. One possibility is that $S^* = -1$ which is not a valid solution as it does not satisfy the requirement that $S^* > \frac{1}{D^*}$.

The other solutions are found at

$$S^* = \frac{D^* \pm \sqrt{D^*(D^*-8)}}{2D^*} \qquad (11.10)$$

We see that (11.10) has real roots when the dimensionless depth of the submerged oscillatory source is larger or equal to eight. Giving one additional solution for $D^* = 8$ and two additional solutions as it increases from eight. The additional solution when $D^* = 8$ is $S = 0.5$ and as D^* increases from eight towards infinity the additional solutions goes towards zero and positive one. This means that there can be resonance between the Doppler wave and the critical wave when $D^* \geq 8$.

This makes all the waves going in the negative direction one single wave with infinite amplitude according to linear theory. For lower dimensionless depth than eight there will never be possible to get resonance between the third and fourth dispersive wave as well as the critical wave. For a dimensionless depth of eight, there will be a single point in the ω^*, S^* plane that there will be resonance between the three waves. This is for dimensionless shear flow $S^* = 0.5$ with dimensionless frequency $\omega^* = 0.5625$. For dimensionless depth larger than eight, there will be possible to get resonance for all three waves in two points in the ω^*, S^* plane as given by equations (11.10) and (11.4). Remember that the dimensionless depth is given by $D^* = \frac{g}{U_0^2} D$. This means that the depth required for resonance is larger when the surface velocity increases.

12. The origin of the critical layer

We now want to give an argument on why Laplace's equation cannot be used to solve this problem. This is because the singularity of the source creates vorticity that again creates the critical layer. This argument was given in private communication with Tyvand (2016) and tells in a relatively simple way why the critical layer exists. Earlier we looked at the manifestation of the critical layer to the surface. We will now look at what happens at the depth of the source, when the water flows through the singular source. The arguments of this section was given with zero surface velocity by Ellingsen and Tyvand (2016),

We will first do a coordinate change to (X,Z) where the source is in the origin. This means that (x,z) = (X,Z-D). We look at a very small square of water heading towards what is macroscopically a singular source. Microscopically this will be a continuous ring source. We look at a very small scale such that the shear flow can be seen as equal over the entire length of the square, and will be given as U_q. This is the shear flow at the depth of the source and its value is $U_0 - SD$. A sketch of the microscopic problem can be seen in figure 12.1.

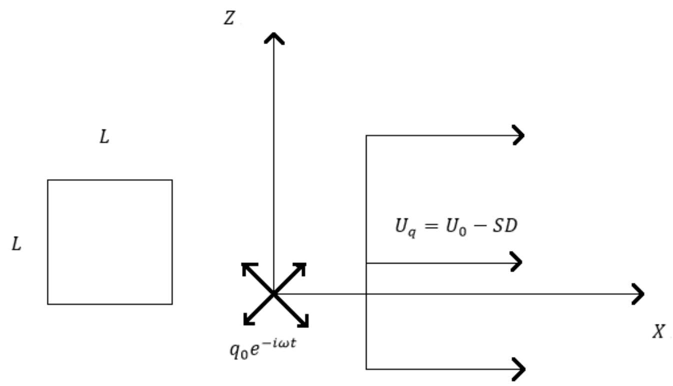

Figure 12.1: Sketch of the microscopic scale system

The length of the material square has to be small enough for the shear flow to be considered equal for the entire square. This means that the length of the material square has to be

$$L \ll \frac{U_q}{S} \qquad (12.1)$$

To make the ring source continuous in the microscopic scale, we introduce a microscopic length scale ε. The radial velocity of the ring source will now be written as

$$\frac{q_0 e^{-i\omega t}}{2\pi\varepsilon} \qquad (12.2)$$

To guarantee small enough velocities that this passes through the material curve and gets inside L^2, ,we must require that the radius of the ring source is

$$\varepsilon > \frac{q_0}{2\pi|U_q|} \qquad (12.3)$$

It is worthwhile to mention that the ring source is only recognizable in our microscopic scale. In the macroscopic scale, the source will still be a singular point. That the flow is nonsingular in this microscopic scale is necessary to guarantee that the material square will be influenced by the ring source. An additional requirement is that the length of the square has to follow: $L \ll \frac{|U_q|}{\omega}$. This is a requirement because we must assure that the time interval a contour spends within the source is much shorter than an oscillation of the source. We include the restriction stated in (12.1)

$$L \ll \frac{|U_q|}{\max(S,\omega)} \qquad (12.4)$$

We sum up the restrictions by adding the inequality (12.3) and the requirements for the local microscopic lengths becomes

$$\frac{q_0}{2\pi|U_q|} < \varepsilon \ll L \ll \frac{|U_q|}{\max(S,\omega)} \tag{12.5}$$

We have to add that $|U_q| \neq 0$ because this would mean that the shear flow at the source is zero, and no critical layer would exist. This is because the material square would not flow into the ring source, but rather stand still. The source spends only a tiny amount of time within the ring source. This means we only need to consider one instant $t = \tau$ as representative for the modification caused by the source. We choose this τ to be the midpoint of the time interval that the square is within the ring source. The material will in this time be modified by a constant area flux $q_0\cos(\omega t)$. In addition the size of the time interval is $L/|U_q|$. This means that the area of the material is changed into

$$A = L^2 + \frac{q_0 L}{|U_q|}\cos(\omega\tau)\Theta(\tfrac{L}{2} + Z)\Theta(\tfrac{L}{2} - Z) \tag{12.6}$$

Where the midpoint of the material curve we consider is called Z. This formula is only valid for the material curve that has already passed the source. This means that it is only valid for *X* satisfying $U_q > \mathrm{sign}(U_q)L/2$. The area of the material curve that has not passed the source is L^2 by definition.

Kelvin's circulation theorem now implies that

$$\Gamma = SL^2 = \Omega A \tag{12.7}$$

Where we applied Stoke's theorem for vorticity inside a small contour. Ω is the modified vorticity of the material curve caused by the source and Γ is here the circulation. Inserting the A we found in () gives us the approximate vorticity inside the material curve

$$\Omega = S(1 - \frac{q_0}{|U_q L|}\mathrm{e}^{-\mathrm{i}\omega\tau}\Theta(\tfrac{L}{2} + Z)\Theta(\tfrac{L}{2} - Z) \tag{12.8}$$

Valid to the first order of the linearization parameter $q_0/(U_q L)$. That the flux amplitude q_0 is small is a requirement in this local analysis. This is because we want *L* to approach zero, but only after q_0 has done so. This leads to a limit vorticity where

$$\Omega = S(1 - \frac{q_0}{|U_q|}\mathrm{e}^{-\mathrm{i}\omega\tau}\delta(Z)\Theta(U_q X) \tag{12.9}$$

Here we have introduced the Heaviside unit step function to identify the downstream direction, in addition the Dirac's delta function appears in the limit $L \to 0$. We now go back to $|U_q| = U_0 - SD$ and we also go back to the (x,z) coordinate system and we get the following formula for the limit vorticity

$$\Omega(x,z,t) = S(1 - \frac{q_0}{|U_0 - SD|D}\mathrm{e}^{-\mathrm{i}\omega t + \mathrm{k_c}x}\delta(\tfrac{z}{D} + 1)\Theta(((U_0 - SD)x) \tag{12.10}$$

Where again emphasize that $U_0 \neq SD$, cause then the critical layer would not exist and the potential theory used by Tyvand and Lepperød (2015) gives the correct solution. Tyvand (2016) showed that when the surface velocity is zero ($U_0 = 0$) these results coincided with the full analysis of Ellingsen and Tyvand (2016). The latter done in the same way as this article. This shows that the arguments done above has general validity in linear theory, even though we had to make quite strict restrictions.

We see that the vorticity perturbation travels with the velocity of the shear flow at the depth of the source in the direction that is downstream for the flow at this depth.

Helmholtz theorem for vorticity evolution in 2D tells us that this generated perturbation vorticity will stick with the fluid particles. This means that the vorticity is conserved in the entire fluid, except at the singular source point. Giving us a critical layer travelling with the velocity and direction of the shear flow at the depth of the source. This shows that the material square clearly gets additional vorticity from the source and proves that using Laplace's equation and potential theory will not yield the correct results in this situation.

Inserting the vorticity, we found in equation (12.10) into the vorticity equation we find that

$$\frac{D\Omega}{Dt} = \frac{\partial\Omega}{dt} + (U_0 + Sz)\frac{\partial\Omega}{\partial x} = -\frac{|U_0 - SD|}{D} S q_0 e^{-i\omega t} \delta(\frac{z}{D} + 1)\delta((U_0 - SD)x) \qquad (12.11)$$

Where we see that if $U_0 - SD = 0$ there is no change in the vorticity and potential theory gives correct solution. We see from equation 12.11 that there must be an additional vorticity created by the singular source. This means that potential theory is not possible, and we have to use the solution method given by Ellingsen and Tyvand (2016) for zero surface velocity and generalized here to include non-zero surface velocities.

13. References

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